ON TERM-BY-TERM INTEGRATION OF OSCILLATING SERIES

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1. In a previous paper published in these *Proceedings** I introduced a direct method of considering oscillating series of functions. The success of the method there adopted depended essentially on the remarkable way in which certain of the properties of a continuous function are distributed among the two wider classes of upper and lower semi-continuous functions respectively. It was this fact that rendered it possible to enunciate and prove theorems about the upper and lower functions of an oscillating series precisely corresponding to known ones for the sum-function of a convergent series. Moreover, with the introduction of the peak and chasm functions, by myself, in an earlier paper† on non-uniform convergence, it became possible to carry over not only the notion of non-uniformity of convergence, but the analytical machinery itself, to the discussion of the modes of oscillation of non-converging series.

One great advantage of the introduction of the peak and chasm functions, even when employed for the purpose of discussing converging series only, is that they enable us to distinguish both what goes on on the right and on the left, both above and below. This advantage was not exploited to the full in either of the two papers above referred to. In the present paper it will be found that these distinctions are fundamental. As regards what goes on above and below, we have interesting results with respect to each independently of the other, results therefore which from their very nature are no longer mere generalizations of theorems already known for converging series. As regards the distinction of right and left, everything turns on this in the present paper: it is more especially with the right- and left-handed peak and chasm functions that we work, and not the peak

* "On Oscillating Successions of Continuous Functions," *Proc. London Math. Soc.* (1908), Ser. 2, Vol. 6, pp. 298–328.

† "On Uniform and non-Uniform Convergence and Divergence of a Series of Continuous Functions and the Distinction of Right and Left," *Proc. London Math. Soc.* (1907), Ser. 2, Vol. 6, pp. 29–52.

and chasm functions *par excellence*, with functions, in short, whose behaviour is known on one side only.*

For the rest the importance of the distinction of right and left appears in the fundamental theorem with which the paper begins. Just as the property of a continuous function that it assumes its upper and lower bounds is divided between the upper and lower semi-continuous functions, so in the fundamental theorem in question we see that the property a continuous function has of assuming in every interval every value between its values at the ends of the interval is passed on to the functions which are lower semi-continuous on the left and upper semi-continuous on the right, when the value at the left-hand end-point is less than that at the right-hand end-point and *vice versa*, while, on the other hand, the points at which such a function is $\leqslant k$, and $\geqslant k$, form in the first class of functions a set closed on the right and closed on the left respectively, and *vice versa* for the second class of functions.

It will be found that these functions, which are at every point lower semi-continuous on one side and upper semi-continuous on the other, naturally arise in the theory of integration of the series we are considering.

The results obtained take, in the first instance, the form of inequalities, not of equalities as in the case of converging series, and thus are not directly available for purposes of calculation. In estimating the value, however, of results of this kind from the point of view of their utility in other parts of analysis, it should be borne in mind that it frequently happens that we are presented with, or confronted by, a series or an integral in the course of our work which we may, or may not, know to be converging, and which may, or may not, happen to be known to be summable in the Cesàro or Borel sense. In such a case we should be landed in an *impasse*, if our only machinery consisted of theorems which assumed the convergence of the series and integrals. That the mathematician who uses series as a mere tool will not, in the ordinary course, come across series or integrals which do not converge, is as much and as little pertinent to the question of their utility as the fact that a man who computes builder's quantities does not come across complex numbers, bears on the question of the value of these numbers for mathematical purposes.

In this connexion it should be noted that *we also adopt as a matter of course the generalised definition of integral, first given by Lebesgue.* We

* It follows from a result of mine that their behaviour is also known on the other side, except possibly at a countable set of points, see footnote to § 2.

do not in this way add to the complexity of the theory, but the reverse. On the other hand *we do not specially consider the possibility of the series oscillating more than finitely.* No fresh ideas would present themselves did we do so, and there is no difficulty in following out the consequences of allowing this possibility to present itself.

The first main result of the present paper refers to *the nature of the oscillation of the series obtained by term-by-term integration of an oscillating series of functions, and to the nature of the upper and lower functions.*

Denoting by U and L the upper and lower functions of the integrated series, and by $s_n(x)$ the sum of the first n terms of the original series, we have :

If the point P is such that in its right-hand neighbourhood $s_n(x)$, regarded as a function of x and and n, is bounded above,	then the integral series oscillates uniformly above on the right at P, so that, in particular, $U(x)$ is upper semi-continuous on the right at P;
if P is such that in its left-hand neighbourhood $s_n(x)$ is bounded above,	then the integral series oscillates uniformly below on the left at P, so that $L(x)$ is lower semi-continuous on the left at P;
if P is such that in its right-hand neighbourhood $s_n(x)$ is bounded below,	then the integral series oscillates uniformly below on the right at P, so that $L(x)$ is lower semi-continuous on the right at P;
if P is such that in its left-hand neighbourhood $s_n(x)$ is bounded below,	then the integral series oscillates uniformly above on the left at P, so that $U(x)$ is upper semi-continuous on the left at P.

The second main result of the paper refers to *the error made in taking the upper (lower) function of the integral series as representing the integral of the upper (lower) function of the original series.* Denoting these latter integrals by H and G, we have

$$U \leqslant H,$$

provided $s_n(x)$ is bounded above, while if $s_n(x)$ is bounded below,

$$G \leqslant L,$$

the error in each case increasing as x increases.

Moreover, the results still hold when points in whose neighbourhood respectively $s_n(x)$ is unbounded above or below, exist, provided they are countable in number, and provided further L and U have the characteristics which they would have if such points did not exist, viz., if U is upper semi-continuous on the right and lower semi-continuous on the left, and L is lower semi-continuous on the right and upper semi-continuous on the left.

This extended result is obtained by applying the following extension of Scheeffer's theorem, which is here proved :—

If a function which is

lower semi-continuous on the left,

and upper semi-continuous on the right,

have one of its derivates everywhere less than or equal to the corresponding derivate of a function which is

upper semi-continuous on the left,

and lower semi-continuous on the right,

except possibly at a countable set of points, the excess of the first function over the second function is a monotone non-increasing function of x throughout the interval.

It will be noted that the results given have been stated only for series, but corresponding results are, of course, true for integrals.

2. The special case *when the given series converges* is, of course, of special interest: in this case G and H coincide, but L and U do not necessarily do so, so that *the integral series need not converge.*

Whether or not the original series converge, if its partial summations are bounded above (below), the integral series, if it converges, represents a function which is upper semi-continuous on one side and lower semi-continuous on the other, and therefore is continuous except possibly at a countable set of points ; while, if the infinite peaks of the original series are all on the right (left), and the infinite chasms on the left (right), the integral series, if it converges, represents an upper (lower) semi-continuous function.

As regards the integration theorem, we may note that if both the original series and the integral series converge, provided the points of infinite non-uniform convergence of the original series are countable, *term-by-term integration is allowable only if the sum of the integral series is a continuous function.* If, for example, it is only

lower semi-continuous on the right,

and upper semi-continuous on the left,

its value will, in general, be too great; while, if it is

upper semi-continuous on the right,

and lower semi-continuous on the left,

its value will be too small.

If, on the other hand, only the original series is known to converge, and not the integral series, we shall have, provided the points of infinite non-uniform convergence of the original series are countable, *the integral of the sum-function of the original series* $= L$, *if* L *is continuous*, $= U$, *if* U *is continuous;* so that, if L and U are both continuous, the integral series is bound to converge, and term-by-term integration is allowable. The same is true if U is known to be

lower semi-continuous on the left,

and upper semi-continuous on the right,

while L is known to be

upper semi-continuous on the left,

and lower semi-continuous on the right,

so that again L and U would be equal and continuous.

Other results follow if we suppose the original series not to converge, but the integral series to do so.

3. THEOREM 1.—*If a function* $f(x)$ *is, at every point of a closed interval,*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and if the value at the left-hand end-point a *is less than the value at the right-hand end-point* b, *then* $f(x)$ *assumes* ALL *values between the two values in question.*

Let q be any value between $f(a)$ and $f(b)$ non-inclusive.

Since $f(x)$ is lower semi-continuous on the left, the points at which

$$f(x) \leqslant q$$

form a set closed on the right. Therefore there is a point at which this inequality holds, and which bounds the whole set on the right, and this point is not b, since q is less than $f(b)$; let it therefore be denoted by x_q.

It follows that at every point to the right of x_q,

$$f(x) > q,$$

so that all the right-hand limits of $f(x)$ at q are $\geqslant q$. But $f(x)$ is upper semi-continuous on the right, hence

$$f(x_q) \geqslant q;$$

but we already had $$f(x_q) \leqslant q,$$

hence $$f(x_q) = q,$$

which shews that the value q is assumed, and so proves the lemma.

Note.—Bearing in mind the fact that there is no distinction of right and left in the nature of the discontinuities of a function except at most at a countable set of points,* we see that the type of function here considered is continuous except at a countable set of points.

4. THEOREM 2.—*If $f(x)$ is, at every point of a closed interval (a, b),*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and if the value at the left-hand end-point a is less than the value at the right-hand end-point b, then the points at which any one of the four derivates of $f(x)$ is positive (not zero) have the potency of the continuum.

Let A and B be the points of the locus

$$y = f(x),$$

corresponding to the values a and b of x.

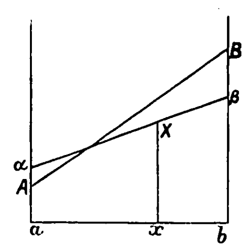

FIG. 1.

Take any point α on the ordinate of A, higher than A and lower than B, and any point β on the ordinate of B, higher than α but lower than B.

Then the intercept between this line and the locus is positive at $x = b$, and negative at $x = a$; also, since it only differs from $f(x)$ by a linear function of x, it is a function of the same type as f itself. Hence, by the preceding theorem, it assumes the value zero between a and b, and the set of these zeros

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* Cf. W. H. Young, "On the Distinction of Right and Left at Points of Discontinuity," *Quar. Jour.*, Vol. XXXIX, 1907, pp. 67–83; cf. also W. H. Young, "Sulle due funzioni a più valori costituite dai limiti d'una funzione d'una variabile reale a destra e a sinistra di ciascun punto," *Rend. d. R. Acc. dei Lincei*, 1908. Vol. XVII, Serie 5, pp. 582–7.

possesses an extreme right-hand point x, which also bounds on the right the points at which the intercept is $\leqslant 0$. Thus the locus cuts the line $\alpha\beta$ at the corresponding point X, and to the right of this point lies above that line. It follows that the limiting directions of chords of the locus through X and on the right of it are at least as steep as $\alpha\beta$, so that the right-hand derivates of $f(x)$ at this point x are positive.

Now it is evident that, corresponding to each different choice of the point β, we get a different point x, since X cannot lie on more than one line through α. Since the point β was any point on the ordinate of B, higher than α and lower than B, this shews that the potency of the points x, and therefore of the points at which either right-hand derivate is positive, is that of the continuum.

On the other hand, the zeros in question possess an extreme left-hand point x', which also bounds on the left the points at which the intercept is $\geqslant 0$. It follows that, at the corresponding point X' of the locus, the limiting directions on the left are at least as steep as $\alpha\beta$, so that the left-hand derivates of $f(x)$ at this point are positive. As before, the potency of those points x' is that of the continuum, so that the same is true of the points at which either left-hand derivate is positive.

COR.—*If $f(x)$ is, at every point of a closed interval,*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and if one of the four derivates is known to be everywhere $\leqslant 0$, except at a countable set of points, then $f(x)$ is a monotone non-increasing function of x throughout the interval, so that at the exceptional points also the derivates are $\leqslant 0$.

5. The two preceding theorems have, of course, their correlatives interchanging greater and less. These theorems, which are stated below, result most simply from those already proved by putting

$$f(x) = -F(x).$$

THEOREM 1′.—*If a function $f(x)$ is, at every point of a closed interval,*

upper semi-continuous on the left,

and lower semi-continuous on the right,

and if the value at the left-hand end-point is greater than the value at *the right-hand end-point, then $f(x)$ assumes all values between the two values in question.*

THEOREM 2′.—*Under the same hypothesis as in the preceding theorem, the points at which any one of the four derivates of $f(x)$ is negative have the potency of the continuum.*

COR.—*If $f(x)$ is, at every point of a closed interval,*

upper semi-continuous on the left,

and lower semi-continuous on the right,

and if one of the four derivates is known to be everywhere $\geqslant 0$, except at a countable set of points, then $f(x)$ is a monotone non-decreasing function of x throughout the interval, so that at the exceptional points also the derivates are $\geqslant 0$.

6. An immediate consequence of the results of the preceding articles is the following:—

The set of points, if any, at which any one of the four derivates of a continuous function is $> a$, and the set of points, if any, at which it is $< a$, have for all values of a each of them the potency of the continuum.

It will be noted that this theorem does not demand that the derivates should be bounded or even finite, and that therefore in every case the upper and lower bounds of the derivates are, whether these bounds be finite or infinite, unaltered, if in determining them we omit the derivates at a countable set of points.

7. THEOREM 3 (Extension of Scheeffer's Theorem).—*If $g(x)$ and $h(x)$ be two functions, of which the first is, throughout a given interval,*

lower semi-continuous on the left,

and upper semi-continuous on the right,

and the latter is upper semi-continuous on the left,

and lower semi-continuous on the right,

and if one of the four derivates of $g(x)$ be everywhere less than the corresponding derivate of $h(x)$, or equal to it and finite, except possibly at a countable set of points, then

$$g(x) - h(x),$$

is a monotone non-increasing function of x throughout the interval.

Put
$$f(x) = g(x) - h(x),$$
so that $f(x)$ is lower semi-continuous on the left,

and upper semi-continuous on the right.

Then, since denoting by $f^+(x)$ and $f_+(x)$ the upper and lower right-hand derivates of $f(x)$,
$$f_+(x) \leqslant g_+(x) - h_+(x),$$
and also
$$\leqslant g^+(x) - h^+(x),$$
with similar inequalities on the left, we see that, whether it is the upper or lower derivates alluded to in the statement of the theorem, we have, in either case,
$$f_+(x) \leqslant 0,$$
except at a doubtful countable set, whence the result follows by Theorem 2, Cor.

8. LEMMA.—*If $S(x)$ denote the integral (generalised or Lebesgue integral) of $s(x)$, the upper and lower bounds of the derivates of $S(x)$ in any interval lie between the upper and lower bounds of $s(x)$ in the same interval*

For
$$\frac{S(x+h) - S(x)}{h} = \frac{1}{h} \int_x^{x+h} s(x)\, dx,$$
so that, if B is the upper bound of $s(x)$ in the interval considered, the left-hand side of the preceding equation is less than or equal to the right-hand side when we replace $s(x)$ by B. That is,
$$\frac{S(x+h) - S(x)}{h} \leqslant B.$$

Hence the upper bound of the derivates of $S(x)$, being the same as that of the incrementary ratio on the left of the preceding inequality, is less than or equal to B. This proves the statement about the upper bounds, and similar reasoning proves the statement about the lower bounds.

COR.—*Hence, for each fixed value of n, the upper bound of $s_n(x)$ is greater than or equal to the upper bound of the derivates of $S_n(x)$.*

Hence also $s_n(x)$ regarded as a function of x and n has necessarily an upper bound which is not less than that of the derivates of $S_n(x)$, when these derivates are regarded as functions of x and n.

9. THEOREM 4.—*If a sequence of functions of a single variable x, $s_1, s_2, \ldots$ is such that $s_n(x)$ regarded as a function of n and x for all values*

of x in a certain closed interval (a, b), and all positive integral values of n has a finite upper bound, then the corresponding sequence of integrals, taken from a to x, viz., the continuous functions S_1, S_2, ... oscillates uniformly above on the right and uniformly below on the left, so that in particular its upper function U is upper semi-continuous on the right, and its lower function L is lower semi-continuous on the left.

A corresponding statement, mutatis mutandis, holds, of course, when $s_n(x)$ has a finite lower bound.

By the definition of the right-hand peak function of S_1, S_2, ..., it follows that, if P were a point at which it were greater than $U(x)+A$, there would be a sequence of points to the right of P, x_1, x_2, ... with P as limiting point, and a corresponding sequence of constantly increasing integers n, such that

$$S_{n_i}(x_i) > U(x)+A. \tag{1}$$

But, from the definition of the upper function $U(x)$, it follows that we can determine an integer m, such that, for all values of $n \geqslant m$,

$$U(x) > S_n(x)-\tfrac{1}{2}A,$$

so that

$$S_{n_i}(x_i) > S\ \ (x)+\tfrac{1}{2}A;$$

and therefore, by the Theorem of the Mean, if applicable*

$$(x_i-x)\frac{d}{dy}S_{n_i}(y) > \tfrac{1}{2}A, \tag{2}$$

at some point $y = y_i$ internal to the interval (x, x_i).

That is, since x_i-x is positive,

$$\frac{d}{dy_i}S_{n_i}(y_i) > \frac{\tfrac{1}{2}A}{x_i-x},$$

where y_1, y_2, ..., like x_1, x_2, ..., form a sequence on the right of x, having x as limit.

If the Theorem of the Mean is not applicable, we only have to interpret the symbol $\frac{d}{dy_i}S_{n_i}(y_i)$ to mean one of the derivates of S_{n_i}, and the same holds.†

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* For instance, if $s_n(x)$ is a continuous function of x.

† E. W. Hobson, *Functions of a Real Variable*, § 219, p. 290, Camb. Univ. Press, 1907. W. H. and G. Chisholm Young, "On Derivates and the Theory of the Mean," *Quar. Jour.*, 1908, p. 12, *et alt. loc.*

Hence, in the neighbourhood of the point P this differential coefficient, or derivate, $\frac{d}{dx}S_n(x)$, regarded as a function of n and x, has no finite upper bound, so that, by the lemma, $s_n(x)$, regarded as a function of n and x, has in the neighbourhood of the point P, no finite upper bound, contrary to the data.

Hence for no positive value of A does the inequality (1) hold, so that the right-hand peak function is not greater than the upper function. But the right-hand peak function is never less than the upper function, whence the right-hand peak function is equal to the upper function.

Q. E. D.

Since the right-hand peak function is upper semi-continuous on the right, it follows, in particular, that the same is true of the upper function $U(x)$. Q. E. D.

Similarly, if there were a point at which the chasm function were less than $L(x)-A$, where $L(x)$ is the lower function, we should at such a point have a sequence of n's for which

$$S_{n_i}(x_i) < L(x)-A,$$

whence, as before, we get

$$(x_i-x)\frac{d}{dy_i}S_{n_i}(y_i) < -\tfrac{1}{2}A\ ;$$

and therefore, bearing in mind that (x_i-x) is negative,

$$\frac{d}{dy_i}S_{n_i}(y_i) > \frac{\frac{1}{2}A}{x-x_i}.$$

Thus, as before, we get to a contradiction, and so prove the remainder of the statement of the theorem. Q. E. D.

COR.—*If the functions $s_n(x)$ are continuous functions of x, and the sequence has no points at which the peak (chasm) function is infinite, the statement in the theorem holds.*

For, by the definition of the peak function, it is the limit, when the interval PQ shrinks up to the point P, of M_Q, where M_Q is the upper limit of $M_{n,Q}$, the upper bound of $s_n(x)$ in the interval PQ. Hence it follows that round each point x we can, if the peak function is always finite, describe an interval throughout which $s_n(x)$ is less than B, if B is greater than the upper bound of the peak function, provided $n \geqslant m$, where m is a certain integer, varying in general with the point P.

By the Heine-Borel theorem a finite number of these intervals suffices to cover the whole closed continuum in which we are working. Choosing

M to be the largest of the corresponding values of m, $s_n(x)$ is less than B, for all integers $n \geqslant M$, and all points x of the closed continuum considered. Hence, since $s_1, s_2, \ldots, s_{M-1}$ are bounded functions, $s_n(x)$, regarded as a function of n and x, has a finite upper bound, and therefore the result of the theorem holds.

10. The preceding theorem deals with uniform oscillation and semi-continuity throughout an interval, but it is clear from the proof that we can enunciate a corresponding theorem which deals with these concepts at a point only.

THEOREM 5.—*If a sequence of functions of a single variable $s_1, s_2, \ldots,$ be such that $s_n(x)$, regarded as a function of n and x, has a finite upper double limit at the point P on the right (left), then the corresponding sequence of integrals oscillates uniformly above (below) on the right (left) at the point P.*

COR.—*If the functions $s_n(x)$ are continuous functions of x, and the right (left) hand peak function is finite at P, the sequence of integrals oscillates uniformly above (below) on the right (left) at the point P.*

We have here two alternate statements, and we get two more by assuming at P a finite lower double limit, or chasm function, interchanging the words above and below, viz.,

THEOREM 5′.—*If $s_n(x)$ has a finite lower double limit at the point P on the right (left), then the corresponding sequence of integrals oscillates uniformly below (above) on the right (left) at the point P.*

COR.—*If the functions $s_n(x)$ are continuous, and the right (left) hand chasm function is finite at P, the sequence of integrals oscillates uniformly below (above) on the right (left) at the point P.*

Corresponding to these results, we have in particular the following table, giving the characteristics of the upper and lower functions of the integrals.

When there are no points at which

	$\pi_R = +\infty$,	$\Pi_R = U$	*is upper*	*semi-continuous on*	*right;*
„	$\pi_L = +\infty$,	$X_L = L$	„ *lower*	„	*left;*
„	$X_R = -\infty$,	$X_R = L$	„ *lower*	„	*right;*
„	$X_L = -\infty$,	$\Pi_L = U$	„ *upper*	„	*left.*

11. THEOREM 6.—*If the partial summations $s_n(x)$, regarded as functions of n and x, have a finite upper (lower) bound, the limits of their integrals are less (greater) than or equal to the integral of the upper (lower) function.*

We proceed to give the proof of the first of these alternative theorems; the proof of the other merely requires the change of "lower" into "upper" and "less" into "greater."

For, denoting by w_n the function whose value at any point is the upper bound of all the functions $s_n(x)$, $s_{n+1}(x)$, ..., we clearly have

$$s_n(x) \leqslant w_n(x), \tag{1}$$

so that

$$\int s_n(x)\,dx \leqslant \int w_n(x)\,dx;$$

and therefore

$$\text{upper limit of}_{n=\infty} \int s_n(x)\,dx \leqslant \text{upper limit of}_{n=\infty} \int w_n(x)\,dx. \tag{2}$$

But the quantities $w_n(x)$ form a monotone decreasing sequence, and their integrals are finite, since $w_n(x)$ is a bounded function of x; for, by (1) the lower bound of w_n is not less than that of s_n and is therefore finite, and, since $s_n(x)$, regarded as a function of n and x, is, by hypothesis, less than a finite quantity, say B, it follows from the definition that $w_n(x)$ is also always less than or equal to B.

Now, the generalised integral of the limit of a monotone sequence is the unique limit of the integrals of the constituent functions, so that

$$\operatorname*{Lt}_{n=\infty} \int w_n(x)\,dx = \int u(x)\,dx,$$

since $u(x)$ is the limit of the sequence $w_1(x)$, $w_2(x)$, ...

Hence, by (2), upper limit of $\int s_n(x)\,dx \leqslant \int u(x)\,dx$,

which proves the theorem.*

COR. 1.—*If the functions $s_n(x)$ are continuous, and the sequence has no points at which the peak (chasm) function is infinite positively (negatively), the result of the theorem holds*

* The following example shews that, without further assumptions, the theorem states the utmost that can be proved. Divide the interval (0, 1) by continued bisection and call the successive intervals d_1, d_2, ... Let $f_n(x) = 1$ at every point of the closed interval d_n, and be zero elsewhere. Here $u(x) = 1$, but $U(x) = L(x) = 0$. Here $f^n(x)$ is discontinuous, but can easily be made continuous by rounding off the end-points of d_n suitably.

COR. 2.—*If both the peak and the chasm functions are bounded, the limits of the integrals lie between the integrals of the lower and upper functions.*

This statement, which is an immediate result of the above theorem, can be proved independently by a method already employed to prove the corresponding result* in the case of convergent series, using one of my theorems in the theory of sets of points.†

12. In Theorems 4 and 5 we have shewn that the upper function of the integrals $S_n(x)$ is, under the conditions there enumerated, upper semi-continuous on the right. If we suppose further that the upper function of the original sequence has a finite integral,‡ we can go a step further and assert that the upper function U of the integrals is not only upper semi-continuous on the right, but also lower semi-continuous on the left.

We have, in fact, the following theorem :—

THEOREM 7.—*If in addition to the condition of Theorem 4, that $s_n(x)$ has a finite upper bound, we assume that the upper function $u(x)$ has a finite integral, then the upper function of the integrals, $U(x)$, is lower semi-continuous on the left as well as upper semi-continuous on the right, and the same is true of the lower function $L(x)$.*

If on the other hand the lower function $l(x)$ has a finite integral, while $s_n(x)$ has a finite lower bound, the lower function of the integrals $L(x)$ is upper semi-continuous on the left and lower semi-continuous on the left, and the same is true of the upper function.

If therefore $s_n(x)$ is bounded both above and below, both $U(x)$ and $L(x)$ are continuous functions.

For

$$\text{lower limit of } (A-B) \leqslant \text{upper limit of } A - \text{upper limit of } B$$
$$\leqslant \text{upper limit of } (A-B),$$

since we can get a limit of $(A-B)$ which is greater (less) than the middle member of this inequality, by proceeding along a sequence which gives for A (or B) its upper limit.

* E. W. Hobson, *Theory of Functions of a Real Variable*, pp. 539 *et seq.*

† *Idem*, § 93, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 25.

‡ If the upper function u had infinite negative discontinuities, as may well be the case even when the sequence oscillates finitely everywhere, and $s_n(x)$ has a finite upper bound, u might not have a finite integral. As to this restriction see, however, a forthcoming paper by the author "On Homogeneous Oscillation of Successions of Functions," where also further results in the theory of the integration of oscillating series will be found.

Hence

$$U(x)-U(x-h) \leqslant \text{upper limit of}_{n=\infty} \int_{x-h}^{x} s_n(x)\,dx \leqslant \int_{x-h}^{x} u(x)\,dx,$$

the last inequality resulting from the above theorem.

This shews that, as h decreases towards zero, $U(x)-U(x-h)$ has no positive limit, so that $U(x)$ is, as stated, lower semi-continuous on the left.

Again,

$$\text{lower limit of } (A-B) \leqslant \text{lower limit of } A - \text{lower limit of } B$$
$$\leqslant \text{upper limit of } (A-B).$$

Whence

$$L(x+h)-L(x) \leqslant \text{upper limit of}_{n=\infty} \int_{x}^{x+h} s_n(x)\,dx \leqslant \int_{x}^{x+h} u(x)\,dx,$$

which shews that $L(x)$ is upper semi-continuous on the right. Q.E.D.

Similarly the alternative statements may be proved.

13. In the case *when the functions* $s_n(x)$ *are continuous*, the points at which there is non-uniform oscillation either above or below form a set of the first category, so that *the points at which the peak and chasm functions are infinite,* which form a closed set, are dense nowhere.*

Hence the closed intervals throughout which the peak and chasm functions are bounded fill up a set of open intervals dense everywhere. It follows at once from the preceding article that *throughout the interior of this set of intervals dense everywhere both L and U are continuous, and the integral series oscillates uniformly both above and below.*

14. THEOREM 8.—*If there is at most a countably infinite number of points in the neighbourhood of which* $s_n(x)$, *regarded as a function of x and n, has no finite upper bound,*† *and if further* $U(x)$ *is the same type of function as it would be if there were no such points,*‡ *viz., if* $U(x)$ *be lower semi-continuous on the left and upper semi-continuous on the right everywhere, then*

$$\int u(x)\,dx - U(x)$$

is a positive non-decreasing function of x.

* Or $\geqslant k$, or $\leqslant k$.

† In particular this condition is fulfilled if we know that $s_n(x)$, regarded as a function of n and x, has at each point a finite upper double limit on the right or on the left; in other words, if there are no points in the neighbourhood of which $s_n(x)$ is unbounded above on *both* sides.

‡ Supposing $\int u(x)\,dx$ finite.

It is obvious that the countable set of exceptional points form a closed set. Hence, taking any point x not belonging to that set, it is internal to an interval throughout which the conditions of Theorem 7 hold. As in the proof of Theorem 7, we obtain the inequalities

$$\frac{U(x)-U(x-h)}{h} \leqslant \frac{1}{h}\int_{x-h}^{x} u(x)\,dx,$$

and

$$\frac{U(x+h)-U(x)}{h} \leqslant \frac{1}{h}\int_{x}^{x+h} u(x)\,dx,$$

the point $x-h$ and the point $x+h$ both lying in this interval.

Hence at such a point x each derivate of $U(x)$ is less than or equal to the corresponding derivate of $\int u(x)\,dx$.

Hence, by Theorem 3, the required result follows.

The alternative theorem, in which, instead of making the above assumption about $U(x)$, we assume that $L(x)$ is the same type of function as it would be if there were no such exceptional points, states, of course, that*

$$\int u(x)\,dx - L(x)$$

is a positive non-decreasing function of x.

If, on the other hand, we assume that the lower, instead of the upper, bound of $s_n(x)$, regarded as a function of n and x, is finite except at points not having the potency of the continuum, then according as $U(x)$ or $L(x)$ is the same type of function as it would be if there were no exceptional points,† viz., upper semi-continuous on the left and lower semi-continuous on the right everywhere, we get

$$U(x)-\int l(x)\,dx,$$

or

$$L(x)-\int l(x)\,dx$$

is a positive non-decreasing function of x.

Both these possibilities may, of course, take place simultaneously.

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* Supposing $\int u(x)\,dx$ finite.

† Supposing $\int l(x)\,dx$ finite.

Combining these theorems, we obtain the following corollary:—

If there is at most a countable number of points in the neighbourhood of which $s_n(x)$, regarded as a function of n and x, has not both its upper and its lower bound finite, and if, further, $U(x)$ or $L(x)$, or both, is the same type of function as it would be if there were no exceptional points, viz., continuous everywhere, then*

$$\int u(x)\,dx - U(x) \quad and \quad U(x) - \int l(x)\,dx,$$

or

$$\int u(x)\,dx - L(x) \quad and \quad L(x) - \int l(x)\,dx,$$

or all these, are positive non-decreasing functions of x. Hence, in particular, $U(x)$ or $L(x)$ or both, lie between the integrals of the upper and lower functions $u(x)$ and $l(x)$.

15. Denoting $\int l(x)\,dx$ by G and $\int u(x)\,dx$ by H, we easily see that any one of the following scheme of inequalities may hold:—

$$G < L < U < H,$$
$$G < L < H < U,$$
$$G < H < L < U,$$
$$L < G < U < H,$$
$$L < G < H < U,$$
$$L < U < G < H,$$

where the sign of inequality may anywhere become the sign of equality.

16. From the preceding theorems we at once deduce the following, which, on account of its importance, is here stated as a separate theorem.

THEOREM 9.—*If the points in the neighbourhood of which $s_n(x)$, regarded as a function of x and n, is unbounded, are at most countably infinite, and if $u(x)$ and $l(x)$ differ only at a set of content zero, while $U(x)$ and $L(x)$ are continuous, the integral series converges to the continuous function constituted by the common integral of $l(x)$ and $u(x)$.*

* Supposing $l(x)$ and $u(x)$ to have finite integrals, which will of itself be fulfilled if both bounds are finite.

This result is, of course, a very special case of those we have just given; it includes, however, all previous results of the kind given by other writers.

We notice, moreover, that in this case, when $u(x)$ and $l(x)$ differ only at a set of content zero, if the points in the neighbourhood of which $s_n(x)$, regarded as a function of x and n, is unbounded, have the potency of the continuum, but those in the neighbourhood of which the upper bound is $+\infty$ are not of this potency, then the result of term-by-term integration, if not what we should wish it to be, is too small. On the other hand, if those points where the lower bound is $-\infty$ are not of the potency of the continuum, the result, if not the desired one, is too great.

It is only when both these types of points have the potency of the continuum that we can assert nothing.