The Spherical Catenary. *By* A. G. GREENHILL.

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The properties of the Spherical Catenary, the curve assumed by a chain wrapped on a globe or resting in a spherical bowl, have been investigated by

- Minding, *Crelle*, 11 and 12;
- Gudermann, *Crelle*, 33, "De curvis catenariis sphæricis dissertatio";
- Biermann, "Problemata quædam mechanica functionum ellipticarum ope soluta." Dissertatio inauguralis, 1865;
- Clebsch, *Crelle*, 57, "Ueber die Gleichgewicht eines biegsamen Fadens";
- Fischer, "Die Kettenlinie auf der Kügel," Brill's *Catalogue of Mathematical Models*, No. 156;
- Max Schlegel, *Jahresbericht der K. Wilhelms Gymnasium in Berlin*, 1884;
- Appell, *Bulletin de la Société Mathématique de France*, III., 1884, *Traité de mécanique rationnelle*, I., p. 202.;
- Routh, *Analytical Statics*, 1891;
- Venske, "Behandlung einiger Aufgaben der Variationsrechnung." Inaugural-Dissertation, Göttingen, 1891.
- Marcolongo, *Rendiconti della R. Accademia della Scienze Fisiche e Matematiche*, Napoli, 1892.

The object of the present paper is to introduce a special form of the elliptic integral of the third kind, required in the solution of this problem, and to discuss the particular cases which arise when this integral becomes *pseudo-elliptic*, in consequence of the parameter being made equal to an aliquot part of the periods.

In this manner the only elliptic transcendent which remains in the solution is the elliptic integral of the first kind; and, when by a special numerical choice of the constants this term can be made to disappear, the spherical catenary becomes a closed algebraical curve.

1. Suppose the chain is wrapped upon a terrestrial globe, suspended from its North Pole; then the general equation connecting ψ, the longitude, with z, the sine of the latitude (south), can be expressed by the integral

$$\psi = \int \frac{A dz}{(1-z^2)\sqrt{Z}} \qquad (1),$$

where

$$Z = (1-z^2)(h-z)^2 - A^2 \qquad (2),$$

and A, h are the arbitrary constants of the problem.

For, if T denotes the tension of the chain, in gravitation measure, w its weight per unit length,

$$T = w(h-z) \qquad (3),$$

where h denotes the depth below the centre of the sphere of the *directrix plane* (Routh, *Analytical Statics*, I., p. 357), the tension at any point being equal to the weight of the length of the chain which will reach the directrix plane hanging vertically downwards.

Again, the moment of the tension round the vertical diameter being constant,

$$Tr^2 \frac{d\psi}{ds} \text{ is constant } = wA, \text{ suppose} \qquad (4),$$

where s denotes the length of the chain measured from a fixed point, and r denotes the distance from the vertical diameter; so that

$$r^2 + z^2 = 1 \qquad (5),$$

if the radius of the sphere is taken as unity.

Taking equation (4), which holds for any system of forces which have no moment about the axis Oz, it may be transformed into

$$\frac{Tr^2}{w^2A^2} = \frac{ds^2}{r^2 d\psi^2} = 1 + \frac{dr^2 + dz^2}{r^2 d\psi^2} = 1 + \left(\frac{dr^2}{dz^2} + 1\right)\frac{dz^2}{r^2 d\psi^2},$$

or

$$\frac{d\psi^2}{dz^2} = \frac{\left(\frac{dr^2}{dz^2} + 1\right)A^2}{r^2 Z} \qquad (6),$$

where

$$Z = \frac{T^2 r^2}{w^2} - A^2 \qquad (7),$$

equations suitable for any surface of revolution.

In the special case of the sphere given by equation (5),

$$\frac{dr^2}{dz^2}+1=\frac{1}{r^2} \qquad (8),$$

so that, in conjunction with the value of T in a field of gravity given by (3), equations (6) and (7) become

$$\frac{d\psi^2}{dz^2}=\frac{A^2}{r^4 Z}=\frac{A^2}{(1-z^2)^2 Z},$$

where $$Z=(1-z^2)(h-z)^2-A^2,$$

as in (1) and (2).

2. Also $$\frac{ds}{dz}=\frac{ds}{d\psi}\frac{d\psi}{dz}=\frac{Tr^3}{wA}\frac{\sqrt{\left(\frac{dr^2}{dz^2}+1\right)}A}{r\sqrt{Z}}$$

$$=\frac{Tr}{w}\frac{\sqrt{\left(\frac{dr^2}{dz^2}+1\right)}}{\sqrt{Z}} \qquad (9),$$

reducing for the spherical catenary to

$$\frac{ds}{dz}=\frac{h-z}{\sqrt{Z}} \qquad (10),$$

so that the curve is rectified by the integral

$$s=\int\frac{h-z}{\sqrt{Z}}\,dz \qquad (11).$$

3. If ϕ denotes the angle at which the curve crosses a parallel of latitude, on any surface of revolution,

$$\cos\phi=\frac{r\,d\psi}{ds}=\frac{wA}{Tr} \qquad (12),$$

so that, from (7), $$\sin\phi=\frac{w\sqrt{Z}}{Tr} \qquad (13),$$

$$\tan\phi=\frac{\sqrt{Z}}{A} \qquad (14).$$

The angle ϕ is thus a maximum when $dZ/dz=0$; this leads in the spherical catenary from equation (2) to

$$z=h, \quad \text{or} \quad 2z^2-hz-1=0 \qquad (15).$$

4. Denoting by R the pressure per unit length on the *outer* surface of the sphere, the equations of equilibrium of the chain may be written

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right)+Rx \quad = 0 \qquad (16),$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right)+Ry \quad = 0 \qquad (17),$$

$$\frac{d}{ds}\left(T\frac{dz}{ds}\right)+Rz+w = 0 \qquad (18),$$

where

$$x^2 + y^2 + z^2 = 1 \qquad (19),$$

$$x\frac{dx}{ds}+y\frac{dy}{ds}+z\frac{dz}{ds} = 0 \qquad (20),$$

$$x\frac{d^2x}{ds^2}+y\frac{d^2y}{ds^2}+z\frac{d^2z}{ds^2} = -1 \qquad (21),$$

so that, multiplying (16) by x, (17) by y, and (18) by z, and adding,

$$-T+R+wz = 0,$$

or

$$R = T-wz = w(h-2z) \qquad (22).$$

The pressure R thus changes sign at a depth below the centre greater than $\frac{1}{2}h$; the chain must then be supposed to rest on the interior of the sphere, if it cannot be made to adhere to the exterior surface.

5. Clebsch has shown, in *Crelle* 57, how the quartic Z in (2) can be exhibited as the product of two quadratic or four linear factors, of the form

$$Z = -(z^2+2kz\sin^2\epsilon+k^2\sin^2\epsilon-\cos^2\epsilon)(z^2-2kz\cos^2\epsilon+k^2\cos^2\epsilon-\sin^2\epsilon) \qquad (23),$$

$$= -\left\{z+k\sin^2\epsilon\pm\cos\epsilon\sqrt{(1-k^2\sin^2\epsilon)}\right\}$$
$$\times\left\{z-k\cos^2\epsilon\pm\sin\epsilon\sqrt{(1-k^2\cos^2\epsilon)}\right\} \qquad (24),$$

the arbitrary constants h and A being replaced by k and ϵ, such that

$$h = k\cos 2\epsilon \qquad (25),$$

$$A = \tfrac{1}{2}(1-k^2)\sin 2\epsilon \qquad (26),$$

$$A^2-h^2 = (\cos^2\epsilon-k^2\sin^2\epsilon)(\sin^2\epsilon-k^2\cos^2\epsilon) \qquad (27),$$

and thence the solution of the problem can be given by means of the Jacobian elliptic functions, the integral in equation (1) being composed of two elliptic integrals of the third kind.

By making

$$k \sin \epsilon = 1, \quad \text{or} \quad k \cos \epsilon = 1 \tag{28},$$

two of the roots of the quartic Z become equal, and the elliptic integrals degenerate into circular integrals; in this manner the model No. 156 in the mathematical collection of Brill, of Darmstadt, constructed by Herr Fischer, has been designed.

6. But it is the object of the present paper to bring out the connexion between the integral (1) and the standard form of the elliptic integral of the third kind, employed in my paper on "Pseudo-Elliptic Integrals," *Proc. Lond. Math. Soc.*, Vol. xxv., expressed by the notation, slightly altered by the omission of μ,

$$I(v) = \tfrac{1}{2} \int \frac{\rho (s-\sigma) - \surd(-\Sigma)}{(s-\sigma)\surd S}\, ds \tag{29},$$

where

$$M^2 (s-\sigma) = \wp u - \wp v \tag{30},$$

and

$$M^{-6} \wp'^2 u = S = 4s(s+x)^2 - \{(1+y)s + xy\}^2 \tag{31},$$

$$M^{-6} \wp'^2 v = \Sigma = 4\sigma(\sigma+x)^2 - \{(1+y)\sigma + xy\}^2 \tag{32},$$

x and y being the quantities employed by Halphen (*F.E.*, I., p. 102).

It is our object also to utilize the *pseudo-elliptic* integrals for the construction of degenerate, algebraical, cases of the spherical catenary.

Putting

$$\psi - pu = \chi \tag{33},$$

where p is constant, and

$$u = \int \frac{dz}{\surd Z} \tag{34},$$

the associated elliptic integral of the first kind, so that

$$\chi = \int \frac{A - p(1-z^2)}{(1-z^2)\surd Z}\, dz \tag{35},$$

then it will be shown in the sequel (§ 8) that the integrals (1) and (35) can be made to depend upon the integral (29) by putting

$$A = M(y+1) \tag{36},$$

where

$$M^2 = -\frac{y+1}{2x} \tag{37},$$

and

$$A^2 - h^2 = 2y + 1 \tag{38},$$

$$h^2 = -\frac{(y+1)^3}{2x} - 2y - 1 \tag{39}.$$

We then find that

$$p = \tfrac{1}{2}(M\rho + A) = \tfrac{1}{2}M(\rho + y + 1) \tag{40}$$

and

$$\sigma = 0 \tag{41}$$

so that, when the integral (29) is pseudo-elliptic and the parameter v is an aliquot part, one μ^{th}, of a period, we may put

$$v = \frac{4\omega_3}{\mu} \tag{42}$$

the parameter

$$v = \frac{2\omega_3}{\mu} \tag{43}$$

corresponding to

$$\sigma = -x \tag{44}$$

7. Writing equation (1) in the form

$$\psi = \psi_1 - \psi_2 \tag{45}$$

where

$$\psi_1 = \tfrac{1}{2}A\int\frac{dz}{(\ 1-z)\sqrt{Z}} \tag{46}$$

$$\psi_2 = \tfrac{1}{2}A\int\frac{dz}{(-1-z)\sqrt{Z}} \tag{47}$$

shows that ψ is given by the difference of two elliptic integrals of the third kind, with Jacobian parameters v_1 and v_2, such that

$$u = v_1, \quad \text{when} \quad z = +1;$$

$$u = v_2, \quad \text{when} \quad z = -1;$$

and Legendre's theorem for the addition of these integrals shows that ψ can be made to depend upon an elliptic integral of the third kind with parameter

$$v = v_1 - v_2 \tag{48}$$

The parameters v_1 and v_2, and therefore also v, are each of the form $f\omega_3$, fractions of the imaginary period ω_3; because the real roots of Z must lie between ± 1; and

$$z = \pm 1 \text{ makes } Z = -A^2.$$

It will also be found that

$$z = h \text{ corresponds to } u = \tfrac{1}{2}(v_1 + v_2).$$

8. Comparing the general quartic

$$Z = az^4 + 4bz^3 + 6cz^2 + 4dz + e \tag{49}$$

and its invariants

$$g_2 = ae - 4bd + 3c^2 \quad (50),$$

$$g_3 = ace + 2bcd - ad^2 - eb^2 - c^3 \quad (51),$$

with our quartic Z, as given in equation (2),

$$a = -1,\ b = \tfrac{1}{2}h,\ c = \tfrac{1}{6}(1-h^2),\ d = -\tfrac{1}{2}h,\ e = h^2 - A^2 \quad (52),$$

and we thus find

$$12g_2 = \quad (1-h^2)^2 + 12A^2 \quad (53),$$

$$216g_3 = -(1-h^2)^3 - 18(1-h^2)A^2 + 54A^2 \quad (54),$$

$$1728\Delta = \quad (12g_2)^3 - (216g_3)^2$$

$$= 108A^2\{(1-h^2)^3 - (8+20h^2-h^4)A^2 + 16A^4\} \quad (55).$$

Next taking the cubic S of equation (31), and reducing it to the form in which the coefficient of t^2 is zero,

$$4t^3 - \gamma_2 t - \gamma_3 \quad (56),$$

by putting $$s = t - \tfrac{1}{12}\{8x - (y+1)^2\} \quad (57),$$

$$12\gamma_2 = \{(y+1)^2 + 4x\}^2 - 24x(y+1) \quad (58),$$

$$216\gamma_3 = \{(y+1)^2 + 4x\}^3 - 36x(y+1)\{(y+1)^2 + 4x\} + 216x^2 \quad (59),$$

We can now make $g_2 = M^4\gamma_2,\quad g_3 = M^6\gamma_3$;

and therefore, from (34),

$$u = \int \frac{dz}{\sqrt{Z}} = \frac{1}{M}\int \frac{ds}{\sqrt{S}} \quad (60),$$

on comparison of (53) and (58), (54) and (59), by taking

$$1 - h^2 = -M^2\{(y+1)^2 + 4x\} \quad (61),$$

$$A^2 = -2M^4x(y+1) \quad (62),$$

$$A^2 = 4M^6x^2 \quad (63),$$

provided that $$M^2 = -\frac{y+1}{2x} \quad (37);$$

and therefore $$A^2 = -\frac{(y+1)^3}{2x} = M^2(y+1)^2 \quad (36),$$

$$1 - h^2 = \frac{(y+1)^3}{2x} + 2(y+1) \quad (64),$$

$$A^2 - h^2 = 2y + 1 \quad (38).$$

9. Next, let u_1, u_2 denote the values of u corresponding to values z_1, z_2 of z; then, according to Weierstrass's important formula, first published in Biermann's thesis "Problemata quædam mechanica functionum ellipticarum ope soluta," 1865,

$$\wp(u_1 \pm u_2) = \frac{F(z_1, z_2) \mp \sqrt{Z_1}\sqrt{Z_2}}{2(z_1 - z_2)^2} \qquad (65),$$

where
$$F(z_1, z_2) = (az_1^2 + 2bz_1 + c)\,z_2^2 + 2(bz_1^2 + 2cz_1 + d)\,z_2 + cz_1^2 + 2dz_1 + e$$
$$= (az_2^2 + 2bz_2 + c)\,z_1^2 + 2(bz_2^2 + 2cz_2 + d)\,z_1 + cz_2^2 + 2dz_2 + e \qquad (66).$$

Therefore, putting $z_1 = 1$, $z_2 = -1$ in our special form of Z,

$$F(1, -1) = a - 2c + e$$
$$= -\tfrac{4}{3}(1 - h^2) - A^2 \qquad (67),$$

$$\sqrt{Z_1} = \sqrt{Z_2} = Ai \qquad (68),$$

and
$$\wp(v_1 - v_2) = \frac{-\tfrac{4}{3}(1 - h^2) - A^2 - A^2}{8}$$
$$= -\tfrac{1}{6}(1 - h^2) - \tfrac{1}{4}A^2 \qquad (69),$$

$$\wp(v_1 + v_2) = -\tfrac{1}{6}(1 - h^2) \qquad (70).$$

We also find
$$i\wp'(v_1 - v_2) = \tfrac{1}{4}A(A^2 - h^2 - 1) \qquad (71),$$
$$i\wp'(v_1 + v_2) = \tfrac{1}{2}Ah \qquad (72).$$

It is convenient to denote $v_1 - v_2$ by v, and $v_1 + v_2$ by w; and now equations (36), (61), and (69) show that

$$\frac{12\wp v}{M^2} = -2\,\frac{1 - h^2}{M^2} - 3\,\frac{A^2}{M^2}$$
$$= 2(y+1)^2 + 8x - 3(y+1)^2$$
$$= 8x - (y+1)^2 \qquad (73).$$

But equation (57) shows that the relation between s, t, and $\wp u$ is

$$\frac{\wp u}{M^2} = t = s + \tfrac{1}{12}\{8x - (y+1)^2\} \qquad (74),$$

so that, as σ denotes in (29) the value of s corresponding to

$$u = v = v_1 - v_2,$$

therefore
$$\sigma = 0 \qquad (41).$$

Thus, if the integral in (29) is a *pseudo-elliptic* integral, we may put

$$v = \frac{4\omega_3}{\mu} \tag{42},$$

Also, since $s = -x$ then corresponds to $u = \frac{1}{2}v$ (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 205), therefore

$$\wp\tfrac{1}{2}v = \tfrac{1}{12}(1-h^2) \tag{75},$$

$$i\wp'\tfrac{1}{2}v = \tfrac{1}{2}A \tag{76}.$$

From (69) and (70),

$$\wp w - \wp v = \tfrac{1}{4}A^2 \tag{78},$$

$$\wp w + \wp v = -\tfrac{1}{3}(1-h^2) - \tfrac{1}{4}A^2 \tag{79}.$$

10. If a factor $z-b$ of Z is known, then

$$A^2 = (1-b^2)(h-b)^2 \tag{80},$$

and a well known formula of elliptic functions gives

$$\wp u = \frac{\{\frac{1}{6}(1-h^2)+b(h-b)\}z + \frac{5}{6}(1-h^2)b - h + 2b^2h - b^3}{2(z-b)} \tag{81}.$$

Then $z = h$ makes

$$\wp u = -\tfrac{1}{12}(5+h^2-6b^2), \quad \wp 2u = -\tfrac{1}{6}(1-h^2) \tag{82},$$

so that $u = \frac{1}{2}w = \frac{1}{2}(v_1+v_2)$ corresponds to $z = h$ (83),

$$u = \tfrac{1}{2}v = \tfrac{1}{2}(v_1 - v_2) \text{ corresponds to } z = h - b + \frac{1}{b} \tag{84}.$$

But, if c, the value of u which makes $z = \infty$, is given, when

$$\wp c = \tfrac{1}{12}(1-h^2) + \tfrac{1}{2}b(h-b) \tag{85},$$

the quartic Z can be resolved into four linear factors, in the form

$$z - z_0 = \frac{i\wp' c}{\wp u - \wp c} \tag{86},$$

$$z - z_1 = \frac{i\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_1}{\wp c - e_1} \tag{87},$$

$$z - z_2 = \frac{i\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_2}{\wp c - e_2} \tag{88},$$

$$z - z_3 = \frac{i\wp' c}{\wp u - \wp c}\,\frac{\wp u - e_3}{\wp c - e_3} \tag{89},$$

$i\wp' c$ being real, because c is a fraction of the imaginary period ω_3.

Then
$$\sqrt{Z} = \frac{dz}{du} = -\frac{i\wp' c\, \wp' u}{(\wp u - \wp c)^2}$$
$$= i\wp\,(u+c) - i\wp\,(u-c) \tag{90}$$
and integrating,
$$z - \tfrac{1}{2}h = -i\zeta\,(u+c) + i\zeta\,(u-c) + i\zeta 2c \tag{91}$$
$$(z-\tfrac{1}{2}h)^2 = -\wp\,(u+c) - \wp\,(u-c) - \wp 2c \tag{92}$$

Writing x for $z-\frac{1}{2}h$, and X for the corresponding value of z, then $6\wp 2c$ and $4i\wp' 2c$ are the coefficients of x^2 and x, so that
$$\wp\, 2c = -\tfrac{1}{12}(2+h^2) \tag{93}$$
$$i\wp' 2c = \tfrac{1}{4}h \tag{94}$$

Thus $c = \frac{1}{2}\omega_3$, if $h=0$; and then $w = \frac{1}{2}\omega_3$, or $\omega_1 + \frac{1}{2}\omega_3$, as well; this is what Biermann calls the *parabolic* case of the spherical catenary.

Also
$$\wp'' 2c = \tfrac{1}{8}(1+2h^2) - \tfrac{1}{2}A^2 \tag{95}$$
$$\wp 4c = -\frac{3-4h^2+4h^4}{48h^2} + \tfrac{1}{2}\frac{(1+2h^2)}{h^2}A^2 - \frac{A^4}{h^2} \tag{96}$$

and c is the parameter required in the rectification of the catenary.

11. The values v_1 and v_2 of u make z assume the values ± 1 and $\sqrt{Z}$ the value Ai; therefore, from (90),
$$A = \frac{\wp' c\, \wp' v_1}{(\wp v_1 - \wp c)^2} = \frac{\wp' c\, \wp' v_2}{(\wp v_2 - \wp c)^2} \tag{97}$$
and
$$1 - z = \frac{i\wp' c\,(\wp u - \wp v_1)}{(\wp v_1 - \wp c)(\wp u - \wp c)} \tag{98}$$
$$-1 - z = \frac{i\wp' c\,(\wp u - \wp v_2)}{(\wp v_2 - \wp c)(\wp u - \wp c)} \tag{99}$$

Therefore
$$\frac{d\psi_1 i}{du} = \frac{\frac{1}{2}iA}{1-z} = \frac{-\frac{1}{2}\wp' v_1\,(\wp u - \wp c)}{(\wp v_1 - \wp c)(\wp u - \wp v_1)}$$
$$= \frac{-\frac{1}{2}\wp' v_1}{\wp v_1 - \wp c} + \frac{-\frac{1}{2}\wp' v_1}{\wp u - \wp v_1}$$
$$= -\tfrac{1}{2}\zeta\,(v_1 - c) - \tfrac{1}{2}\zeta\,(v_1 + c) + \zeta v_1$$
$$-\tfrac{1}{2}\zeta\,(u - v_1) + \tfrac{1}{2}\zeta\,(u + v_1) - \zeta v_1 \tag{100}$$
and integrating,
$$\psi_1 i = -\tfrac{1}{2}\left\{\zeta(v_1 - c) + \zeta(v_1 + c)\right\}u + \tfrac{1}{2}\log\frac{\sigma\,(u+v_1)}{\sigma\,(u-v_1)} \tag{101}$$

Similarly,

$$\psi_2 i = -\tfrac{1}{2}\{\zeta(v_2-c)+\zeta(v_2+c)\}u+\tfrac{1}{2}\log\frac{\sigma(u+v_2)}{\sigma(u-v_2)} \qquad (102);$$

and therefore

$$\psi i = -\tfrac{1}{2}Ru+\tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u-v_2)}{\sigma(u-v_1)\,\sigma(u+v_2)} \qquad (103),$$

where $R = \zeta(v_1-c)+\zeta(v_1+c)-\zeta(v_2-c)-\zeta(v_2+c)$ (104).

12. But the formula

$$\frac{\sigma(u+v_1)\,\sigma(u+v_2)\,\sigma(v_1+v_2)}{\sigma(u+v_1+v_2)\,\sigma u\,\sigma v_1\,\sigma v_2} = \begin{vmatrix} 1, & \wp u, & \wp^2 u \\ 1, & \wp v_1, & \wp^2 v_1 \\ 1, & \wp v_2, & \wp^2 v_2 \end{vmatrix} \div \begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v_1, & \wp' v_1 \\ 1, & \wp v_2, & \wp' v_2 \end{vmatrix}$$

$$= \zeta(u+v_1+v_2)-\zeta u-\zeta v_1-\zeta v_2 \qquad (105)$$

shows that, changing v_2 into $-v_2$, and writing v for v_1-v_2,

$$\tfrac{1}{2}\log\frac{\sigma(u+v_1)\,\sigma(u+v_2)}{\sigma(u-v_1)\,\sigma(u-v_2)} = \tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)}\,\Omega \qquad (106),$$

where Ω is a rational algebraical function of $\wp u$ and $\wp' u$, and therefore also of z and $\sqrt{Z}$, of the form

$$\Omega = \frac{C+Bi\,\wp' u}{C-Bi\,\wp' u} \qquad (107),$$

so that $\tfrac{1}{2}\log\Omega = i\tan^{-1}\frac{B}{C}\,\wp' u$ (108).

Equation (29), expressed by elliptic functions of u and v, gives

$$M^2(s-\sigma) = \wp u-\wp v \qquad (109)$$

$$I(v) = \tfrac{1}{2}\rho Mu-iu\zeta v+\tfrac{1}{2}i\log\frac{\sigma(u+v)}{\sigma(u-v)} \qquad (110),$$

so that, with $v = v_1-v_2$,

$$\tfrac{1}{2}\log\frac{\sigma(u+v)}{\sigma(u-v)} = -u\zeta v+\tfrac{1}{2}\rho Miu-iI(v) \qquad (111),$$

and thus $\psi = -(i\zeta v-\tfrac{1}{2}Ri-\tfrac{1}{2}\rho M)\,u-I(v)+\tan^{-1}\frac{B}{C}\,\wp' u$ (112).

13. Now

$$i\zeta v-\tfrac{1}{2}Ri = \tfrac{1}{2}i\left\{\zeta(v_1-v_2)-\zeta(v_1-c)-\zeta(v_2-c)\right\}$$

$$+\tfrac{1}{2}i\left\{\zeta(v_1-v_2)-\zeta(v_1+c)-\zeta(v_2+c)\right\}$$

$$= \tfrac{1}{2}\sqrt{\left\{-\wp(v_1-v_2)-\wp(v_1-c)-\wp(v_2-c)\right\}}$$

$$+\tfrac{1}{2}\sqrt{\left\{-\wp(v_1-v_2)-\wp(v_1+c)-\wp(v_2+c)\right\}} \qquad (113).$$

But, taking $z=\pm 1$, $z_2=\infty$, in equation (65),

$$\wp(v_1\pm c) = \frac{-1+h+\frac{1}{6}(1-h^2)\pm A}{2} \qquad (114),$$

$$\wp(v_2\pm c) = \frac{-1-h+\frac{1}{6}(1-h^2)\pm A}{2} \qquad (115);$$

and therefore

$$-\wp(v_1-v_2)-\wp(v_1-c)-\wp(v_2-c) = 1+A+\tfrac{1}{4}A^2$$

$$= (1+\tfrac{1}{2}A)^2 \qquad (116),$$

$$-\wp(v_1-v_2)-\wp(v_1+c)-\wp(v_2+c) = 1-A+\tfrac{1}{4}A^2$$

$$= (1-\tfrac{1}{2}A)^2 \qquad (117),$$

and taking the square roots of opposite signs, equation (113) gives

$$i\zeta v-\tfrac{1}{2}Ri = -\tfrac{1}{2}A \qquad (118),$$

so that

$$\psi = \tfrac{1}{2}(A+\rho M)\,u - I(v) + \tan^{-1}\frac{B}{C}\,\wp' u \qquad (119),$$

or, as stated at the outset,

$$\psi = pu+\chi \qquad (33),$$

where $$p = \tfrac{1}{2}(A+\rho M) \qquad (40),$$

and then

$$\chi = -I(v)+\tan^{-1}\frac{B}{C}\,\wp' u \qquad (120).$$

One great difficulty in these calculations is the determination of the proper sign to employ when a square root is taken, or with an imaginary quantity; in most cases this can be settled only by a verification, or by an appeal to a special case.

14. When $I(v)$ is pseudo-elliptic, then $\mu\chi$ is an angle such that $\sin\mu\chi$ and $\cos\mu\chi$ are algebraical functions of z and $\sqrt{Z}$; and now, putting

$$P = \mu p = \tfrac{1}{2}\mu\,(A+M\rho) \qquad (121),$$

$$Q = \mu q = \tfrac{1}{2}\mu\,(A-M\rho) \qquad (122),$$

so that

$$P+Q = \mu A \qquad (123),$$

then

$$\mu\chi = \mu\psi - Pu \qquad (124),$$

and the equation of the catenary can be written in either of the forms

$$r^{\mu}\cos\mu\chi = Hz^{\mu}+H_1 z^{\mu-1}+\ldots+H_{\mu} \qquad (125),$$

$$r^{\mu}\sin\mu\chi = (Lz^{\mu-2}+L_1 z^{\mu-3}+\ldots+L_{\mu-2})\sqrt{Z} \qquad (126),$$

where

$$r^2+z^2 = 1 \qquad (5),$$

both leading to the differential elation, equivalent to (1),

$$\mu\frac{d\chi}{dz} = \frac{Pz^2+Q}{(1-z^2)\sqrt{Z}} \qquad (127),$$

or

$$\mu\,(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \qquad (128).$$

15. Squaring and adding (125) and (126), and equating coefficients, leads to

$$H^2-L^2 = (-1)^{\mu}$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots,$$

$$H_{\mu}^2-(2y+1)\,L_{\mu-2}^2 = 1 \qquad (129),$$

and to other relations, theoretically sufficient, in conjunction with the differentiations of (125) and (126), leading to (128), to determine the other coefficients H and L in terms of P, Q, A, h, and ρ; all functions of a single parameter, when once the pseudo-elliptic form of integral (29) for an assigned order μ has been introduced.

But for values of μ above 6 the complication of this method became so formidable that it was absolutely necessary to seek for some other method of determining the leading coefficients H and L; this was effected by a consideration of the form assumed when $z = \infty$, in a manner to be explained in the sequel (§ 30).

16. Consider first the simplest case of

$$\mu = 3.$$

Then $x = 0$, and we must take $y = -1$ to keep M finite (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 210).

As the general expression in (29) then assumes an indeterminate form, start *ab initio* with

$$v = \tfrac{4}{3}\omega_3 \qquad (130),$$

and

$$3I(v) = \tfrac{1}{2}\int \frac{-s-3c}{s\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{s+c}{2s^{\frac{3}{2}}} = \sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} \qquad (131),$$

where

$$S = 4s^3-(s+c)^2 \qquad (132).$$

Then, as in (58) and (59),

$$12\gamma_2 = 1+24c \qquad (133),$$

$$216\gamma_3 = 1+36c+216c^2 \qquad (134),$$

so that we must take, as in (61), (62), (63),

$$1-h^2 = -M^2 \qquad (135),$$

$$A^2 = -2M^4c \qquad (136),$$

$$A^2 = 4M^6c^2 \qquad (137),$$

and therefore

$$M^2 = A^2 = -\frac{1}{2c} \qquad (138),$$

$$1-h^2 = -A^2, \quad \text{or} \quad h^2 = A^2+1 \qquad (139),$$

and

$$Z = -z^4+2hz^3+(1-h^2)z^2-2hz+1 \qquad (140).$$

Equation (131), on comparison with (29), shows that we must take

$$3\rho = -1;$$

and thus

$$P = \tfrac{3}{2}(A+M\rho) = A \qquad (141),$$

$$Q = \tfrac{3}{2}(A-M\rho) = 2A \qquad (142),$$

and we have now to determine the values of the coefficients H and L which will make the equations

$$(1-z^2)^{\frac{3}{2}}\cos 3\chi = Hz^3+H_1z^2+H_2z+H_3 \qquad (143),$$

$$(1-z^2)^{\frac{3}{2}}\sin 3\chi = (Lz+L_1)\sqrt{Z} \qquad (144),$$

consistent, and make them both lead to the differential relation

$$3(1-z^2)\sqrt{Z}\frac{d\chi}{dz} = A(z^2+2) \qquad (145).$$

17. Differentiating (143) logarithmically,

$$\frac{-3z}{1-z^2}-3\tan 3\chi\frac{d\chi}{dz}=\frac{3Hz^2+2H_1z+H_2}{Hz^3+H_1z^2+H_2z+H_3},$$

or $$3\frac{Lz+L_1}{Hz^3+H_1z^2+H_2z+H_3}\sqrt{Z}\frac{d\chi}{dz}=\frac{3z}{z^2-1}-\frac{3Hz^2+2H_1z+H_2}{Hz^3+H_1z^2+H_2z+H_3},$$

or $$3(Lz+L_1)(1-z^2)\sqrt{Z}\frac{d\chi}{dz}$$

$$=-3z(Hz^3+H_1z^2+H_2z+H_3)+(3Hz^2+2H_1z+H_2)(z^2-1)$$
$$=-H_1z^3-(3H+2H_2)z^2-(2H_1+3H_3)z-H_2 \qquad (146),$$

and this must

$$=(Lz+L_1)A(z^2+2) \qquad (147);$$

and therefore, equating coefficients

$$0-H_1=AL \qquad (148),$$
$$-3H-2H_2=AL_1 \qquad (149),$$
$$-2H_1-3H_3=2AL \qquad (150),$$
$$-H_2-0=2AL_1 \qquad (151).$$

Differentiating (144) logarithmically, we find in a similar manner

$$3(Hz^3+H_1z^2+H_2z+H_3)(1-z^2)\sqrt{Z}\frac{d\chi}{dz},$$

or $$3(Hz^3+H_1z^2+H_2z+H_3)A(z^2+2)$$

$$=3z(Lz+L_1)Z-L(z^2-1)Z-(Lz+L_1)(z^2-1)\tfrac{1}{2}\frac{dZ}{dz}$$
$$=(2Lz^2+3L_1z+L)\{-z^4+2hz^3+(1-h^2)z^2-2hz+1\}$$
$$-(Lz+L_1)(z^2-1)\{-2z^3+3hz^2+(1-h^2)z-h\} \qquad (152),$$

and therefore, equating coefficients,

$$3AH=hL-L_1 \qquad (153),$$
$$\dots\quad\dots\quad\dots\quad\dots$$
$$6AH_3=L-hL_1 \qquad (154).$$

From these equations we find

$$H=A,\quad H_1=-Ah,\quad H_2=-2A,\quad H_3=0,$$
$$L=h,\quad L_1=1; \qquad (155);$$

so that the equation of the catenary may be written

$$(1-z^2)^{\frac{3}{2}} e^{3\chi i} = A\,(z^3-hz^2-2z)+i\,(hz+1)\sqrt{Z} \qquad (156),$$

with $\qquad 3\chi = 3\psi - Au, \text{ and } h = \sqrt{(A^2+1)},$

and now squaring and adding equations (143) and (144) will lead to a verification, when the conditions obtained above are satisfied.

Equation (139) gives, in conjunction with Clebsch's notation of (25), (26), (27),

$$\cos 2\epsilon = \frac{\sqrt{(k^4-2k^2+5)}}{k^2+1}, \quad \sin 2\epsilon = 2\,\frac{\sqrt{(k^2-1)}}{k^2+1},$$

$$h^2 = \frac{k^6-2k^4+5k^2}{(k^2+1)^2},$$

$$A^2 = \frac{(k^2-1)^3}{(k^2+1)^2},$$

$$Z = \left\{-z^2+\frac{k^2+1+\sqrt{(k^4-2k^2+5)}}{k^2+1}\,kz-\frac{k^2-1+\sqrt{(k^4-2k^2+5)}}{2}\right\}$$

$$\left\{z^2+\frac{k^2+1-\sqrt{(k^4-2k^2+5)}}{k^2+1}\,kz+\frac{k^2-1-\sqrt{(k^4-2k^2+5)}}{2}\right\}.$$

$$\mu = 4.$$

18. Here $y = 0$, and $v = \omega_3$;

but, from (38), $\qquad A^2 = h^2+1 \qquad (157),$

$$Z = -z^4+2hz^3+(1-h^2)\,z^2-2hz-1$$

$$= -(z^2-hz-1)^2-z^2 \qquad (158),$$

so that $\sqrt{Z}$ is imaginary, and the catenary also.

$$\mu = 5.$$

19. Here we must put

$$x = y = -c,$$

suppose (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 213); and now

$$M^2 = \frac{1+y}{-2x} = \frac{1-c}{2c} \qquad (159),$$

$$A^2 = \frac{(1-c)^3}{2c} \qquad (160),$$

$$h^2 = A^2 - 2y - 1$$

$$= \frac{(1-c)^3}{2c} - 1 + 2c$$

$$= \frac{1-5c+7c^2-c^3}{2c} \qquad (161).$$

Now, with $\qquad v = \frac{4}{5}\omega_3,$

we may write (29) in the form

$$5I(v) = \frac{1}{2}\int \frac{(1+3c)\,s-5c^2}{s\sqrt{S}}\,ds$$

$$= \tan^{-1}\frac{(s-c^2)\sqrt{S}}{(1+3c)\,s^2-(2c^2+c^3)\,s+c^4} \qquad (162),$$

where $\qquad S = 4s\,(s-c)^2 - \{(1-c)\,s+c^2\}^2 \qquad (163),$

$$5\rho = 1+3c \qquad (164),$$

so that $\qquad P = \frac{5}{2}(A+M\rho) = M(3-c) \qquad (165),$

$$Q = \frac{5}{2}(A-M\rho) = 2M(1-2c) \qquad (166),$$

$$P+Q = 5M(1-c) = 5A \qquad (167).$$

The equation of the catenary is now of either of the forms

$$(1-z^2)^{\frac{5}{2}}\cos 5\chi = Hz^5+H_1z^4+H_2z^3+H_3z^2+H_4z+H_5 \qquad (168),$$

$$(1-z^2)^{\frac{5}{2}}\sin 5\chi = (Lz^3+L_1z^2+L_2z+L_3)\sqrt{Z} \qquad (169),$$

leading to the relation

$$5(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \qquad (170),$$

also $\qquad L^2 - H^2 = 1 \qquad (171).$

20. A straightforward verification of (170) by logarithmic differentiation of (168) and (169) leads to the following values of the coefficients

$$H = \frac{2-5c+c^2}{c}M \qquad (172),$$

$$H_1 = \frac{(3-c)(2-c)}{c}\,Mh \qquad (173),$$

$$H_2 = \frac{3-15c+20c^2-4c^3}{c^3} M \qquad (174),$$

$$H_3 = -\frac{1-9c+4c^2}{c^2} Mh \qquad (175),$$

$$H_4 = -2\frac{(1-2c)^2}{c^2} M \qquad (176),$$

$$H_5 = -\frac{2}{c} Mh \qquad (177),$$

$$L = -\frac{2-c}{c} h \qquad (178),$$

$$L_1 = -\frac{2-6c+3c^2}{c^2} \qquad (179),$$

$$L_2 = \frac{1-2c}{c^2} h \qquad (180),$$

$$L_3 = \frac{1-2c}{c^2} \qquad (181).$$

The verification is rather long, but the work is given herewith, as a type of the calculations required.

Differentiating (168) logarithmically,

$$\frac{-5z}{1-z^2} - 5\tan 5\chi \frac{d\chi}{dz} = \frac{5Hz^4+4H_1z^3+\ldots}{Hz^5+H_1z^4+\ldots} \qquad (182),$$

that is, $5\tan 5\chi \frac{d\chi}{dz}$ or $5\frac{Lz^3+L_1z^2+\ldots}{Hz^5+H_1z^4+\ldots}\sqrt{Z}\frac{d\chi}{dz}$

$$= \frac{-5z}{1-z^2} - \frac{5Hz^4+4H_1z^3+\ldots}{Hz^5+H_1z^4+\ldots} \qquad (183),$$

$$5(Lz^3+L_1z^2+\ldots)(1-z^2)\sqrt{Z}\frac{d\chi}{dz} \text{ or } (Lz^3+L_1z^2+\ldots)(Pz^2+Q)$$

$$= -5z(Hz^5+H_1z^4+\ldots)+(z^2-1)(5Hz^4+4H_1z^3+\ldots) \qquad (184);$$

and therefore, equating coefficients of z^6, z^5, ...,

$$0+PL = 0 - H_1 \qquad (185),$$

$$0+PL_1 = -5H-2H_2 \qquad (186),$$

$$QL+PL_2 = -4H_1-3H_3 \qquad (187),$$

$$QL_1+PL_3 = -3H_2-4H_4 \qquad (188),$$

$$QL_2+\;\;0\;\; = -2H_3-5H_5 \qquad (189),$$

$$QL_3+\;\;0\;\; = -\;\;H_4-\;0 \qquad (190).$$

Differentiating (169) logarithmically,

$$\frac{-5z}{1-z^2}-5\cot 5\chi\frac{d\chi}{dz} = \frac{3Lz^2+2L_1z+L_2}{Lz^3+L_1z^2+L_2z+L_3}+\frac{-2z^3+3hz^2+(1-h^2)\,z-h}{Z} \qquad (191),$$

$$5\,\frac{Hz^5+H_1z^4+\dots}{Lz^3+L_1z^2+\dots}\;\frac{1}{\sqrt{Z}}\;\frac{d\chi}{dz} = \frac{5z}{1-z^2}+\frac{3Lz^2+\dots}{Lz^3+\dots}+\frac{-2z^3+3hz^2+\dots}{Z} \qquad (192),$$

$$5\,(Hz^5+H_1z^4+\dots)(1-z^2)\sqrt{Z}\frac{d\chi}{dz} \quad \text{or} \quad (Hz^5+H_1z^4+\dots)(Pz^2+Q)$$

$$= \left\{5Lz^4+5L_1z^3+5L_2z^2+5L_3z-(z^2-1)(3Lz^2+2L_1z+L_2)\right\}Z$$

$$-(z^2-1)(Lz^3+L_1z^2+L_2z+L_3)\left\{-2z^3+3hz^2+(1-h^2)\,z-h\right\} \qquad (193),$$

and equating coefficients,

$$0\;+PH\;\; = hL-L_1 \qquad (194),$$

$$0\;+PH_1 = -(4+h^2)\,L+3hL_1-2L_2 \qquad (195),$$

$$QH\;+PH_2 = 6hL-2\,(1+h^2)\,L_1+5hL_2-3L_3 \qquad (196),$$

$$QH_1+PH_3 = \dots\;\dots\;\dots\;\dots\;\dots\;\dots \qquad (197),$$

$$QH_2+PH_4 = \dots\;\dots\;\dots\;\dots\;\dots\;\dots \qquad (198),$$

$$QH_3+PH_5 = \dots\;\dots\;\dots\;\dots\;\dots\;\dots \qquad (199),$$

$$QH_4+\;\;0\;\; = -\,2(A^2-h^2)\,L_1-3hL_2-(5A^2-4h^2-1)\,L_3 \qquad (200),$$

$$QH_5+\;\;0\;\; = -(A^2-h^2)\,L_2-hL_3 \qquad (201).$$

Put

$$\frac{H}{L} = x \qquad (202),$$

so that, when x is found, the values of H and L can be inferred from (171),

$$H = \frac{x}{\sqrt{(1-x^2)}} \qquad (203),$$

$$L = \frac{1}{\sqrt{(1-x^2)}} \qquad (204).$$

21. Then, from (185), $\frac{H_1}{L} = -P$ (205),

from (194), $\frac{L_1}{L} = h - Px$ (206),

from (186), $2\frac{H_2}{L} = -5x - P\frac{L_1}{L} = (P^2-5)\,x - Ph$ (207),

from (195),
$$2\frac{L_2}{L} = -4 - h^2 + 3h\frac{L_1}{L} - P\frac{H_1}{L}$$
$$= -3Phx + P^2 + 2h^2 - 4 \quad (208),$$

from (187),
$$3\frac{H_3}{L} = -4\frac{H_1}{L} - Q - P\frac{L_2}{L}$$
$$= \tfrac{3}{2}P^2hx - \tfrac{1}{2}P^3 + 6P - Q - Ph^3 \quad (209),$$

from (196),
$$3\frac{L_3}{L} = 6h - 2\,(1+h^2)\frac{L_1}{L} + 5h\frac{L_2}{L} - Qx - P\frac{H_2}{L}$$
$$= 6h - 2\,(1+h^2)(h-Px)$$
$$+\tfrac{5}{2}h\,(-3Phx + P^2 + 2h^2 - 4)$$
$$-Qx - \tfrac{1}{2}(P^3 - 5P)\,x + \tfrac{1}{2}P^2h \quad (210),$$

from (188),
$$4\frac{H_4}{L} = -3\frac{H_3}{L} - Q\frac{L_1}{L} - P\frac{L_3}{L}$$
$$= -\tfrac{3}{2}(P^2-5)\,x - Q\,(h-Px) - P\frac{L_3}{L} \quad (211),$$

and from (190), $\frac{H_4}{L} + Q\frac{L_3}{L} = 0$ (212),

so that
$$-9\,(P^2-5)\,x - 6Q\,(h-Px)$$
$$+(4Q-P)\left[h\left\{12 - 4(1+h^2) + 5\,(P^2+2h^2-4) + P^2\right\}\right.$$
$$\left.-x\left\{P^3 - 5P - 4(1+h^2)\,P + 15Ph^2\right\}\right] = 0 \quad (213),$$

the equation to determine x; thence

$$x = \frac{(4Q-P)(6P^2+6h^2-12) - 6Q}{9\,(P^2-5) - 6PQ + (4Q-P)(P^2-9+11h^2)\,P}\,h \quad (214),$$

where, from the values of M, h, P, Q, ... given in (159)—(167), with

$$4Q - P = 5\,(1-3c)\,M \quad (215),$$

we find, after reduction and cancelling a common factor,

$$5 - 14c + 6c^2,$$

of the numerator and denominator,

$$x = \frac{2-5c+c^2}{2-c}\,\frac{M}{h} \tag{216}$$

so that, from (203) and (204),

$$H = \frac{2-5c+c^2}{c} M \tag{172}$$

$$L = \frac{2-c}{c} h \tag{178}$$

and thence the values of H_1, H_2, H_3, H_4, H_5, L_1, L_2, L_3 are readily inferred from equations (185)—(201).

Putting $z = \pm 1$ in (184), we obtain

$$(\ \ L+L_1+L_2+L_3)(P+Q) = -5\,(\ \ H+H_1+\ldots+H_5) \tag{179}$$

$$(-L+L_1-L_2+L_3)(P+Q) = \ \ 5\,(-H+H_1-\ldots+H_5) \tag{180}$$

so that, in conjunction with (167),

$$A\,(L_1+L_3)+H+H_2+H_4 = 0 \tag{181}$$

$$A\,(L\ +L_2)+H_1+H_3+H_5 = 0 \tag{182}$$

useful as verifications; and the same can be obtained by putting $z = \pm 1$ in (193).

$$\mu = 6.$$

22. The equation of the catenary can be reduced to the same form as for $\mu = 3$, namely, combining (143) and (144),

$$(1-z^2)^{\frac{3}{2}}\, e^{3xi} = Hz^3+H_1z^2+H_2z+H_3+i\,(Lz+L_1)\sqrt{Z} \tag{217}$$

Referring to the *Proc. Lond. Math. Soc.*, Vol. xxv., p. 216,

$$\gamma_6 = 0 \tag{218}$$

or

$$y-x-y^2 = 0 \tag{219}$$

is satisfied by taking

$$y = -c, \quad x = -c-c^2 \tag{220}$$

and then

$$M^2 = \frac{1-c}{2c\,(1+c)} \tag{221}$$

$$A^2 = \frac{(1-c)^3}{2c\,(1+c)} \tag{222}$$

$$h^2 = \frac{(1-3c)(1-2c-c^2)}{2c\,(1+c)} \tag{223}$$

The pseudo-elliptic form of (29) can now be written, with

$$v = \tfrac{2}{3}\omega_3 \qquad (224),$$

$$3I(v) = \tfrac{1}{2}\int \frac{(1+3c)\,s - 3c^2(1+c)}{s\sqrt{S}}\,ds$$

$$= \cos^{-1}\frac{(1+3c)\,s - c^2(1+c)}{2s^{\frac{3}{2}}}$$

$$= \sin^{-1}\frac{\sqrt{S}}{2s^{\frac{3}{2}}} \qquad (225),$$

and thus

$$3\rho = 1+3c \qquad (226),$$

$$P = \tfrac{3}{2}(A+M\rho) = 2M \qquad (227),$$

$$Q = \tfrac{3}{2}(A-M\rho) = (1-3c)\,M \qquad (228),$$

$$P+Q = 3A \qquad (229),$$

and the differential relation to be satisfied by (217) is

$$3(1-z^2)\sqrt{-}\frac{d\chi}{dz} = Pz^2+Q \qquad (230).$$

23. Differentiating (217) logarithmically and equating coefficients, as in the case of $\mu = 3$, the resulting equations are

$$0 - H_1 = PL \qquad (231),$$

$$-3H - 2H_2 = PL_1 \qquad (232),$$

$$-2H_1 - 3H_3 = QL \qquad (233),$$

$$-H_2 - 0 = QL_1 \qquad (234),$$

$$PH = hL - L_1 \qquad (235),$$

$$PH_1 = \dots \quad \dots \qquad (236),$$

$$QH + PH_2 = \dots \quad \dots \qquad (237),$$

$$QH_1 + PH_3 = \dots \quad \dots \qquad (238),$$

$$QH_2 + 0 = \dots \quad \dots \qquad (239),$$

$$QH_3 + 0 = \dots \quad \dots \qquad (240).$$

From (231) and (233), eliminating H_1,

$$3H_3 = (2P-Q)\,L,$$

or

$$H_3 = (1+c)\,ML \qquad (241).$$

From (232) and (234), eliminating H_2,

$$3H = (2Q-P)\,L_1,$$

or
$$H = -2cML_1 \qquad (242).$$

Substituting in (235),

$$\frac{hL}{L_1} = 1 + \frac{PH}{L_1} = 1 - 2cPM$$

$$= 1 - 4cM^2 = 1 - 2\frac{1-c}{1+c} = -\frac{1-3c}{1+c} \qquad (243);$$

and therefore
$$\frac{H}{L} = -2cM\frac{L_1}{L} = 2c\frac{1+c}{1-3c}Mh$$

$$= 2c\,\frac{1+c}{1-3c}\,\frac{\sqrt{\{(1-c)(1-3c)(1-2c-c^2)\}}}{2c\,(1+c)}$$

$$= \sqrt{\left\{\frac{(1-c)(1-2c-c^2)}{1-3c}\right\}} \qquad (244).$$

Also, by squaring and adding (143) and (144),

$$L^2 - H^2 = 1 \qquad (245),$$

and thence

$$H^2 = \frac{H^2}{L^2-H^2} = \frac{(1-c)(1-2c-c^2)}{1-3c-(1-c)(1-2c-c^2)} = \frac{(1-c)(1-2c-c^2)}{-c^2\,(1+c)},$$

$$H = \frac{1}{c}\sqrt{\left\{\frac{(1-c)(1-2c-c^2)}{-1-c}\right\}} \qquad (246),$$

$$L = \frac{1}{c}\sqrt{\left(\frac{1-3c}{-1-c}\right)} \qquad (247),$$

$$H_1 = -PL = -2ML = -\frac{1}{c\,(1+c)}\sqrt{\left\{\frac{2\,(1-c)(1-3c)}{-c}\right\}} \qquad (248),$$

$$L_1 = -\frac{H}{2cM} = -\frac{1}{c}\sqrt{\left(\frac{1-2c-c^2}{-2c}\right)} \qquad (249),$$

$$H_2 = -QL_1 = \frac{1-3c}{2c^2}\sqrt{\left\{\frac{(1-c)(1-2c-c^2)}{-1-c}\right\}} \qquad (250),$$

$$H_3 = (1+c)\,ML = \frac{1}{c}\sqrt{\left\{\frac{(1-c)(1-3c)}{-2c}\right\}} \qquad (251).$$

The verifications obtained by putting $z = \pm 1$ in (146) are

$$AL + H_1 + H_3 = 0 \qquad (252),$$

$$AL_1 + H + H_2 = 0 \qquad (253),$$

and these are found to be satisfied.

24. When μ is even, the cubic S can be resolved into factors; and therefore Z also can be factorized.

Proceeding in Clebsch's manner (*Crelle*, 57, p. 105), and writing k for Clebsch's ρ, we put

$$Z = -\left(z^2 - hz + \frac{k^2-1}{2}\right)^2 + \left(kz - \frac{h}{k}\,\frac{k^2+1}{2}\right)^2$$

$$= -z^4 + 2hz^3 + (1-h^2)\,z^2 - 2hz$$

$$-\left(\frac{k^2-1}{2}\right)^2 + \frac{h^2}{k^2}\left(\frac{k^2+1}{2}\right)^2 \qquad (254),$$

and this is the case provided that

$$A^2 - h^2 = \left(\frac{k^2-1}{2}\right)^2 - \frac{h^2}{k^2}\left(\frac{k^2+1}{2}\right)^2$$

or

$$(k^2-1)^2\,(k^2-h^2) - 4A^2k^2 = 0 \qquad (255),$$

a cubic equation for k^2; and Z then breaks up into the quadratic factors Z_1 and Z_2, where

$$Z_1 = -z^2 + (h+k)\,z - \frac{k^2-1}{2} - \frac{h}{k}\,\frac{k^2+1}{2} \qquad (256),$$

$$Z_2 = \quad z^2 - (h-k)\,z + \frac{k^2-1}{2} - \frac{h}{k}\,\frac{k^2+1}{2} \qquad (257).$$

But, putting, with Halphen's notation of x and y,

$$k^2 = 4M^2s + h^2 - A^2 = -2\,\frac{1+y}{x}\,s - 1 - 2y \qquad (258),$$

$$k^2 - 1 = -2\,(1+y)\,\frac{s+x}{x} \qquad (259),$$

$$k^2 - h^2 = 4M^2s - A^2 = M^2\left\{4s - (1+y)^2\right\} \qquad (260),$$

the cubic equation (255) for k^2 becomes transformed into

$$4s\,(s+x)^2 - \left\{(1+y)\,s + xy\right\}^2 = 0, \quad \text{or} \quad S = 0 \qquad (31);$$

and therefore the quartic Z can be resolved when we know a root, say s_3, of equation (31).

25. Clebsch's $\sin 2\epsilon$ or x (which we change into ξ to avoid confusion of notation) is connected with k by the relation

$$\cos^2 2\epsilon = 1-\xi^2 = \frac{h^2}{k^2} \qquad (261),$$

and it is connected with the s above in (258)—(260) by the relation

$$\xi = M(-2\sqrt{s}+1+y) = -2M\sqrt{s}+A \qquad (262).$$

For Clebsch's m is our h, and his b is our A; so that his resolvent cubic (*Crelle*, 57, p. 105)

$$(x-2b)(1-x^2)-m^2x = 0 \qquad (263)$$

becomes
$$(\xi-2A)(1-\xi^2)-h^2\xi = 0,$$

or
$$\xi^3-2A\xi^2+(1-h^2)\xi+2A = 0 \qquad (264),$$

and, putting
$$\xi = M(-2t+1+y) \qquad (265),$$
this reduces (264) to

$$2t^3-(1+y)t^2+2xt-xy = 0 \qquad (266),$$

or, putting $t = \sqrt{s}$, to

$$4s(s+x)^2-\{(1+y)s+xy\}^2 = 0 \qquad (31).$$

26. In the case of $\mu = 6$, we find that

$$s_3 = y^2 = c^2 \qquad (267),$$

and then
$$k^2 = -1+2c+2\frac{1-c}{c+c^2}c^2 = \frac{1-3c}{-1-c} \qquad (268),$$

$$\frac{k^2-1}{2} = \frac{1-c}{-1-c} \qquad (269),$$

$$\frac{k^2+1}{2} = \frac{2c}{1+c} \qquad (270),$$

$$\frac{h^2}{k^2} = \frac{1-2c-c^2}{-2c} \qquad (271),$$

and the quadratic factors Z_1 and Z_2 of Z are thus determined.

Putting
$$Z_1 = (z_1-z)(z-z_0) \qquad (272),$$
so that $z = z_1$ and $z = z_0$ give parallels of latitude between which a branch of the catenary lies, then, as z grows from z_0 to z_1, the

variable u may be taken to grow from 0 to ω_1, the real period of the elliptic functions, such that

$$\omega_1 = \int_{z_0}^{z_1} \frac{dz}{\sqrt{Z}} \qquad (273).$$

If K denotes the corresponding quarter-period of the associated Jacobian elliptic functions, then (Klein, *Math. Ann.*, XIV., p. 118)

$$\sqrt[4]{(12g_2)}\, \omega_1 = \sqrt[4]{(1-\kappa^2\kappa'^2)}\, 2K \qquad (274).$$

At the same time $\mu\chi$ grows from 0 to $\frac{1}{2}\pi$, or $\mu\psi$ from 0 to $P\omega_1 + \frac{1}{2}\pi$; so that, if $P\omega_1$ can be made an exact multiple of $\frac{1}{2}\pi$, the catenary will close in upon itself, and form a closed curve.

27. The simplest mode of effecting this closure is to make P vanish; but, in the case of $\mu = 6$, this requires A to vanish also, and the catenary degenerates into a vertical great circle.

Calculating the invariants g_2 and g_3 of the quartic Z in this case of $\mu = 6$, we find, from (53) and (54),

$$12g_2 = \frac{(1-c)^2 (1+3c)(1+9c+3c^2+3c^3)}{4c^2 (1+c)^2} \qquad (275),$$

$$216g_3 = \frac{(1-c)^3 (1+6c-3c^2)(1+12c+30c^2+36c^3+9c^4)}{8c^3 (1+c)^3} \qquad (276)$$

$$\begin{aligned} S &= 4s(s-c-c^2)^2 - \{(1-c)\, s + c^2 + c^3\}^2 \\ &= \{4s^2 - (1+c)(1+5c)\, s + c^2 (1+c)^2\}\, (s-c^2) \\ &= 4(s-s_1)(s-s_2)(s-s_3) \qquad (277), \end{aligned}$$

suppose; and, with $s_1 > s_3 > s_2$,

$$\kappa^2\kappa'^2 = \frac{(s_1-s_3)(s_3-s_2)}{(s_1-s_2)^2} = \frac{16c^3}{(1+c)^3(1+9c)} \qquad (278),$$

$$1-\kappa^2\kappa'^2 = \frac{(1+3c)(1+9c+3c^2+3c^3)}{(1+c)^3(1+9c)} \qquad 279),$$

and thus, from (274),

$$\frac{\omega_1}{2K} = \sqrt[4]{\left\{\frac{4c^2}{(1-c)^2(1+c)(1+9c)}\right\}} \qquad (280),$$

$$P\omega_1 = 2M\omega_1 = \frac{4K}{\sqrt[4]{\{(1+c)^3(1+9c)\}}} = \frac{2K\sqrt{(\kappa\kappa')}}{c^{\frac{3}{4}}} \qquad (281).$$

To construct a closed catenary we must search, by a tentative process, for values of c which will make

$$\frac{2K\sqrt{(\kappa\kappa')}}{\frac{1}{2}\pi c^{\frac{1}{2}}} = 1, 2, 3, \dots \qquad (282),$$

where $\kappa\kappa'$ is given by (278).

The formula given by Klein (*Math. Ann.*, XIV., p. 119),

$$12g_2\left(\frac{\omega_1}{\pi}\right)^4 = 1+240\left(\frac{q^2}{1-q^2}+\frac{2^3q^4}{1-q^4}+\dots\right)$$

$$\approx 1+\frac{240q^2}{1-q^2} \quad \text{or} \quad 1+240q^2 \qquad (283),$$

will perhaps assist in obtaining a first approximation; or otherwise the curve which is the graph of (282) must be plotted, preferably with logarithmic coordinates.

28. With negative discriminant

$$\Delta = g_2^3-27g_3^2 = \frac{(1-c)^6(1+9c)}{64(1+c)^3} \qquad (284),$$

two of the roots of the quartic Z and also of the cubic S, are imaginary; and now

$$4\kappa^2\kappa'^2 = \frac{(1+c)^3(1+9c)}{64c^3} \qquad (285),$$

the reciprocal of the preceding value of $4\kappa^2\kappa'^2$ in (278); and

$$\kappa^2 = \tfrac{1}{2}-\frac{1+6c-3c^2}{16(-c)^{\frac{3}{2}}} \qquad (286),$$

$$\kappa'^2 = \tfrac{1}{2}+\frac{1+6c-3c^2}{16(-c)^{\frac{3}{2}}} \qquad (287),$$

so that c is now negative, or $y=-c$ positive.

With negative discriminant, suppose

$$Z_2 = (z-m)^2+n^2 \qquad (288),$$

and Z_1 as before in (272); then, from (256) and (257),

$$z_0+z_1 = h+k \qquad (289),$$

$$z_0z_1 = \frac{k^2-1}{2}+\frac{h}{k}\,\frac{k^2+1}{2} \qquad (290),$$

$$2m = h-k \tag{291}$$

$$m^2+n^2 = \frac{k^2-1}{2} - \frac{h}{k}\,\frac{k^2+1}{2} \tag{292}$$

so that, in the general case,

$$\{(z_0-m)^2+n^2\}\{(z_1-m)^2+n^2\} = (k^2-1)^2+\left(1-\frac{h^2}{k^2}\right)(k^4-1) \tag{293}$$

In the special case of $\mu = 6$, from (269), (270), and (271), with $c = -y$,

$$\{(z_0-m)^2+n^2\}\{(z_1-m)^2+n^2\} = 4\left(\frac{1+y}{1-y}\right)^2 - 4\,\frac{(1+y)^2}{1-y}$$

$$= 4y\left(\frac{1+y}{1-y}\right)^2 \tag{294}$$

and then

$$P\omega_1 = 2M\int_{z_0}^{z_1}\frac{dz}{\sqrt{Z}} = 2M\,\frac{2K}{\sqrt{\{(z_0-m)^2+n^2\,.\,(z_1-m)^2+n^2\}}}$$

$$= \frac{2K}{y}\sqrt{\left(\frac{1-y}{1+y}\right)} \tag{295}$$

But since $(1-y)(1-9y)$ is now negative, or $1>y>\frac{1}{9}$, M^2 is negative, and the catenary is imaginary.

$\mu = 7$.

29. Here the first relation to be satisfied is

$$\gamma_7 = 0 \tag{296}$$

or

$$(y-x)\,x-y^3 = 0 \tag{297}$$

the equation of a unicursal cubic, in which we can put

$$x = -c\,(1+c)^2 \tag{298}$$

$$y = -c\,(1+c) \tag{299}$$

and now

$$M^2 = \frac{1-c-c^2}{2c\,(1+c)^2} \tag{300}$$

$$A^2 = \frac{(1-c-c^2)^3}{2c\,(1+c)^2} \tag{301}$$

$$A^2-h^2 = 1-2c-2c^2 \tag{302}$$

$$h^2 = \frac{1-5c+0+15c^3+12c^4+c^5-c^6}{2c(1+c)^2} \qquad (303).$$

The integral (29) is now (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 226), with

$$v = \tfrac{4}{7}\omega_3 \qquad (304),$$

$$7I(v) = \tfrac{1}{2}\int \frac{(3+9c+5c^2)\,s+7c^3(1+c)^3}{s\sqrt{S}}\,ds$$

$$= \tan^{-1}\frac{(s^2+Cs+D)\sqrt{S}}{7\rho s^3+\sigma s^2+\tau s+\phi} \qquad (305),$$

where

$$7\rho = 3+9c+5c^2 \qquad (306),$$

$$\sigma = -(1+c)^3(1+6c+13c^2+5c^3) \qquad (307),$$

$$\tau = c^2(1+c)^6(2+6c+c^2) \qquad (308),$$

$$\phi = -c^4(1+c)^9 \qquad (309),$$

$$C = -(1+c)^3(1+3c) \qquad (310),$$

$$D = c^2(1+c)^6 \qquad (311),$$

$$P = \tfrac{7}{2}(A+M\rho) = (5+c-c^2)\,M \qquad (312),$$

$$Q = \tfrac{7}{2}(A-M\rho) = 2(1-4c-3c^2)\,M \qquad (313),$$

$$P+Q = 7A \qquad (314).$$

The equation of the catenary is now of the form

$$(1-z^2)^{\frac{7}{2}}e^{7\chi i} = Hz^7+H_1z^6+\ldots+H_7+i\,(Lz^5+L_1z^4+\ldots+L_5)\sqrt{Z} \qquad (315),$$

but when it was attempted to employ the differential relation

$$7(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^2+Q \qquad (316)$$

for the determination of the H's and L's, in the manner illustrated above, the complication became so formidable that another method had to be sought for to determine the leading H and L, connected by the relation

$$L^2-H^2 = 1 \qquad (317),$$

upon which the other coefficients depend.

30. The clue was obtained by noticing that the value $z=\infty$ makes, in the general case, in (125) and (126),

$$\tan\mu\chi = \frac{Li}{H} \qquad (318),$$

or
$$\mu\chi i = i\tan^{-1}\frac{Li}{H} = \tfrac{1}{2}\log\frac{L+H}{L-H} = \log(L+H) \qquad (319),$$

$$L+H = e^{\mu\chi i},\quad \frac{L+H}{L-H} = e^{2\mu\chi i} \qquad (320).$$

By means of the formulas

$$\wp(u+v_1)-\wp(u+v_2) = -\frac{\sigma(2u+v_1+v_2)\,\sigma(v_1-v_2)}{\sigma^2(u+v_1)\,\sigma^2(u+v_2)} \qquad (321),$$

$$\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} = \frac{\sigma(2u-v_1-v_2)}{\sigma(2u+v_1+v_2)}\,\frac{\sigma^2(u+v_1)\,\sigma^2(u+v_2)}{\sigma^2(u-v_1)\,\sigma^2(u-v_2)} \qquad (322),$$

equation (103) may be replaced by

$$\psi i = -\tfrac{1}{2}Ru+\tfrac{1}{4}\log\frac{\sigma(2u+v)}{\sigma(2u-v)}+\tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} \qquad (323),$$

and now, supposing that

$$M^2s = \wp 2u-\wp v \qquad (324),$$

in equation (29), then

$$\psi i = -\tfrac{1}{2}Ru+u\zeta v+\tfrac{1}{2}\rho Mui-\tfrac{1}{2}iI(v)+\tfrac{1}{4}\log\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)} \qquad (325),$$

or
$$\mu\chi i = \tfrac{1}{4}\log\left\{\frac{\wp(u-v_1)-\wp(u-v_2)}{\wp(u+v_1)-\wp(u+v_2)}\right\}^{\mu}-\tfrac{1}{2}\mu iI(v) \qquad (326).$$

With the special value of $z=\infty$ and $u=c$, suppose S becomes C, and in (29),

$$\mu I(v) = \tan^{-1}\frac{G}{F}\sqrt{C}$$

$$= \tfrac{1}{2}i\log\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}} \qquad (327).$$

Then

$$\mu\chi i,\ \text{or}\ \tfrac{1}{2}\log\frac{L+H}{L-H} = \tfrac{1}{4}\log\left\{\frac{\wp(v_1-c)-\wp(v_2-c)}{\wp(v_1+c)-\wp(v_2+c)}\right\}^{\mu}$$

$$+\tfrac{1}{4}\log\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}} \qquad (328).$$

But, from (114) and (115),

$$\wp(v_1-c)-\wp(v_2-c) = h+A \tag{329}$$

$$\wp(v_1+c)-\wp(v_2+c) = h-A \tag{330}$$

so that

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^{\frac{1}{2}\mu} \left(\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}}\right)^{\frac{1}{2}} \tag{331}$$

$$L+H = \frac{(h+A)^{\frac{1}{2}\mu}\,\{F-G\sqrt{(-C)}\}^{\frac{1}{2}}}{(h^2-A^2)^{\frac{1}{2}\mu}\,(F^2+G^2C)^{\frac{1}{2}}} \tag{332}$$

31. If H denotes the Hessian of the general quartic in (49),

$$\wp\, 2u = -\frac{H}{Z} \tag{333}$$

by Hermite's transformation; also, from (49) and (52),

$$H = -\{\tfrac{1}{6}(1-h^2)+\tfrac{1}{4}h^2\}\,x^4+\dots \tag{334}$$

so that, taking $z=\infty$ and $u=c$,

$$\wp\, 2c = -\tfrac{1}{6}(1-h^2)-\tfrac{1}{4}h^2 \tag{335}$$

and from (69),

$$\wp\, 2c-\wp v = \tfrac{1}{4}(A^2-h^2) \tag{336}$$

so that the corresponding value of s is given by

$$s = \tfrac{1}{4}\,\frac{A^2-h^2}{M^2} = -x\,\frac{1+2y}{2+2y} \tag{337}$$

$$s+x = \frac{x}{2+2y} \tag{338}$$

$$(1+y)\,s+xy = -\tfrac{1}{2}x \tag{339}$$

and

$$C = -\frac{x^2h^2}{4A^2},\quad \sqrt{(-C)} = \frac{xh}{2A} \tag{340}$$

32. In a similar manner we may employ the special value $z=0$, when

$$Z = \sqrt{(h^2-A^2)} = \sqrt{(-1-2y)} \tag{341}$$

to determine the ratio of the final coefficients H_μ and $L_{\mu-2}$.

Now, from (65), denoting by e the value of u corresponding to $z=0$,

$$\wp(v_1 \pm e) = \frac{c+2d+e \pm A\sqrt{(1+2y)}}{2} \qquad (342),$$

$$\wp(v_2 \pm e) = \frac{c-2d+e \mp A\sqrt{(1+2y)}}{2} \qquad (343),$$

so that $\wp(v_1-e)-\wp(v_2-e) = -h-A\sqrt{(1+2y)}$ (344),

$$\wp(v_1+e)-\wp(v_2+e) = -h+A\sqrt{(1+2y)} \qquad (345).$$

Also (noting the two meanings of the letters c and e)

$$\wp 2e = -\frac{ce-d^2}{e} = -\tfrac{1}{6}(1-h^2) - \frac{\tfrac{1}{4}h^2}{1+2y} \qquad (346),$$

$$\wp 2e - \wp v = \tfrac{1}{4}A^2 - \frac{\tfrac{1}{4}h^2}{1+2y} \qquad (347),$$

and the value of s corresponding to $z=0$ is given by

$$s = \tfrac{1}{4}\left(A^2 - \frac{h^2}{1+2y}\right)\frac{-2x}{1+y} = \frac{-x(1+2y)+y(1+y)^3}{2(1+y)(1+2y)} \qquad (348),$$

$$s+x = \tfrac{1}{2}x\frac{1+2y}{1+y} + \frac{y(1+y)^2}{2(1+2y)} \qquad (349),$$

$$(1+y)s+xy = -\tfrac{1}{2}x(1-2y) + \frac{y(1+y)^3}{2(1+2y)} \qquad (350),$$

and, if E denotes the corresponding value of S, we shall find, after reduction,

$$S = -\frac{\{x(1+2y)^2-y(1+y)^3\}^2}{(1+2y)^3}\,\frac{h^2}{A^2} \qquad (351).$$

Thence, as in equations (331), (332), we shall find

$$\frac{H_\mu+L_{\mu-2}\sqrt{(1+2y)}}{H_\mu-L_{\mu-2}\sqrt{(1+2y)}} = \left\{\frac{h+A\sqrt{(1+2y)}}{h-A\sqrt{(1+2y)}}\right\}^{\frac{1}{2}\mu}\left\{\frac{F-G\sqrt{(-E)}}{F+G\sqrt{(-E)}}\right\}^{\frac{1}{2}} \qquad (352),$$

or, since $$H_\mu^2 - L_{\mu-2}^2(1+2y) = 1 \qquad (353),$$

$$H_\mu+L_{\mu-2}\sqrt{(1+2y)} = \frac{\{h+A\sqrt{(1+2y)}\}^{\frac{1}{2}\mu}\{F-G\sqrt{(-E)}\}^{\frac{1}{2}}}{\{h^2-A^2(1+2y)\}^{\frac{1}{4}\mu}(F^2+G^2E)^{\frac{1}{4}}} \qquad (354).$$

33. As a preliminary test of these new methods, we apply them to the cases of $\mu = 3$ and $\mu = 5$, already worked out independently.

Thus, for instance, with $\mu = 3$, we find $z = \infty$ makes

$$s = -\tfrac{1}{2}x, \quad \surd(-C) = \frac{xh}{2A} \tag{355}$$

$$F = s + x = \tfrac{1}{2}x, \quad G = 1 \tag{356}$$

and therefore
$$\frac{F - G\surd(-C)}{F + G\surd(-C)} = \frac{h-A}{h+A} \tag{357}$$

so that, in (331),
$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^{\frac{3}{2}} \left(\frac{h-A}{h+A}\right)^{\frac{1}{2}}$$

$$= \frac{h+A}{h-A} \tag{358}$$

thus giving $H = A$, and $L = h$,
as before, in (155).

34. With $\mu = 5$, and taking equation (162),

$$\frac{F - G\surd(-C)}{F + G\surd(-C)} = \frac{(1+3c)\,s^2 - (2c^2+c^3)\,s + c^4 - (s-c^2)\surd(-C)}{(1+3c)\,s^2 - (2c^2+c^3)\,s + c^4 + (s-c^2)\surd(-C)} \tag{359}$$

with
$$s = \frac{c-2c^2}{2-2c}, \quad \surd(-C) = -\frac{ch}{2A} \tag{360}$$

so that, from (160) and (161),

$$\frac{h+A}{h-A} = \frac{\surd(1-5c+7c^2-c^3) + (1-c)\surd(1-c)}{\surd(1-5c+7c^2-c^3) - (1-c)\surd(1-c)} \tag{361}$$

$$\frac{F - G\surd(-C)}{F + G\surd(-C)}$$

$$= \frac{(1-5c+6c^2+2c^3)\surd(1-c) + (1-4c+2c^2)\surd(1-5c+7c^2-c^3)}{(1-5c+6c^2+2c^3)\surd(1-c) - (1-4c+2c^2)\surd(1-5c+7c^2-c^3)} \tag{362}$$

and thus equation (331) gives

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^2 \left\{\frac{h+A}{h-A}\,\frac{F - G\surd(-C)}{F + G\surd(-C)}\right\}^{\frac{1}{2}}$$

$$= \left(\frac{h+A}{h-A}\right)^3 \frac{(1-3c)\surd(1-c) + \surd(1-5c+7c^2-c^3)}{(1-3c)\surd(1-c) - \surd(1-5c+7c^2-c^3)}$$

$$= \frac{(2-c)\surd(1-5c+7c^2-c^3) + (2-5c+c^2)\surd(1-c)}{(2-c)\surd(1-5c+7c^2-c^3) - (2-5c+c^2)\surd(1-c)} \tag{363}$$

$$\frac{H}{L} = \frac{(2-5c+c^2)\sqrt{(1-c)}}{(2-c)\sqrt{(1-5c+7c^2-c^3)}} \qquad (364),$$

so that

$$H = \frac{2-5c+c^2}{c} M \qquad (172),$$

$$L = \frac{2-c}{c} h \qquad (178),$$

as before.

35. The preceding verifications for $\mu = 3$ and $\mu = 5$ having served to settle the doubtful signs in the expressions, we now resume the case of

$$\mu = 7,$$

employing this new procedure.

With $z = \infty$, equations (298)–(303) show that

$$s = -x\frac{1-2y}{2+2y} = \tfrac{1}{2}c\,(1+c)^3\,\frac{1-2c-2c^2}{1-c-c^2} \qquad (365),$$

$$\sqrt{(-C)} = \frac{xh}{2A} = -\tfrac{1}{2}\frac{c\,(1+c)}{1-c-c^2}\,\frac{h}{M} \qquad (366),$$

$$F = (3+9c+5c^2)\,s^3 - (1+c)^3(1+6c+13c^2+5c^3)\,s^2 + c^2\,(1+c)^6\,(2+6c+c^2)\,s - c^4\,(1+c)^9 \qquad (367),$$

$$G = s^2-(1+c)^3(1+3c)\,s+c^2\,(1+c)^6 \qquad (368),$$

and proceeding as before, we shall find, after considerable reduction,

$$\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}}$$

$$= \left\{\frac{(1+c)(1-4c+c^2+3c^3) + \sqrt{(1-c-c^2)}\sqrt{(1-5c+0+15c^3+12c^4+c^5-c^6)}}{(1+c)(1-4c+c^2+3c^3) - \sqrt{(1-c-c^2)}\sqrt{(1-5c+0+15c^3+12c^4+c^5-c^6)}}\right\}^2 \qquad (369),$$

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^3 \left\{\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-C)}}{F+G\sqrt{(-C)}}\right\}^4 \qquad (370),$$

which reduces to

$$\frac{L+H}{L-H} = \frac{\begin{array}{c}(4-2c-5c^2+0+c^4)\sqrt{(1-5c+0+15c^3+12c^4+c^5-c^6)}\\ +(4-10c-15c^2+5c^3+10c^4+c^5-c^6)\sqrt{(1-c-c^2)}\end{array}}{\begin{array}{c}(4-2c-5c^2+0+c^4)\sqrt{(1-5c+0+15c^3+12c^4+c^5-c^6)}\\ -(4-10c-15c^2+5c^3+10c^4+c^5-c^6)\sqrt{(1-c-c^2)}\end{array}} \qquad (371),$$

or
$$\frac{H}{L} = \frac{4-10c-15c^2+5c^3+10c^4+c^5-c^6}{4-2c-5c^2+0+c^4}\,\frac{M}{h} \qquad (372).$$

Also
$$L^2-H^2=1$$

and
$$(4-2c-5c^2+0+c^4)\,h^2-(4-10c-15c^2+5c^3+10c^4+c^5-c^6)^2M^2$$
$$= c^4(1+c)^4 \qquad (373),$$

so that
$$H = \frac{4-10c-15c^2+5c^3+10c^4+c^5-c^6}{c^2(1+c)^2}M \qquad (374),$$

$$L = \frac{4-2c-5c^2+0+c^4}{c^2(1+c)^2}h \qquad (375).$$

36. The leading coefficients H and L being now determined, the remainder are readily found from the identities obtained by the logarithmic differentiation of (315) and a comparison with (316), namely,

$$0+PL = 0-H_1 \qquad (376),$$

$$0+PL_1 = -7H-2H_2 \qquad (377),$$

$$QL+PL_2 = -6H_1-3H_3 \qquad (378),$$

$$QL_1+PL_3 = -5H_2-4H_4 \qquad (379),$$

$$QL_2+PL_4 = -4H_3-5H_5 \qquad (380),$$

$$QL_3+PL_5 = -3H_4-6H_6 \qquad (381),$$

$$QL_4+0 = -2H_5-7H_7 \qquad (382),$$

$$QL_5+0 = -H_6-0 \qquad (383),$$

as in (185) to (190), and

$$0 + PH = hL - L_1 \qquad (384),$$

$$0 + PH_1 = -(6+h^2)\,L + 3hL_1 - 2L_2 \qquad (385),$$

$$QH + PH_2 = 10hL - 2\,(2+h^2)\,L_1 + 5hL_2 - 3L_3 \qquad (386),$$

$$QH_1 + PH_3 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (387),$$

$$QH_2 + PH_4 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (388),$$

$$QH_3 + PH_5 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (389),$$

$$QH_4 + PH_6 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (390),$$

$$QH_5 + PH_7 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (391),$$

$$QH_6 + \;\; 0 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (392),$$

$$QH_7 + \;\; 0 = \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (393).$$

I am indebted to Mr. T. I. Dewar for the calculation of these coefficients, and for a general verification of the work; his results are

$$H_1 = -PL = -\frac{(5+c-c^2)(4-2c-5c^2+0+c^4)}{c^2\,(1+c)^2}\,Mh \qquad (394),$$

$$H_2 = \frac{20-65c-93c^2+190c^3+307c^4+20c^5-151c^6-63c^7+8c^8+6c^9}{c^3\,(1+c)^4}\,M \qquad (395),$$

$$H_3 = \frac{-20+35c+139c^2+28c^3-167c^4-134c^5-7c^6+23c^7+c^8}{c^3\,(1+c)^4}\,Mh \qquad (396),$$

$$H_4 = \frac{5-45c+52c^2+217c^3-80c^4-459c^5-276c^6+59c^7+99c^8+24c^9}{c^4\,(1+c)^4}\,M \qquad (397),$$

$$H_5 = \frac{-1+19c-33c^2-46c^3+21c^4+35c^5+9c^6}{c^4\,(1+c)^3}\,Mh \qquad (398),$$

$$H_6 = -QL_5 = -\frac{2\,(1-4c-3c^2)(1-4c+0+6c^3+3c^4)}{c^4\,(1+c)^2}\,M \qquad (399),$$

$$H_7 = 2\,\frac{-1+2c+c^2}{c^3}\,Mh \qquad (400),$$

$$L_1 = \frac{-8+22c+34c^2-35c^3-60c^4-10c^5+14c^6+5c^7}{c^3\,(1+c)^4} \qquad (401),$$

$$L_2 = \frac{12-c-36c^2-21c^3+16c^4+17c^5+4c^6}{c^3\,(1+c)^4}\,h \qquad (402),$$

$$L_3 = \frac{-4+20c-3c^2-64c^3-19c^4+60c^5+51c^6+12c^7}{c^4(1+c)^4} \quad (403),$$

$$L_4 = \frac{1-8c+4c^2+10c^3+3c^4}{c^4(1+c)^4} \quad (404),$$

$$L_5 = \frac{1-4c+0+6c^3+3c^4}{c^4(1+c)^2} \quad (405),$$

and the verifications obtained by putting $z = \pm 1$,

$$A(L+L_2+L_4)+H_1+H_3+H_5+H_7 = 0 \quad (406),$$

$$A(L_1+L_3+L_5)+H+H_2+H_4+H_6 = 0 \quad (407),$$

are found to be satisfied.

Try putting $P = 0$; then

$$c^2-c-5 = 0, \quad c = \tfrac{1}{2}-\tfrac{1}{2}\sqrt{(21)} \quad (408),$$

taking the negative root, as this makes

$$1-c-c^2 \text{ and } 1-5c+0+15c^3+12c^4+c^5-c^6 \text{ negative,}$$

and therefore M^2 and h^2 positive; and now, from (299)–(303),

$$x = \frac{-39+9\sqrt{(21)}}{2}, \quad y = -6+\sqrt{(21)},$$

$$M^2 = \frac{1+\sqrt{(21)}}{30},$$

$$A^2 = \frac{-82+18\sqrt{(21)}}{15}, \quad h^2 = \frac{83-12\sqrt{(21)}}{15}$$

$$\mu = 8.$$

37. In this case the parameter

$$v = \tfrac{1}{2}\omega_3 \quad (409),$$

and the equation of the catenary will reduce to either of the forms

$$(1-z^2)\cos 2\chi = (Hz-H_1)\sqrt{Z_1} \quad (410),$$

$$(1-z^2)\sin 2\chi = (Lz-L_1)\sqrt{Z_2} \quad (411),$$

leading to the differential relation

$$2(1-z^2)\sqrt{Z}\frac{d\chi}{dz} = Pz^2+Q \quad (412),$$

where Z_1, Z_2, the quadratic factors of the quartic Z, are given in (256), (257).

The pseudo-elliptic form of (29) to be employed here must be taken from p. 229 of the article on "Pseudo-Elliptic Integrals," and, referring to p. 226, replacing the z employed there by $\frac{1}{2}(1+c)$, we must take

$$x = -\tfrac{1}{2}c(1+c) \qquad (413),$$

$$y = -c\frac{1+c}{1-c} \qquad (414),$$

$$1+y = \frac{1-2c-c^2}{1-c} \qquad (415),$$

$$M^2 = \frac{1-2c-c^2}{c(1-c^2)} \qquad (416),$$

$$A^2 = \frac{(1-2c-c^2)^3}{c(1+c)(1-c)^3} \qquad (417),$$

$$h^2 = \frac{(1-c-c^2)(1-6c+8c^2+6c^3-c^4)}{c(1+c)(1-c)^3} \qquad (418),$$

$$s_3 = \tfrac{1}{4}(1+c)^2 \qquad (419),$$

$$k^2 = 4M^2s_3-1-2y = \frac{1-c-c^2}{c} \qquad (420),$$

$$\frac{h^2}{k^2} = \frac{1-6c+8c^2+6c^3-c^4}{(1+c)(1-c)^3} \qquad (421),$$

$$8\rho = \frac{2c^2}{1-c} \qquad (422),$$

$$P = \tfrac{1}{2}(2A+M\rho) = \frac{1-2c}{1-c}M \qquad (423),$$

$$Q = \tfrac{1}{2}(2A-M\rho) = \frac{1-2c-2c^2}{1-c}M \qquad (424),$$

$$P+Q = 2A \qquad (425).$$

The coefficients H, H_1, L, L_1 can now be determined in a straightforward manner by differentiation and verification; in this way we find

$$L^2-H^2 = 1 \qquad (426),$$

$$L^2+H^2 = \frac{2-3c-c^2}{c(1+c)}\,\frac{h}{k} \qquad (427),$$

$$\frac{H}{H_1} = -\frac{(1+c)(1-3c)}{(1-c)^2}\,\frac{L_1}{L} \qquad (428),$$

$$\frac{L_1}{L} = -\frac{1-c}{2c\,(1+c)(1-3c)}\;\frac{1-4c-c^2+(1-c^2)\dfrac{h}{k}}{M} \qquad (429),$$

$$\frac{L}{H_1} = \frac{1}{2c\,(1-c)}\;\frac{1-4c-c^2+(1-c^2)\dfrac{h}{k}}{M} \qquad (430).$$

If we try to make this catenary a purely algebraical curve, by putting $P=0$, $c=\frac{1}{2}$, we find

$$M^2 = -\tfrac{2}{3} \qquad (431),$$

so that the catenary becomes imaginary.

$$\mu = 9.$$

38. This case has been worked out by Mr. T. I. Dewar, making use of the second method of § 30 for the determination of the leading coefficients H and L; the numerical calculations were extremely laborious, and the leading steps only are indicated here; the results satisfy the tests of accuracy that have been applied so far.

The equations are now of the form

$$(1-z^2)^{\frac{9}{2}}\cos 9\chi = Hz^9+H_1z^8+\ldots+H_9 \qquad (432),$$

$$(1-z^2)^{\frac{9}{2}}\sin 9\chi = (Lz^7+L_1z^6+\ldots+L_7)\sqrt{Z} \qquad (433),$$

leading to the relation

$$9\,(1-z^2)\sqrt{Z}\,\frac{d\chi}{dz} = Pz^3+Q \qquad (434)$$

Referring to "Pseudo-Elliptic Integrals," p. 232, we take

$$x = p^2\,(1-p)(1-p+p^2) \qquad (435),$$

$$y = p^2\,(1-p) \qquad (436),$$

$$1+y = 1+0+p^2-p^3 \qquad (437),$$

and the pseudo-elliptic form employed for (29) is

$$v = \tfrac{4}{9}\omega_3 \qquad (438),$$

$$I(v) = \tfrac{1}{3}\int \frac{\rho s - xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{9}\tan^{-1}\frac{s^3+Cs^2+Ds+E}{9\rho s^4+\sigma s^3+\tau s^2+Ts+V}\sqrt{S} = \tfrac{1}{9}\tan^{-1}\frac{G}{F}\sqrt{S} \quad (439),$$

where $9\rho = 1+0-3p^2+7p^3$ (440),

$$\sigma = -p^4(4-11p+11p^2+4p^3-17p^4+14p^5) \qquad (441),$$

$$\tau = \quad p^8(1-p+p^2)(6-23p+37p^2-24p^3-2p^4+7p^5) \qquad (442),$$

$$T = -p^{12}(1-p)^2(1-p+p^2)^2(4-9p+6p^2+p^3) \qquad (443),$$

$$V = \quad p^{16}(1-p)^4(1-p+p^2)^3 \qquad (444),$$

$$C = -3p^4(1-2p+2p^2) \qquad (445),$$

$$D = \quad p^8(1-p+p^2)(3-7p+5p^2) \qquad (446),$$

$$E = -p^{12}(1-p)^2(1-p+p^2)^2 \qquad (447),$$

$$P = \tfrac{1}{2}(9A+M\rho) = \quad (5+0+3p^2-p^3)\,M \qquad (448),$$

$$Q = \tfrac{1}{2}(9A-M\rho) = 2(2+0+3p^2-4p^3)\,M \qquad (449),$$

$$P+Q = 9A \qquad (450).$$

39. When $z = \infty$,

$$s = -x\frac{1-2y}{2+2y} = -\frac{p^2(1-p)(1-p+p^2)(1+0+2p^2-p^3)}{2(1+0+p^2-p^3)} \qquad (451),$$

$$\sqrt{(-S)} = \frac{x}{2+2y}\,\frac{h}{M}$$

$$= \frac{p^2(1-p)(1-p+p^2)}{2(1+0+p^2-p^3)}$$

$$\times\sqrt{\left(\frac{-1+0-5p^2+7p^3-11p^4+20p^5-20p^6+15p^7-7p^8+p^9}{-1+0+p^2-p^3}\right)} \qquad (452),$$

and $$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^4\left\{\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-S)}}{F+G\sqrt{(-S)}}\right\}^4 \qquad (453),$$

the expression requiring the enormous algebraical labour for its reduction.

Mr. Dewar first calculated

$$\frac{h+A}{h-A}\,\frac{F-G\surd(-S)}{F+G\surd(-S)} \qquad (454),$$

and found that it was a perfect square; its root was then multiplied by

$$\frac{h+A}{h-A} \qquad (455)$$

four times in succession, large common factors making their appearance each time in the numerator and denominator, and thus he found finally that, using detached coefficients,

$$\frac{L+H}{L-H} = \frac{\begin{matrix}(4-0+14-16+21-26+19-13+6-p^9)\surd(-1+0-5+7-11+20-20+15-7+p^9)\\ +(4-0+22-28+55-96+114-124+101-63+31-9+p^{12})\surd(-1+0+p^2-p^3)\end{matrix}}{\begin{matrix}(4-0+14-16+21-26+19-13+6-p^9)\surd(-1+0-5+7-11+20-20+15-7+p^9)\\ -(4-0+22-28+55-96+114-124+101-63+31-9+p^{12})\;(-1+0+p^2-p)\end{matrix}} \qquad (456),$$

$$\frac{H}{L} = \frac{\begin{matrix}4-0+22-28+55-96+114-124+101-63+31-9+p^{12}\surd(-1+0+p^2-p^3)\\ -1+0-5+7-11+20-20+15-7+p^9\end{matrix}}{4-0+14-16+21-26+19-13+6-p^9}$$

$$= \frac{4-0+22-28+55-96+114-124+101-63+31-9+p^{12}}{4-0+14-16+21-26+19-13+6-p^9}\,\frac{M}{h} \qquad (457),$$

and, since $L^2-H^2=1$,

$$(4-0+14-16+21-26+19-13-6-p^9)^2 h^2$$
$$-(4-0+22-28+55-96+114-124+101-63+31-9+p^{12})\,M^2 = p^8(1-p)^6(1-p+p^2)^2 \qquad (458);$$

therefore, finally,

$$H = \frac{4-0+22-28+55-96+114-124+101-63+31-9+p^{12}}{p^4(1-p)^3(1-p+p^2)}\,M \qquad (459),$$

$$L = \frac{4-0+14-16+21-26+19-13+6-p^9}{p^4(1-p)^3(1-p+p^2)}\,h \qquad (460).$$

40. The determination of the remaining coefficients is now comparatively an easy matter, and Mr. Dewar finds

$$H_1 = -PL = -\frac{(5-0+3-1)(4-10+14-16+21-26+19-13+6-p^9)}{p^4(1-p)^3(1-p+p^2)}\,Mh \qquad (461),$$

$$H_2 = \frac{\begin{matrix}-20+0-165+190-626+1169-1946+3141-4161+5063\\ -5262+4610-3453+2092-1011+371-84+8p^{17}\end{matrix}}{p^6(1-p)^4(1-p+p^2)^3}\,M \qquad (462),$$

$$H_3 = \frac{\begin{array}{r}20-10+175-210+642-1123+1842-2619+3043-3177\\ +2738-2010+1247-608+237-65+8p^{16}\end{array}}{p^6(1-p)^4(1-p+p^2)^2} Mh \qquad (463),$$

$$H_4 = \frac{\begin{array}{r}5+5+90-45+489-877+1296-3946+6401-10002+13532\\ -16302+17394-15762+12194-7806+3974-1558+406-48p^{19}\end{array}}{p^8(1-p)^5(1-p+p^2)^2} M \qquad (464),$$

$$H_5 = \frac{\begin{array}{r}-1-2-39-8-220+283-575+1206-1582+2080-2194\\ +1852-1378+772-330+114-20p^{16}\end{array}}{p^8(1-p)^5(1-p+p^2)} Mh \qquad (465),$$

$$H_6 = \frac{\begin{array}{r}-4-4-64+40-288+636-1076+2320-3580+4980-6515\\ +7091-6698+5294-3258+1551-512+80p^{17}\end{array}}{p^8(1-p)^6(1-p+p^2)} M \qquad (466),$$

$$H_7 = \frac{12-0+68-120+160-400+524-520+517-348+157-65+16p^{12}}{p^6(1-p)^6} Mh \qquad (467),$$

$$H_8 = -QL_7 = \frac{2(2-0+3-4)(2-0+8-14+16-30+34-26+15-4p^9)}{p^6(1-p)^6} M \qquad (468),$$

$$H_9 = -\frac{2(1-1+1)(2-0+4-8+4-2+p^6)}{p^4(1-p)^5} Mh \qquad (469),$$

$$L_1 = \frac{8-0+54-60+164-284+389-542+575-544+440-274+139-48+7p^{14}}{p^6(1-p)^4(1-p+p^2)^2} \qquad (470),$$

$$L_2 = 3\,\frac{-4+0-23+22-57+82-109+132-120+100-64+30-11+2p^{13}}{p^6(1-p)^4(1-p+p^2)^2} h \qquad (471),$$

$$L_3 = \frac{\begin{array}{r}-4-4-48+16-185+299-541+1009-1365+1807-2000\\ +1860-1516+974-487+175-30p^{16}\end{array}}{p^8(1-p)^5(1-p+p^2)^2} \qquad (472),$$

$$L_4 = \frac{1+1+23-15+97-161+286-436+560-654+598-470+294-130+46-10p^{15}}{p^8(1-p)^5(1-p+p^2)^2} h \qquad (473),$$

$$L_5 = \frac{3+3+30-15+87-192+246-468+618-660+672-516+306-138+30p^{14}}{p^8(1-p)^6(1-p+p^2)} \qquad (474),$$

$$L_6 = \frac{-6+0-16+30-20+42-46+22-11+4p^9}{p^6(1-p)^6} h \qquad (475),$$

$$L_7 = \frac{-2+0-8+14-16+30-34+26-15+4p^9}{p^6(1-p)^6} \qquad (476).$$

Mr. Dewar has also performed the verification of showing that

$$A(L+L_2+L_4+L_6)+H_1+H_3+H_5+H_7+H_9 = 0 \qquad (477),$$

$$A(L_1+L_3+L_5+L_7)+H+H_2+H_4+H_6+H_8 = 0 \qquad (478).$$

$$\mu = 10.$$

41. This investigation was interesting as affording the first case of a purely algebraical Spherical Catenary, shown in the stereographic and stereoscopic projections of the accompanying diagrams (pp. 170, 171), drawn to scale and in perspective by Mr. T. I. Dewar.

The equations to be satisfied are

$$(1-z^2)^{\frac{5}{2}} e^{\chi i} = (Hz^5 + H_1 z^4 + \ldots + H_5) + i(Lz^3 + L_1 z^2 + \ldots + L_3)\sqrt{Z} \quad (479),$$

leading to

$$5(1-z^2)\sqrt{Z}\frac{d\chi}{dz} = Pz^2 + Q \quad (480),$$

as in the case of $\mu = 5$.

Referring to the *Proc. Lond. Math. Soc.*, Vol. xxv., p. 235, the relation

$$\gamma_{10} = 0 \quad (481),$$

the equation of a unicursal quintic curve, is satisfied by

$$x = \frac{-a(1+a)}{(1-a)(1-a-a^2)^2} \quad (482),$$

$$y = \frac{-a(1+a)}{(1-a)(1-a-a^2)} \quad (483),$$

$$1+y = \frac{1-3a-a^2+a^3}{(1-a)(1-a-a^2)} \quad (484),$$

and thence

$$M^2 = \frac{(1-a-a^2)(1-3a-a^2+a^3)}{2a(1+a)} \quad (485),$$

$$A^2 = \frac{(1-3a-a^2+a^3)^3}{2a(1+a)(1-a)^2(1-a-a^2)} \quad (486),$$

$$h^3 = \frac{(1-5a-a^2+a^3)(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}{2a(1+a)(1-a)^2(1-a-a^2)} \quad (487).$$

The pseudo-elliptic integral to employ for (29) is

$$v = \tfrac{2}{5}\omega_3,$$

$$I(v) = \tfrac{1}{2}\int \frac{\rho s - xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{5}\tan^{-1}\frac{s-C}{5\rho s^2 - \sigma s + \tau}\sqrt{S} = \tfrac{1}{5}\tan^{-1}\frac{G\sqrt{S}}{F} \quad (488),$$

where $$5\rho = \frac{3+3a+a^2-a^3}{(1-a)(1-a-a^2)} \qquad (489),$$

$$\sigma = \frac{3(1+a)^2(1+a+2a^2-a^3)}{(1-a)^3(1-a-a^2)^3} \qquad (490),$$

$$\tau = \frac{a^2(1+a)^4}{(1-a)^4(1-a-a^2)^5} \qquad (491),$$

$$C = \frac{(1+a)^2}{(1-a)^2(1-a-a^2)^2} \qquad (492).$$

Thence $$P = \tfrac{5}{2}(A+M\rho) = 2\frac{2-a}{1-a}M \qquad (493),$$

$$Q = \tfrac{5}{2}(A-M\rho) = \frac{1-9a-3a^2+3a^3}{(1-a)(1-a-a^2)}M \qquad (494),$$

$$P+Q = 5A \qquad (495).$$

We take $$s_3 = \frac{a^2}{(1-a)^2(1-a-a^2)^2} \qquad (496),$$

$$k^2 = 4M^2s_3-1-2y = \frac{-1+5a+a^2-a^3}{(1+a)(1-a)^2} \qquad (497),$$

$$\frac{h^2}{k^2} = \frac{-1+6a-3a^2-8a^3+3a^4+2a^5-a^6}{2a(1-a-a^2)} \qquad (498).$$

42. Corresponding to $z = \infty$,

$$s = -x\frac{1+2y}{2+2y} = \frac{a(1+a)(1-4a-2a^2+a^3)}{2(1-a)(1-a-a^2)(1-3a-a^2+a^3)} \qquad (499),$$

$$\sqrt{(-S)} = \frac{xh}{2A} = -\frac{a(1+a)}{2(1-a)(1-a-a^2)^2}\,\frac{h}{A} \qquad (500).$$

Working with these values, Mr. Dewar found, after a long calculation,

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^2 \left\{\frac{h+A}{h-A}\,\frac{F-G\sqrt{(-S)}}{F+G\sqrt{(-S)}}\right\}^{\frac{1}{2}}$$

$$= \frac{\begin{array}{l}(1-a-a^2)(2-6a+0+3a^3-a^4)\sqrt{(-1+5a+a^2-a^3)} \\ \quad +(2-4a-a^2+a^3)\sqrt{(-1+3a+a^2-a^3)}\sqrt{(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}\end{array}}{\begin{array}{l}(1-a-a^2)(2-6a+0+3a^3-a^4)\sqrt{(-1+5a+a^2-a^3)} \\ \quad -(2-4a-a^2+a^3)\sqrt{(-1+3a+a^2-a^3)}\sqrt{(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}\end{array}} \qquad (501),$$

$$\frac{H}{L} = \frac{(2-4a-a^2+a^3)\sqrt{(-1+3a+a^2-a^3)}\sqrt{(1-6a+3a^2+8a^3-3a^4-2a^5+a^6)}}{(1-a-a^2)(2-6a+0+3a^3-a^4)\sqrt{(-1+5a+a^2-a^3)}} \quad (502);$$

and therefore

$$H = \frac{2-4a-a^2+a^3}{a^2(1+a)(1-a)}\sqrt{\left\{\frac{(-1+3a+a^2-a^3)(1-6a-3a^2+8a^3-3a^4-2a^5+a^6)}{1+a}\right\}} \quad (503),$$

$$L = \frac{(1-a-a^2)(2-6a+0+3a^3-a^4)}{a^2(1+a)(1-a)}\sqrt{\left(\frac{-1+5a+a^2-a^3}{1+a}\right)} \quad (504).$$

Thence the other coefficients follow, and writing

α for $-1+3a+a^2-a^3$,

β for $-1+5a+a^2-a^3$,

γ for $-1+a+a^2$,

δ for $1-6a+3a^2+8a^3-3a^4-2a^5+a^6$,

$$H_1 = -\frac{2(2-a)(1-a-a^2)(2-6a+0+3a^3-a^4)}{a^2(1+a)^2(1-a)^2}\sqrt{\left(\frac{\alpha\beta\gamma}{2a}\right)} \quad (505),$$

$$H_2 = \frac{12-60a+32a^2+91a^3-23a^4-30a^5+8a^6+3a^7-a^8}{2a^3(1+a)^2(1-a)}\sqrt{\left(\frac{\alpha\delta}{1+a}\right)} \quad (506),$$

$$H_3 = \frac{-4+28a-28a^2-64a^3+37a^4+36a^5-18a^6-6a^7+3a^8}{a^3(1+a)^3(1-a)}\sqrt{\left(\frac{\alpha\beta\gamma}{2a}\right)} \quad (507),$$

$$H_4 = -QL_3 = \frac{(1-9a-3a^2+3a^3)(1-3a-2a^2)}{2a^4(1+a)^2}\sqrt{\left(\frac{\alpha\delta}{1+a}\right)} \quad (508),$$

$$H_5 = \frac{1-4a-a^2+2a^3}{a^3(1+a)^2}\sqrt{\left(\frac{\alpha\beta\gamma}{2a}\right)} \quad (509),$$

$$L_1 = \frac{-6+16a+10a^2-7a^3-2a^4+a^5}{a^2(1+a)^2}\sqrt{\left(\frac{\gamma\delta}{2a}\right)} \quad (510),$$

$$L_2 = \frac{(1-a-a^2)(3-9a-6a^2+8a^3+2a^4-2a^5)}{a^3(1+a)^2}\sqrt{\left(\frac{\beta}{1+a}\right)} \quad (511),$$

$$L_3 = \frac{-(1-a)(1-3a-2a^2)}{a^3(1+a)^2}\sqrt{\left(\frac{\gamma\delta}{2a}\right)} \quad (512),$$

and these values verify the equations

$$A(L+L_2)+H_1+H_3+H_5 = 0 \quad (513),$$

$$A(L_1+L_3)+H+H_2+H_4 = 0 \quad (514).$$

43. Here we can make P vanish by taking

$$a = 2,$$

and this gives real numerical values to the coefficients, namely,

$$H = \frac{1}{6}\sqrt{\left(\frac{17}{3}\right)}, \quad H_1 = 0, \quad H_2 = -\frac{5}{12}\sqrt{\left(\frac{17}{3}\right)}, \quad H_3 = -\frac{25}{108},$$

$$H_4 = \frac{65}{288}\sqrt{\left(\frac{17}{3}\right)}, \quad H_5 = \frac{25}{144};$$

$$L = -\frac{5}{6}\sqrt{\frac{5}{3}}, \quad L_1 = -\frac{5}{36}\sqrt{(85)}, \quad L_2 = \frac{35}{72}\sqrt{\frac{5}{3}}, \quad L_3 = \frac{13}{144}\sqrt{(85)},$$

$$h = \frac{1}{2}\sqrt{\left(\frac{17}{3}\right)}, \quad A = -\frac{1}{2\sqrt{(15)}};$$

and the equation of the catenary can be written in either of the forms

$$r^5 \cos 5\psi = \frac{48\sqrt{(51)}\,z^5 - 120\sqrt{(51)}\,z^3 - 200z^2 + 62\sqrt{(51)}\,z + 150}{864} \quad (515),$$

$$r^5 \sin 5\psi = \frac{-120\sqrt{(15)}\,z^4 - 60\sqrt{(85)}\,z^3 + 70\sqrt{(15)}\,z - 39\sqrt{(85)}}{432}$$

$$\times\sqrt{\left\{(1-z^2)\left[\frac{1}{2}\sqrt{\left(\frac{17}{3}\right)} - z\right]^2 - \frac{1}{60}\right\}} \quad (516),$$

so that its projection on the equatorial plane is a closed algebraical curve, with pentagonal symmetry.

With $\qquad a = 2, \quad k^2 = \frac{5}{3},$

$$k = -\tfrac{1}{3}\sqrt{(15)}, \quad \frac{h}{k} = -\frac{\sqrt{(85)}}{10} \quad (517),$$

$$Z_1 = -z^2 + \frac{\sqrt{(51)} - 2\sqrt{(15)}}{6}z - \tfrac{1}{3} + \frac{2\sqrt{(85)}}{15} \quad (518),$$

$$Z_2 = \quad z^2 + \frac{\sqrt{(51)} + 2\sqrt{(15)}}{6}z - \tfrac{1}{3} - \frac{2\sqrt{(85)}}{15} \quad (519).$$

The roots of $Z_2 = 0$ are imaginary, but the roots of $Z_1 = 0$ are

$$z_0 = -0{\cdot}9982585 \quad (520),$$

$$z_1 = \quad 0{\cdot}8975022 \quad (521),$$

giving the limits of latitude between which the catenary lies, namely,

86° 37′ 5″ N. and 63° 49′ 54″ S.

The curve crosses the twenty prime meridians, at intervals of 18°, at the points corresponding to the roots h_1, h_2, h_3, h_4, of the quintic

$$Hz^5+H_1z^4+\ldots+H_5=0 \qquad (522),$$

and to the roots, l_1, l_2, l_3, of the cubic

$$Lz^3+L_1z^2+L_2z+L_3=0 \qquad (523).$$

These roots have been calculated by Mr. Dewar, employing Horner's method, and the results are tabulated in the following columns; it will be observed that, near the upper N. pole, the roots are crowded together, and the disentanglement of these roots caused some trouble.

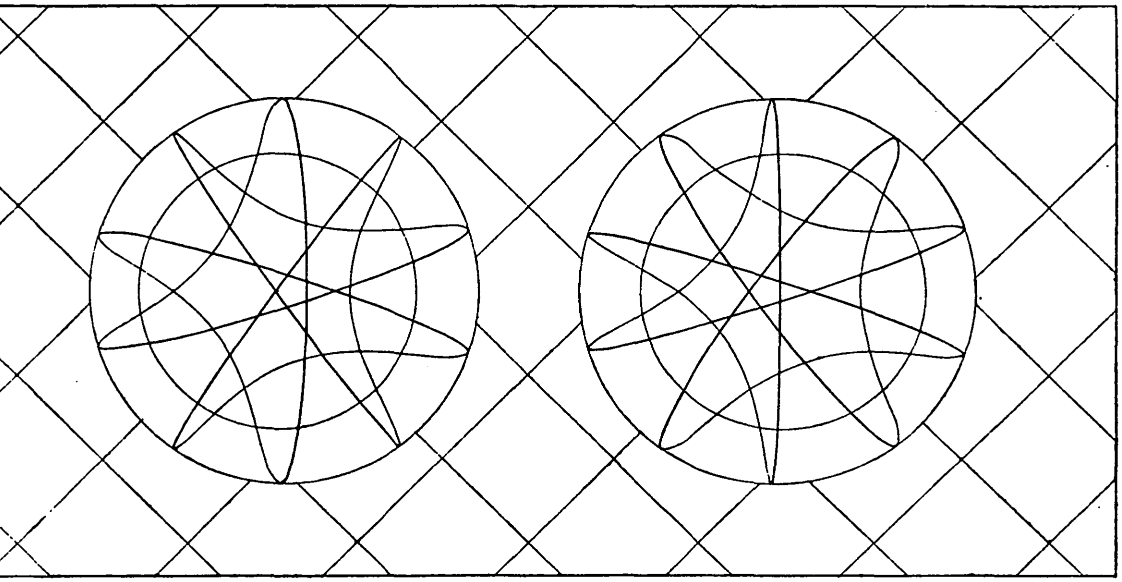
Some intermediate points were also calculated, for the purpose of plotting the curve with accuracy on a globe; and, denoting the latitude (South) by λ, the following table embodies the numerical results :—

z	$\sin\lambda$	$\cos\lambda$	$\tan\frac{1}{2}(90°-\lambda)$	λ	ψ
z_0	−0·9982585	0·0589918	−0·0295	86° 37′ 5″ N.	0° 0′ 0″
h_1	9980756	0619599	0311	86 26 42	18 0 0
l_1	9973452	0728175	0365	85 49·27	36 0 0
h_2	9949993	0998841	0501	84 16 3	54 0 0
l_2	9824577	1864859	0941	79 15 8	72 0 0
h_3	3436408	9391010	6989	20 5 56 N.	90 0 0
0	0·	1·	1·	0 0 0	91 59 50
$\frac{1}{2}h$	+0·5951190	0·8036694	0·5038	36 31 12 S.	98 45 0
l_3	7895648	6136251	3430	52 8 42	108 0 0
h_4	8805984	4738620	2520	61 42 53	126 0 0
z_1	8975022	4410099	2324	63 49 54	144 0 0
h_5	1·4561185				

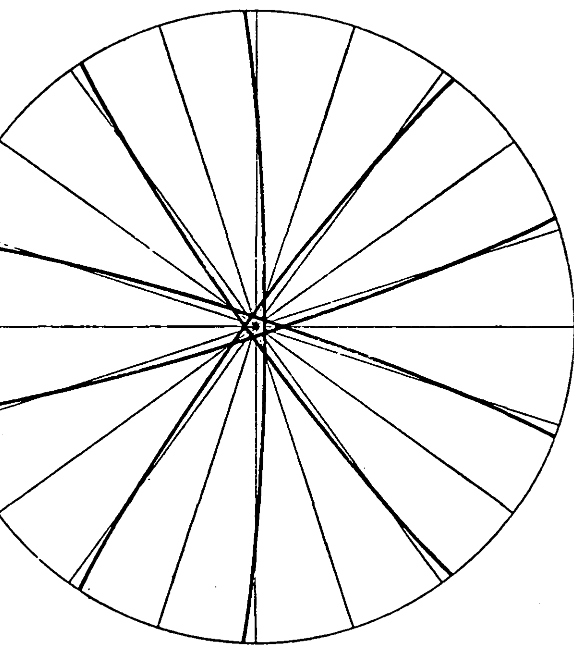
The curve crosses the equator at an angle 83° 46′ 24″, and it makes a maximum angle, 86° 56′ 45″, with the parallel of latitude 28° 0′ 31″ N.

Below the depth $\frac{1}{2}h$ from the centre, that is, in latitude greater than 36° 31′ 12″ S., the pressure changes sign, and the chain must be supposed to rest on the inside of a spherical surface.

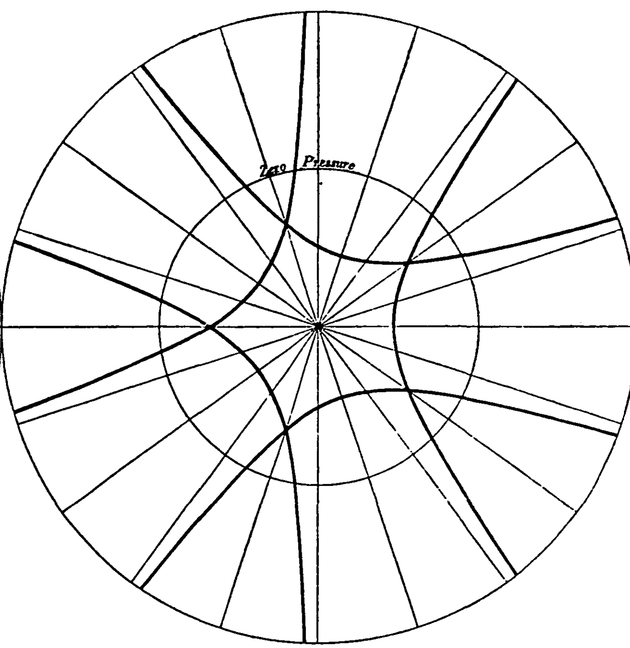
Stereoscopic View of Catenary.



Upper Hemisphere.



Lower Hemisphere.



The stereoscopic photograph can be made to show the solid figure, in the absence of a stereoscope, either by fixing the eyes on a distant object, and then raising the card into the line of sight at the distance of distinct vision; or else by holding the card at a distance of about the arm's length, and focussing the eyes on a point half way; a smaller image is now formed, and the tesselated background, representing the directrix plane, is now seen as a network in front of the solid sphere.

$$\mu = 12.$$

44. Here the parameter of (29) is

$$v = \tfrac{1}{3}\omega_3 \qquad (524),$$

and the equation of the catenary may be written

$$(1-z^2)^{\frac{3}{2}} e^{3\chi i} = (Hz^2+H_1z+H_2)\sqrt{Z_2}+i\,(Lz^2+L_1z+L_2)\sqrt{Z_1} \qquad (525),$$

leading to

$$(1-z^2)\sqrt{Z}\frac{d\chi}{dz} = pz^2+q \qquad (526).$$

Referring to "Pseudo-Elliptic Integrals," p. 248,

$$x = -\frac{a(1+a)(1+a^2)(1+a+a^2)}{(1-a)^2} \qquad (527),$$

$$y = -\frac{a(1+a)(1+a+a^2)}{1-a} \qquad (528),$$

$$1+y = \frac{1-2a-2a^2-2a^3-a^4}{1-a} \qquad (529),$$

$$M^2 = \frac{(1-a)(1-2a-2a^2-2a^3-a^4)}{2a(1+a)(1+a^2)(1+a+a^2)} \qquad (530),$$

Clebsch's x or ξ is given by (265), so that, with

$$t = -\frac{a(1+a+a^2)}{1-a} \qquad (531),$$

$$\xi = M(1+a)(1+a^2) \qquad (532),$$

and

$$\frac{h^2}{k^2} = 1-\xi^2 = -\frac{1-4a-4a^2-4a^3-2a^4+2a^5+2a^6+2a^7+a^8}{2a(1+a+a^2)} \qquad (533).$$

But

$$k^2 = A^2 - 1 - 2y$$

$$= \frac{(1-4a-4a^2-4a^3-a^4)(1-4a-4a^2-4a^3-2a^4+2a^5+2a^6+2a^7+a^8)}{2a\,(1-a^4)(1+a+a^2)} \qquad (534),$$

so that

$$k^2 = -\frac{1-4a-4a^2-4a^3-a^4}{1-a^4} \qquad (535),$$

and the same value of k^2 is obtained from formula (258) with

$$s = t^2 = \frac{a^2\,(1+a+a^2)^2}{(1-a)^2} \qquad (536),$$

so that

$$\frac{k^2-1}{2} = -\frac{1-2a-2a^2-2a^3-a^4}{1-a^4} \qquad (537),$$

$$\frac{k^2+1}{2} = 2a\frac{1+a+a^2}{1-a^4} \qquad (538).$$

45. The chief difficulty here is the determination of the appropriate value of ρ to employ in (29), which we distinguish as $\rho\,(\frac{1}{3}\omega_3)$; but this is given by the general formula

$$\rho\left(\frac{4\omega_3}{\mu}\right)+2\rho\left(\frac{2\omega_3}{\mu}\right) = -(1+y) \qquad (539),$$

and with

$$\mu = 12,$$

$$12\rho\,(\tfrac{1}{6}\omega_3) = -2\,(5+3a+3a^2+a^3) \qquad (540),$$

(*Proc. Lond. Math. Soc.*, Vol. xxv., p. 251); so that

$$12\rho\,(\tfrac{1}{3}\omega_3) = 4\,(5+3a+3a^2+a^3) - 12\frac{1-2a-2a^2-2a^3-a^4}{1-a}$$

$$= 8\frac{(1+a+a^2)^2}{1-a} \qquad (541),$$

Next

$$p = \tfrac{1}{2}\,(A+M\rho) = \tfrac{1}{2}M\left\{\frac{1-2a-2a^2-2a^3-a^4}{1-a}+\tfrac{2}{3}\,\frac{(1+a+a^2)^2}{1-a}\right\}$$

$$= \tfrac{1}{6}M\,(5+3a+3a^2+a^3) \qquad (542),$$

$$q = \tfrac{1}{2}\,(A-M\rho) = \tfrac{1}{6}M\frac{1-10a-12a^2-10a^3-5a^4}{1-a} \qquad (543),$$

$$p+q = A \qquad (544).$$

The integral (29) now assumes the pseudo-elliptic form

$$I(\tfrac{1}{3}\omega_3) = \tfrac{1}{2}\int \frac{\tfrac{2}{3}\dfrac{(1+a+a^2)^2}{1-a} - 3\dfrac{a^2(1+a)^2(1+a^2)(1+a+a^2)^2}{1-a}}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{3}\tan^{-1}\frac{F\sqrt{(s-s_3)}}{G\sqrt{\{4(s-s_1)(s-s_2)\}}} \qquad (545),$$

where $$F = 2s - \frac{(1+a)^2(1+a^2)(1+a+a^2)^2}{(1-a)^2} \qquad (546),$$

$$G = \frac{(1+a+a^2)^2}{1-a} \qquad (547).$$

46. Proceeding to the value $z = \infty$,

$$\frac{L+H}{L-H} = \left(\frac{h+A}{h-A}\right)^{\frac{1}{2}} \left\{\frac{F\sqrt{(s-s_3)} - G\sqrt{(-4\,.\,s-s_1\,.\,s-s_2)}}{F\sqrt{(s-s_3)} + G\sqrt{(-4\,.\,s-s_1\,.\,s-s_2)}}\right\}^{\frac{1}{2}} \qquad (548),$$

where

$$s = \frac{1+2y}{2+2y} = \frac{a(1+a)(1+a^2)(1+a+a^2)(1-3a-4a^2-4a^3-2a^4)}{2(1-a)^2(1-2a-2a^2-2a^3-a^4)} \qquad (549).$$

Writing α for $1-2a-2a^2-2a^3-a^4$ (550),

β for $1-4a-4a^2-4a^3-a^4$ (551),

γ for $1-4a-4a^2-4a^3-2a^4+2a^5+2a^6+2a^7+a^8$ (552),

so that $$M^2 = \frac{(1-a)\alpha}{2a(1+a)(1+a^2)(1+a+a^2)} \qquad (553),$$

$$A^2 = \frac{\alpha^3}{2a(1-a^4)(1+a+a^2)} \qquad (554),$$

$$h^2 = \frac{\beta\gamma}{2a(1-a^4)(1+a+a^2)} \qquad (555),$$

$$\frac{A^2}{h^2} = \frac{\alpha^3}{\beta\gamma} \qquad (556),$$

$$\frac{h+A}{h-A} = \frac{\sqrt{(\beta\gamma)}+\alpha\sqrt{\alpha}}{\sqrt{(\beta\gamma)}-\alpha\sqrt{\alpha}} \qquad (557).$$

Then $$s-s_3 = \frac{a(1+a+a^2)}{2(1-a)^2}\,\frac{\beta}{\alpha} \tag{558},$$

$$-4(s-s_1)(s-s_2) = \frac{a(1+a)^2(1+a^2)^2(1+a+a^2)\gamma}{2(1-a)^2\alpha^2} \tag{559},$$

$$\frac{F\surd(s-s_3)}{G\surd(-4\,.\,s-s_1\,.\,s-s_2)} = \frac{1-a-a^2-5a^3-7a^4-6a^5-4a^6-a^7}{(1-a)^3}\sqrt{\left(\frac{\beta}{\alpha\gamma}\right)} \tag{560},$$

and finally, after considerable reduction, Mr. Dewar finds

$$\frac{L+H}{L-H} = \sqrt{\left\{\frac{m\surd\gamma+n\surd(\alpha\beta)}{m\surd\gamma-n\surd(\alpha\beta)}\right\}} \tag{561},$$

where $$m = 4-10a-3a^2-3a^3+9a^4+12a^5+9a^6+5a^7+a^8 \tag{562},$$

$$n = (1-a)(4-2a-5a^2-10a^3-11a^4-7a^5-4a^6-a^7) \tag{563},$$

and thence the coefficients H and L can be inferred.

Putting $$L = \cosh\lambda,\quad H = \sinh\lambda \tag{564},$$

then $$e^{4\lambda} = \frac{m\surd\gamma+n\surd(\alpha\beta)}{m\surd\gamma-n\surd(\alpha\beta)} \tag{565},$$

and $$\tanh 2\lambda = \frac{n\surd(\alpha\beta)}{m\surd\gamma} \tag{566},$$

$$\cosh 2\lambda = \frac{m\surd\gamma}{l\surd\delta},\quad \sinh 2\lambda = \frac{n\surd(\alpha\beta)}{l\surd\delta} \tag{567},$$

where $$l = a^2(1+a)^2(1+a^2)(1+a+a^2) \tag{568},$$

$$\delta = -2a(1+a+a^2) \tag{569},$$

and $$L^2 = \tfrac{1}{2}(\cosh 2\lambda+1) = \frac{m\surd\gamma+l\surd\delta}{2l\surd\delta} \tag{570},$$

$$H^2 = \tfrac{1}{2}(\cosh 2\lambda-1) = \frac{m\surd\gamma-l\surd\delta}{2l\surd\delta} \tag{571},$$

$$L = \frac{\surd\left[m\surd\gamma+n\surd(\alpha\beta)\right]+\surd\left[m\surd\gamma-n\surd(\alpha\beta)\right]}{2\surd l\sqrt[4]{\delta}} \tag{572},$$

$$H = \frac{\surd\left[m\surd\gamma+n\surd(\alpha\beta)\right]-\surd\left[m\surd\gamma-n\surd(\alpha\beta)\right]}{2\surd l\sqrt[4]{\delta}} \tag{573}.$$

These values of H and L have been checked by Mr. Dewar by a straightforward verification, but the work was very long.

This catenary will be an algebraical curve if we can make $P = 0$, and then

$$4+(1+a)^3 = 0, \quad a = -1-\sqrt[3]{4} \qquad (574);$$

this makes

$$k^2 = 1-\tfrac{1}{2}\sqrt[3]{4} \qquad (575),$$

but

$$a = -\tfrac{2}{3}(\sqrt[3]{2}+1),$$

so that the catenary is imaginary.

47. It will be noticed now that, in general,

$$P = \tfrac{1}{2}\mu M\rho\left(\frac{2\omega_3}{\mu}\right) \qquad (576),$$

so that an algebraical catenary requires as a first condition that

$$\rho\left(\frac{2\omega_3}{\mu}\right) = 0 \qquad (577);$$

afterwards we must examine the reality of the catenary by finding out if M^2, A^2, h^2, k^2, ... are positive.

Taking, for example, the case of

$$\mu = 14$$

(*Proc. Lond. Math. Soc.*, Vol. xxv., p. 257), and the formula (p. 206)

$$\mu\rho\wp' v = \tfrac{1}{2}(q_1q_2+q_2q_3+\ldots+q_{\mu-4}q_{\mu-3})-(\mu-2)\wp'' v,$$

or

$$-\mu\rho x = \tfrac{1}{2}(q_1q_2+\ldots)-(\mu-2)\,x\,(1+y) \qquad (578),$$

where

$$q_{r-1} = 2(s_r-s_1) = -2x^{\frac{1}{2}}\frac{\gamma_{r-1}\gamma_{r+1}}{\gamma_r^2} \qquad (579),$$

$$q_rq_{r-1} = 4x^{\frac{1}{2}}\frac{\gamma_{r-1}\gamma_{r+2}}{\gamma_r\gamma_{r+1}} \qquad (580),$$

we find (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 199)

$$\tfrac{1}{4}q_1q_2 = xy \qquad (581),$$

$$\tfrac{1}{4}q_2q_3 = x - \frac{x^2}{y} \qquad (582),$$

$$\tfrac{1}{4}q_3q_4 = \frac{x^2}{y} - \frac{x^2y}{y-x} \qquad (583),$$

$$\tfrac{1}{4}q_4q_5 = \frac{x^2y}{y-x} - \frac{xy^3}{y-x-y^2} \qquad (584),$$

$$\tfrac{1}{4}q_5q_6 = \frac{xy^3}{y-x-y^2} - \frac{xy\{(y-x)(y-2x)+y^3\}}{xy-x^2-y^3} \qquad (585),$$

$$\tfrac{1}{4}q_6q_7 = \frac{xy\{(y-x)(y-2x)+y^3\}}{xy-x^2-y^3} - x\,\frac{(y-x)^3+y^2(y-x)^2-2xy^2(y-x-y^2)}{y\{x(y-x-y^2)-(y-x)^2\}} \qquad (586),$$

$$\tfrac{1}{4}q_7q_8 = x\,\frac{(y-x)^3+y^2(y-x)^2-2xy^2(y-x-y^2)}{y\{x(y-x-y^2)-(y-x)^2\}} - \frac{N_9}{y^3(y-x-y^2)-(y-x)^3} \qquad (587).$$

where $$N_9 = x\{(y-x)^3-3xy(y-x)^2+2y^6\}$$

48. With $\mu = 14$, $q_{12} = 0$ (Abel), and

$$q_{11} = q_1,\quad q_{10} = q_2,\quad q_9 = q_3,\quad q_8 = q_4,\quad q_7 = q_5 \qquad (588),$$

$$\begin{aligned} -14\rho x &= q_1q_2+q_2q_3+q_3q_4+q_4q_5+q_5q_6-12x(1+y) \\ &= 4x(1+y) - 4xy\,\frac{(y-x)(y-2x)+y^3}{xy-x^2-y^3} - 12x(1+y) \qquad (589), \end{aligned}$$

$$\begin{aligned} \tfrac{7}{2}\rho &= 2(1+y)+y\,\frac{(y-x)(y-2x)+y^3}{xy-x^2-y^3} \\ &= y+2+\frac{y(y-x)^2}{xy-x^2-y^3} \\ &= y+2+c(z-p) \qquad (590), \end{aligned}$$

and, with the values of p, z, y given on p. 257, *Proc. Lond. Math. Soc.*, Vol. xxv., this reduces to

$$7\rho = \frac{4+4c+3c^2+2c^3+c\sqrt{C}}{1+c} \qquad (591),$$

where $$C = c(1+2c)(4+5c+2c^2) \qquad (592).$$

Then ρ, and therefore P, vanishes if

$$(4+4c+3c^2+2c^3)^2 - c^3(1+2c)(4+5c+2c^2) = 0,$$

or
$$(1+c)(4+4c+6c^2+3c^3) = 0,$$

or
$$(c+1)\left\{(3c+2)^3+28\right\} = 0 \qquad (593),$$
$$3c+2 = -\sqrt[3]{(28)}.$$

With
$$a = \sqrt[3]{(28)} = 2^{\frac{2}{3}}.\,7^{\frac{1}{3}},$$
$$b = \sqrt[3]{(98)} = 2^{\frac{1}{3}}.\,7^{\frac{2}{3}},$$

we find
$$\sqrt{C} = \frac{14+5a-2b}{9},$$

and
$$p = \frac{a}{7} \text{ or } \frac{28+2b+7a}{21} \qquad (594);$$

but these values make M^2 negative, and the catenary is imaginary.

49. So also, with
$$\mu = 18, \quad v = \tfrac{2}{9}\omega_3,$$
$$q_{16} = 0, \quad q_{15} = q_1, \quad q_{14} = q_2, \quad q_{15} = q_3 \qquad (595),$$
$$-18\rho x = q_1q_2+q_2q_3+\ldots+q_7q_8-16x(1+y)$$
$$= 4x(1+y)-4\frac{N_9}{y^3(y-x-y^2)-(y-x)^3}-16x(1+y) \qquad (596),$$
$$\tfrac{9}{2}\rho = 3(1+y)+\frac{N_9}{y^3(y-x-y^2)-(y-x)^3} \qquad (597),$$

reducing ultimately to (*Proc. Lond. Math. Soc.*, Vol. xxv., p. 265)
$$\tfrac{9}{2}\rho = 3+y-z\left(1-\frac{c}{t}\right)$$
$$= -q^3+3q^2+16q+13-3\sqrt{Q} \qquad (598).$$

Therefore ρ and P vanish, if
$$(q^3-3q^2-6q-13)^2-9Q = 0 \qquad (599),$$

or
$$q^6+3q^5+6q^4+10q^3-3q^2-15q-20 = 0 \qquad (600).$$

This sextic equation has only two real roots
$$+1{\cdot}21921268, \text{ and } -2{\cdot}301874537 \qquad (601),$$

which were calculated by Mr. Dewar, with Horner's method; he has also calculated the remaining quartic factor
$$q^4+1{\cdot}917338143q^3+6{\cdot}730647q^2+8{\cdot}09394q+7{\cdot}1264 \qquad (602),$$

and has resolved the sextic into its cubic factors in the two ways

$$(q^3+3\cdot914721q^2+5\cdot278q+3\cdot464)(q^3-0\cdot914721q^2+4\cdot3629q-5\cdot772) \qquad (603),$$

$$(q^3+2\cdot60635q^2+5\cdot43504q+10\cdot898)(q^3+0\cdot39365q^2-0\cdot46117q-1\cdot8352) \qquad (604).$$

If we had obtained a simple rational root, as in the case of $\mu = 10$, it would have been worth while to go on with the calculation of this algebraical catenary, which would possess the symmetry of the nonagon, the prime meridians being 10° apart.

$$\mu = 16.$$

50. The parameter of the associated pseudo-elliptic integral assumed by (29) is in this case

$$v = \tfrac{1}{4}\omega_3 \qquad (605),$$

and the catenary will be given by an equation of the form

$$(1-z^2)^2 e^{4\chi i} = (Hz^3+H_1z^2+H_2z+H_3)\sqrt{Z_1}+i\,(Lz^3+L_1z^2+L_2z+L_3)\sqrt{Z_2} \qquad (606),$$

leading to

$$4\,(1-z^2)\sqrt{Z}\frac{d\chi}{dz} = Pz^2+Q \qquad (607).$$

According to p. 262 of " Pseudo-Elliptic Integrals," we must now take

$$x = -\frac{(a-1)(a^2+1)(a^4-a^3+a^2+3a+1)+(a^4-a^3+a^2-a-1)\sqrt{A'}}{2a^3(a^2-2a-1)} \qquad (608),$$

$$y = \frac{(a+1)(a^4-1)(a^2-2a-1)+(a^4-2a-1)\sqrt{A'}}{2a^5\,(a+1)(a^2-2a-1)} \qquad (609),$$

where

$$A' = (a^4-1)(a^2-2a-1) \qquad (610),$$

$$M^2 = \frac{a^7-a^6-3a^5+a^4-3a^3+3a^2+5a+1+(a^4+4a+1)\sqrt{A'}}{4a\,(a^4-1)} \qquad (611),$$

$$s_3 = \left\{\frac{a^3-a^2-3a-1+(2a+1)\sqrt{A'}}{2a^4\,(a+1)(a^2-2a-1)}\right\}^2 \qquad (612),$$

$$k^2 = \frac{(a+1)(a^4-1)(a^2-2a-1)+4a\sqrt{A'}}{-(a+1)(a^4-1)(a^2-2a-1)} \qquad (613),$$

$$I(\tfrac{1}{4}\omega_3) = \tfrac{1}{2}\int \frac{\rho_2 s - xy}{s\sqrt{S}}\,ds$$

$$= \tfrac{1}{4}\tan^{-1}\frac{(s+C)\sqrt{(4\,.\,s-s_1\,.\,s-s_2)}}{(4\rho_2 s+R)\sqrt{(s-s_3)}} \qquad (614),$$

where
$$4\rho_2 = -\frac{(a^2+1)(a^2-3)}{a}\sqrt{s_3}$$

$$= -(a^2+1)(a^2-3)\frac{a^3-a^2-3a-1+(2a+1)\sqrt{A'}}{2a^5(a+1)(a^2-2a-1)} \qquad (615),$$

$$R = -\frac{(a^2-1)(a^2+1)^2}{a^3}s_3^{\frac{3}{2}} \qquad (616),$$

$$C = -\frac{a^2+1}{a^2}s_3 \qquad (617),$$

$$4(s-s_1)(s-s_2) = 4s^2 - \frac{(a^4-1)(a^4-4a^2-1)}{a^2}ss_3 + \frac{(a^4-1)^2}{a^2}s_3^2 \qquad (618).$$

Better therefore take

$$\frac{s}{s_3} = s' \qquad (619)$$

as variable; and then

$$xy = \frac{A'}{a(a^2-2a-1)}s_3^{\frac{3}{2}} = \frac{a^4-1}{a}s_3^{\frac{3}{2}} \qquad (620),$$

$$\sqrt{(s_1 s_2)} = \frac{xy}{2s_3^{\frac{1}{2}}} = -\frac{a^4-1}{2a}s_3 \qquad (621).$$

The condition $p=0$ thus gives

$$-4\frac{(a^2+1)(a^2-3)}{a}\sqrt{s_3}+16(1+y)=0 \qquad (622),$$

leading to the sextic equation

$$15a^6+30a^5+15a^4+12a^3+9a^2-2a-7=0 \qquad (623),$$

which, according to Mr. Dewar, has two real roots

$$+0{\cdot}551764 \quad \text{and} \quad -1{\cdot}55711472 \qquad (624),$$

but this does not at present look promising enough to make it worth while to investigate the corresponding catenary.

51. Clebsch extends his investigations to the case where the spherical surface is made to spin about the vertical axis with constant angular velocity n; and now, changing to the absolute unit of force in the C.G.S. system, the tension

$$T = wg\,(h-z)+wn^2\,(b^2-r^2) \qquad (625),$$

together with
$$Tr^2\frac{d\psi}{ds} = H \qquad (626).$$

Equation (6) now becomes perfectly intractable, and of hyper-elliptic character, when gravity and centrifugal whirling are both taken into account; but when gravity is neglected in comparison with the whirling effect, by putting $g = 0$, equation (6) becomes

$$\psi = \int \frac{H\,dz}{r^2\sqrt{\{w^2n^4(b^2-r^2)^2r^2-H^2\}}} \qquad (627),$$

or, putting $H = wn^2A$ (628),

and taking r^2 for independent variable,

$$\psi = \tfrac{1}{2}\int \frac{A\,dr^2}{r^2\sqrt{(1-r^2)}\sqrt{\{(b^2-r^2)^2\,r^2-A^2\}}} \qquad (629),$$

an elliptic integral of the third kind.

Also
$$\frac{ds}{d\psi} = \frac{Tr^2}{H} = \frac{(b^2-r^2)\,r^2}{A} \qquad (630),$$

so that the arc

$$s = \tfrac{1}{2}\int \frac{(b^2-r^2)\,dr^2}{\sqrt{(1-r^2)}\sqrt{\{(b^2-r^2)^2\,r^2-A^2\}}} \qquad (631),$$

introducing elliptic integrals of the second kind.

52. Clebsch (*Crelle*, 57, p. 106) puts

$$R = (b^2-r^2)^2r^2-A^2 = (r^2-\rho^2)(r^2-\sigma^2)(r^2-\tau^2) \qquad (632),$$

where
$$2b^2 = \rho^2+\sigma^2+\tau^2 \qquad (633),$$

$$b^4 = \sigma^2\tau^2+\tau^2\rho^2+\rho^2\sigma^2 \qquad (634),$$

$$A^2 = \rho^2\sigma^2\tau^2 \qquad (635);$$

and therefore $(\rho^2+\sigma^2+\tau^2)^2-4(\sigma^2\tau^2+\tau^2\rho^2+\rho^2\sigma_2)=0$ (636),

or $(\rho+\sigma+\tau)(\rho-\sigma-\tau)(-\rho+\sigma-\tau)(-\rho-\sigma+\tau)=0$ (637),

so that, taking $\rho=\sigma+\tau$ (638),

and therefore $b^2=\sigma^2+\sigma\tau+\tau^2$ (639),

$$R=r^2-(\sigma+\tau)^2 \,.\, r^2-\sigma^2 \,.\, r^2-\tau^2 \qquad (640),$$

$$T=wn^2(\sigma^2+\sigma\tau+\tau^2-r^2) \qquad (641),$$

and the integral is reduced to the standard form (29) by putting

$$r^2=\frac{\wp u-\wp v}{\wp u-\wp c} \qquad (642),$$

$$1-r^2=\frac{\wp v-\wp c}{\wp u-\wp c} \qquad (643),$$

$$R=\lambda\frac{\wp'^2 u}{(\wp u-\wp c)^3} \qquad (644).$$

But we shall find that these integrals are essentially the same as those required for the catenary assumed by a chain wrapped on a vertical paraboloid, whether spinning about its axis, or at rest; the investigation of this new problem had better be reserved for another paper.

[*Additional Note, 7th May*, 1896.

The algebraical case of the Spherical Catenary for $\mu=7$, indicated in (408), p. 159, has now been completed by Mr. Dewar, and the figure obtained is similar to that on p. 170, but having heptagonal instead of pentagonal symmetry.

The numerical data, exhibited in a tabular form, similar to that on p. 169, are given in the table on next page.

The disentanglement of the roots z_0, h_1, l_1, h_2, ... near the North Pole is facilitated by drawing the osculating plane of the catenary at its highest point, and calculating by spherical trigonometry the points where the osculating small circle cuts the prime meridians; these points will be indistinguishable from points on the catenary for some considerable distance.

z	$\sin\lambda$	λ	ψ
z_0	$-0\cdot9970939$	$85^\circ\ 37'\cdot85$ N.	0
h_1	9969428	31·12	$\frac{90^\circ}{7}$
l_1	9964255	9·24	$\frac{180}{7}$
h_2	9952666	84 25·35	$\frac{270}{7}$
l_2	9926029	83 1·60	$\frac{360}{7}$
h_3	9850008	80 3·83	$\frac{450}{7}$
l_3	9479943	71 26·44	$\frac{540}{7}$
h_4	3250447	18 58·13	90°
0	0·	0	92° 20′·30
l_4	$+0\cdot6789442$	42 45·67 S.	$\frac{720^\circ}{7}$
$\frac{1}{2}h$	6832409	43 5·84	
h_5	8414974	57 15·91	$\frac{810}{7}$
l_5	8915966	63 4·47	$\frac{900}{7}$
h	9110921	65 39·40	$\frac{990}{7}$
z_1	9164583	66 24·82	$\frac{1080}{7}$
h_7	1·5496647		

The angular radius θ of the osculating small circle at any point, or $\rho = \sin\theta$, the radius of absolute curvature, is given by the simple expressions

$$\frac{1}{\rho^2} = \frac{1}{\sin^2\theta} = 1 + \frac{A^2}{(h-z)^4},$$

or

$$\tan\theta = \frac{(h-z)^2}{A};$$

this follows from equations (16)–(22), for

$$\frac{d}{ds}(h-z)\frac{dx}{ds} = -x(h-2z),$$

$$\frac{d}{ds}(h-z)\frac{dy}{ds} = -y(h-2z),$$

$$\frac{d}{ds}(h-z)\frac{dz}{ds} = -z(h-2z)-1,$$

so that

$$(h-z)\frac{d^2x}{ds^2} = \frac{dx}{ds}\,\frac{dz}{ds} - x\,(h-2z),$$

$$(h-z)\frac{d^2y}{ds^2} = \frac{dy}{ds}\,\frac{dz}{ds} - y\,(h-2z),$$

$$(h-z)\frac{d^2z}{ds^2} = \frac{dz}{ds}\,\frac{dz}{ds} - z\,(h-2z) - 1.$$

Squaring and adding,

$$(h-z)^2\frac{1}{\rho^2} = \frac{dz^2}{ds^2} + (h-2z)^2 - 2\frac{dz}{ds}\,\frac{dz}{ds} + 2z\,(h-2z) + 1$$

$$= (h-z)^2 + 1 - z^2 - \frac{dz^2}{ds^2},$$

or, from (11),

$$(h-z)^2\,\frac{1}{\rho^2} = (h-z)^2 + 1 - z^2 - \frac{Z}{(h-z)^2}$$

$$= (h-z)^2 + \frac{A^2}{(h-z)^2}.$$

Knowing the angular radius θ of the osculating small circle, and also ϕ, the angle at which the catenary crosses a parallel of latitude, from § 3, it is possible to employ Mr. C. V. Boys's method of drawing the curve by means of a celluloid scale, bent to the radius of the sphere, and pivoted instantaneously at the centre of the osculating small circle.

The angles θ and ϕ are connected with λ, the latitude, by the relations

$$\tan\theta = \frac{(h-\sin\lambda)^2}{A},$$

$$\tan^2\phi = \frac{\cos^2\lambda\,(h-\sin\lambda)^2}{A^2} - 1,$$

or

$$\sec\phi\,\sec\lambda = \frac{h-\sin\lambda}{A},$$

$$\tan\theta = (h-\sin\lambda)\sec\lambda\,\sec\phi$$

$$= A\sec^2\lambda\,\sec^2\phi.$$

Suppose we put, in § 8,

$$x = -\frac{m^3 a}{(a-m)^2},\quad y = -\frac{(1-2m)\,a}{a-m};$$

we now find that a root of equation (31) is

$$s = \frac{m^2 a^2}{(a-m)^2},$$

so that, in § 24,

$$k^2 = 4a-1,$$

$$h^2 = \frac{(4a-1)\,\{2a^2-2\,(m+1)\,a+1\}}{2a\,(a-m)},$$

$$\frac{h^2}{k^2} = \frac{2a^2-2\,(m+1)\,a+1}{2a\,(a-m)},$$

$$\frac{\xi}{M} = -\frac{m}{a-m}.]$$

Thursday, January 9th, 1896.

Major MACMAHON, R.A., F.R.S., President, in the Chair.

Miss Grace Chisholm, Ph.D. Göttingen, and Dr. Robert Bryant, were elected members.

Prof. Elliott, by a method used in connexion with seminvariants, showed how to obtain a criterion as to whether or not a rationa integral homogeneous function of y, a function of x, and its derivatives, is an exact differential, and further showed that, if it is, its integral can be found by differential operations only.

The President announced the title of a paper by Prof. Lloyd Tanner, viz., "On a certain Ternary Cubic." The paper, in the absence of the author, was taken as read.

Mr. S. H. Burbury made a further communication "On Boltzmann's Minimum Function."

Mr. Love communicated "Some Examples illustrating Lord

Rayleigh's Theory of the Stability or Instability of certain Fluid Motions."

Messrs. Cunningham and Larmor spoke on the subject of the papers.

The following presents to the Library were received:—

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XIX., St. 11; Leipzig, 1895.

"Nautical Almanac for the Year 1899," 8vo; London, 1896.

"Bulletin des Sciences Mathématiques," Tome XIX., Nov. et Déc. 1895; Paris.

"Bulletin of the American Mathematical Society," 2nd Series, Vol. II., No. 3; New York, 1895.

"Memorias y Revista de la Sociedad Científica Mexico," Tomo VIII. (1894-5), Nos. 1-2, 3-4; Mexico.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Geschäftliche Mittheilungen, 1895, Heft 2; Mathematisch-Physikalische Klasse, 1895, Heft 3, 1895.

"Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche," Serie 3, Vol. I., Fasc. 11; Napoli, November, 1895.

"Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux," 4e Serie, Tome V.; Paris, 1895.

Rayet, G.—"Observations Pluviométriques et Thermométriques faites dans la Gironde de Juin 1893 à Mai 1894," 8vo; Bordeaux, 1894.

Peter, B.—"Beobachtungen am Sechszölligen Repsoldschen Heliometer der Leipziger Sternwarte," roy. 8vo; Leipzig, 1895.

His, W.—"Anatomische Forschungen über Johann S. Bach's Gebeine und Antlitz," roy. 8vo; Leipzig, 1895.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 2, Vol. IV., Fasc. 10, 11; Roma, 1895.

"Educational Times," January, 1896.

"Indian Engineering," Vol. XVIII., Nos. 21, 22.