A CLASS OF DIFFRACTION PROBLEMS

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IN the solution of the problem of the diffraction of electric waves by a perfectly conducting right circular cone, the constants of the requisite series were determined from the fact that these constants must be identical with the constants in the series which represents the solution of the corresponding potential problem, where both series are expressed in terms of the appropriate harmonic functions.* The solution of the problem of the diffraction of electric waves by a perfectly conducting wedge was also obtained in the same way.† The object of the present communication is to exhibit in analytical form the argument which demonstrates that the constants are the same for the solution of the diffraction problem as for the corresponding potential problem, and also to apply the method to the solution of the diffraction of waves of sound by a rigid wedge. Solutions of the corresponding potential problems are obtained in the form suitable for comparison with the diffraction problem.

The solution of the diffraction problem depends on the solution of linear partial differential equations of the form

$$Ew + \kappa^2 w = 0, \tag{1}$$

where E is some operator, and w satisfies certain boundary conditions, while the solution of the corresponding potential problem depends on the solution of equations of the form

$$Ew = 0, \tag{2}$$

with the same boundary conditions. When the space for which the solutions of (1) and (2) are required is the space bounded by $\xi = \xi_1$, $\xi = \xi_2$; $\eta = \eta_1$, $\eta = \eta_2$; $\zeta = \zeta_1$, $\zeta = \zeta_2$; where ξ, η, ζ are coordinates determining

* *Electric Waves*, p. 90, 1902.
† *Electric Waves*, p. 187.

three families of surfaces, if the equations (1) and (2), when they are transformed to the new variables, take the form

$$D_1 w + D_2 w + D_3 w + \kappa^2 w\,[f_1(\xi) + f_2(\eta) + f_3(\zeta)] = 0, \tag{3}$$

$$D_1 w + D_2 w + D_3 w = 0, \tag{4}$$

where D_1 involves only ξ and operations with respect to ξ, D_2 involves only η and operations with respect to η, and D_3 involves only ζ and operations with respect to ζ, the solutions of (3), when the boundary condition is that w or $\partial w/\partial n$ is given over the boundary, will be of the form

$$w = \Sigma L'M'N', \tag{5}$$

and the solution of (4) will be of the form

$$w = \Sigma LMN, \tag{6}$$

where L', M', N', L, M, N are harmonic functions, and L', M', N' assume the values L, M, N when κ vanishes.

The function L' satisfies a linear differential equation involving ξ only, M' satisfies a linear differential equation involving η only, N' satisfies a linear differential equation involving ζ only, and when only one of these three equations involves κ the value of w which satisfies (3) can be immediately obtained from the value of w which satisfies (4) with the same boundary conditions. If the equation which is satisfied by L' is the one that involves κ, then

$$M' = M, \qquad N' = N$$

for the differential equations, and the boundary conditions are the same in both cases. Hence the solution of the problem is reduced to finding the solution of the linear differential equation satisfied by L', which satisfies the boundary conditions, and takes the value L when κ vanishes; therefore, when the solution of (3) and (4) are expressed in terms of the appropriate harmonic functions the constants in the two series are identical. This applies when the space is that bounded by two concentric spheres, two coaxial right circular cones, and two planes through the axis.

In order to apply the method to the solution of the problem of the diffraction of waves of sound by a wedge it is necessary to obtain the solution of the corresponding potential problem in the appropriate form.

Green's Function for a Wedge.

Choosing polar coordinates r, θ, ϕ, the potential V at any point satisfies the differential equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial V}{\partial\mu}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}+4\pi\rho=0, \qquad (7)$$

where μ is $\cos\theta$, ρ is the density of the inducing distribution at the point, and the condition to be satisfied at the boundary is that V vanishes. Writing $r=ae^{-\lambda}$, equation (7) becomes

$$\frac{\partial^2 V}{\partial\lambda^2}-\frac{\partial V}{\partial\lambda}+\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial V}{\partial\mu}+\frac{1}{\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}+4\pi\rho a^2e^{-2\lambda}=0,$$

and, substituting $$V=Ue^{\frac{1}{2}\lambda},$$

the equation satisfied by U is

$$\frac{\partial^2 U}{\partial\lambda^2}-\tfrac{1}{4}U+\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial U}{\partial\mu}+\frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}+4\pi\rho a^2e^{-\frac{3}{2}\lambda}=0. \qquad (8)$$

When the space for which equation (8) is satisfied is the space bounded by the spheres

$$r=a, \qquad r=ae^{-\lambda_0},$$

and the planes $$\phi=0, \qquad \phi=\phi_0,$$

the harmonic functions of λ and ϕ which vanish at the boundary are

$$\sin\frac{\pi l\lambda}{\lambda_0}, \qquad \sin\frac{\pi m\phi}{\phi_0},$$

where l and m are positive integers, and the harmonic function of μ is the function which satisfies the equation

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial y}{\partial\mu}-\frac{m^2\pi^2}{\phi_0^2}\frac{y}{1-\mu^2}+n(n+1)\,y=0, \qquad (9)$$

where n is determined by the condition that y is finite for all values of μ in the space, that is, when

$$1\geqslant\mu\geqslant-1.$$

Writing $m\pi/\phi_0=m'$, the solution of (9), which is finite when $\mu=1$, is

$$y=P_n^{-m'}(\mu),$$

and n is restricted by the condition that $P_n^{-m'}(\mu)$ is to be finite when $\mu=-1$. Now

$$P_n^{-m'}(\mu)=\frac{1}{\Gamma(1+m')}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{1}{2}m'}F[-n,\,n+1,\,1+m',\,\tfrac{1}{2}(1-\mu)],$$

and $F[-n,\ n+1,\ 1+m',\ \tfrac{1}{2}(1-\mu)]$

$$= \{\tfrac{1}{2}(1+\mu)\}^{m'} F[n+m'+1,\ m'-n,\ 1+m',\ \tfrac{1}{2}(1-\mu)];$$

whence $P_n^{-m'}(\mu) = \dfrac{(1-\mu^2)^{\frac{1}{2}m'}}{2^{m'}\Gamma(1+m')} F[n+m'+1,\ m'-n,\ 1+m',\ \tfrac{1}{2}(1-\mu)]$,

therefore, if $P_n^{-m'}(-1)$ is finite, $F(n+m'+1,\ m'-n,\ 1+m',\ 1)$ must be finite, and since the series diverges, it must consist of a finite number of terms, that is, either $n+m'+1$ or $m'-n$ is a negative integer. Again, $P_{-n-1}^{-m'}(\mu)$ and $P_n^{-m'}(\mu)$ are not different solutions of (9), therefore it is sufficient to take the positive values of n for which $n-m'$ is a positive integer. Hence the solution of equation (8) is

$$U = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} A_{lmn} \sin\frac{l\pi\lambda}{\lambda_0} \sin m'\phi\, P_n^{-m'}(\mu),$$

where $n-m'$ is a positive integer, and

$$\tfrac{1}{4}\lambda_0\phi_0 A_{lmn} \int_{-1}^{1} \{P_n^{-m'}(\mu)\}^2 d\mu \left[(n+\tfrac{1}{2})^2 + \frac{l^2\pi^2}{\lambda_0^2}\right]$$

$$= 4\pi a^2 \int_0^{\lambda_0}\int_0^{\phi_0}\int_{-1}^{1} \rho e^{-\frac{1}{2}\lambda} \sin\frac{l\pi\lambda}{\lambda_0} \sin\frac{m\pi\phi}{\phi_0} P_n^{-m'}(\mu)\, d\lambda\, d\phi\, d\mu,$$

and therefore

$$U = \frac{16\pi a^2}{\lambda_0\phi_0} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \sin\frac{l\pi\lambda}{\lambda_0} \sin\frac{m\pi\phi}{\phi_0} P_n^{-m'}(\mu) \int_0^{\lambda_0}\int_0^{\phi_0}\int_{-1}^{1} \rho e^{-\frac{1}{2}\lambda'}$$

$$\times \sin\frac{l\pi\lambda'}{\lambda_0} \sin\frac{m\pi\phi'}{\phi_0} P_n^{-m'}(\mu')\, d\lambda'\, d\phi'\, d\mu' \Big/ \left[(n+\tfrac{1}{2})^2 + \frac{l^2\pi^2}{\lambda_0^2}\right] \int_{-1}^{1} \{P_n^{-m'}(\mu')\}^2 d\mu', \tag{10}$$

and for the case of a point charge q at the point λ_1, ϕ_1, μ_1, this becomes

$$U = \frac{16\pi q e^{\frac{1}{2}\lambda_1}}{\lambda_0\phi_0 a} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \sin\frac{l\pi\lambda}{\lambda_0} \sin\frac{l\pi\lambda_1}{\lambda_0} \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) P_n^{-m'}(\mu_1)$$

$$\Big/ \left[(n+\tfrac{1}{2})^2 + \frac{l^2\pi^2}{\lambda_0^2}\right] \int_{-1}^{1} \{P_n^{-m'}(\mu')\}^2 d\mu'. \tag{11}$$

It is known that*

$$\int_{\mu_0}^{1} P_n^{-m'}(\mu) P_{n_1}^{-m'}(\mu)\, d\mu$$

$$= \frac{1-\mu_0^2}{(n_1-n)(n_1+n+1)} \left[P_n^{-m}(\mu_0) \frac{dP_{n_1}^{-m'}(\mu_0)}{d\mu_0} - P_{n_1}^{-m'}(\mu_0) \frac{dP_n^{-m'}(\mu_0)}{d\mu_0}\right],$$

* *Proc. London Math. Soc.*, Vol. XXXI, p. 265, 1900.

therefore

$$\int_0^1 \{P_n^{-m'}(\mu)\}^2 d\mu = \frac{1}{2n+1}\left[P_n^{-m'}(\mu)\frac{d^2P_n^{-m'}(\mu)}{d\mu\, dn} - \frac{dP_n^{-m'}(\mu)}{d\mu}\,\frac{dP_n^{-m'}(\mu)}{dn}\right]_{\mu=0}$$

Now

$$P_n^{-m'}(\mu) = 2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(-\frac{1}{2})}(1-\mu^2)^{-\frac{1}{2}m'}$$
$$\times F[\tfrac{1}{2}(n-m'+1),\ -\tfrac{1}{2}(n+m'),\ \tfrac{1}{2},\ \mu^2]$$
$$+2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})}\mu(1-\mu^2)^{-\frac{1}{2}m'}$$
$$\times F[\tfrac{1}{2}(n-m'+2),\ -\tfrac{1}{2}(n+m'-1),\ \tfrac{3}{2},\ \mu^2],$$

whence, when $\mu = 0$,

$$P_n^{-m'}(0) = 2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(-\frac{1}{2})},$$

$$\frac{dP_n^{-m'}(\mu)}{d\mu} = 2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})},$$

$$\frac{dP_n^{-m'}(\mu)}{dn} = -\tfrac{1}{2}\pi\,.\,2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(-\frac{1}{2})}$$
$$+2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{d}{dn}\,\frac{\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi(\frac{1}{2})},$$

$$\frac{d^2P_n^{-m'}(\mu)}{d\mu\, dn} = \tfrac{1}{2}\pi\,.\,2^{-m'}\cos\tfrac{1}{2}(n-m')\pi\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})}$$
$$+2^{-m'}\sin\tfrac{1}{2}(n-m')\pi\,\frac{d}{dn}\,\frac{\Pi\{\frac{1}{2}(n-m')\}}{\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(\frac{1}{2})},$$

and, since $n-m'$ is an integer,

$$\left[P_n^{-m'}(\mu)\frac{d^2P_n^{-m'}(\mu)}{d\mu\, dn} - \frac{dP_n^{-m'}(\mu)}{d\mu}\,\frac{dP_n^{-m'}(\mu)}{dn}\right]_{\mu=0}$$
$$= \tfrac{1}{2}\pi\,.\,2^{-2m'}\frac{\Pi\{\frac{1}{2}(n-m')\}\,\Pi\{\frac{1}{2}(n-m'-1)\}}{\Pi\{\frac{1}{2}(n+m')\}\,\Pi\{\frac{1}{2}(n+m'-1)\}\,\Pi(-\frac{1}{2})\Pi(\frac{1}{2})},$$

hence $$\int_0^1 \{P_n^{-m'}(\mu)\}^2 d\mu = \frac{1}{2n+1}\,\frac{\Pi(n-m')}{\Pi(n+m')}.$$

Again, when $n-m'$ is an integer,

$$P_n^{-m'}(-\mu) = \cos(n-m')\pi\, P_n^{-m'}(\mu),$$

therefore $$\int_{-1}^{1}\{P_n^{-m'}(\mu)\}^2 d\mu = 2\int_0^1 \{P_n^{-m'}(\mu)\}^2 d\mu,$$

whence $$\int_{-1}^{1}\{P_n^{-m'}(\mu)\}^2 d\mu = \frac{2}{2n+1}\frac{\Pi(n-m')}{\Pi(n+m')}.$$

The value of V is therefore given by

$$V = \frac{8\pi q}{a\lambda_0\phi_0} e^{\frac{1}{2}(\lambda+\lambda_1)} \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=m'}^{\infty} \frac{2n+1}{(n+\frac{1}{2})^2+\dfrac{l^2\pi^2}{\lambda_0^2}} \frac{\Pi(n+m')}{\Pi(n-m')}$$
$$\times \sin\frac{l\pi\lambda}{\lambda_0}\sin\frac{l\pi\lambda_1}{\lambda_0}\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)P_n^{-m'}(\mu_1);$$

and effecting the summation for the values of l, this becomes

$$V = \frac{4\pi q}{a\phi_0} e^{\frac{1}{2}(\lambda+\lambda_1)} \sum_{m=0}^{\infty}\sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \frac{\cosh(n+\frac{1}{2})(\lambda_0-\lambda+\lambda_1)-\cosh(n+\frac{1}{2})(\lambda_0-\lambda-\lambda_1)}{\sinh(n+\frac{1}{2})\lambda_0}$$
$$\times \sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)P_n^{-m'}(\mu_1),$$

when $\lambda > \lambda_1$, and

$$V = \frac{4\pi q}{a\phi_0} e^{\frac{1}{2}(\lambda+\lambda_1)} \sum_{m=0}^{\infty}\sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \frac{\cosh(n+\frac{1}{2})(\lambda_0-\lambda_1+\lambda)-\cosh(n+\frac{1}{2})(\lambda_0-\lambda-\lambda_1)}{\sinh(n+\frac{1}{2})\lambda_0}$$
$$\times \sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)P_n^{-m'}(\mu_1),$$

when $\lambda < \lambda_1$.

When the space, for which the solution is required, is the space inside the sphere $r = a$, bounded by the planes $\phi = 0$, $\phi = \phi_0$, the value of λ_0 is infinite, and the corresponding expressions for V are

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty}\sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \left\{\frac{r^n}{r_1^{n+1}} - \frac{r^n r_1^n}{a^{2n+1}}\right\}$$
$$\times \sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)P_n^{-m'}(\mu_1) \quad (r < r_1),$$

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty}\sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \left\{\frac{r_1^n}{r^{n+1}} - \frac{r^n r_1^n}{a^{2n+1}}\right\}$$
$$\times \sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu)P_n^{-m'}(\mu_1) \quad (r > r_1).$$

For the wedge, the space outside it being bounded by the planes

$\phi = 0$, $\phi = \phi_0$, the value of V is obtained by making $a = \infty$ in the above, and therefore

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \frac{r^n}{r_1^{n+1}} \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) P_n^{-m'}(\mu_1) \quad (r<r_1),$$

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} \frac{\Pi(n+m')}{\Pi(n-m')} \frac{r_1^n}{r^{n+1}} \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) P_n^{-m'}(\mu_1) \quad (r>r_1).$$

These expressions can be simplified by observing that the origin of the coordinates can be chosen so that the point charge is in the plane $\theta = \frac{1}{2}\pi$, then $\mu_1 = 0$, and the expressions for V become

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} 2^{m'} \frac{\Pi\{\frac{1}{2}(n+m'-1)\}}{\Pi\{\frac{1}{2}(n-m')\}\,\Pi(-\frac{1}{2})} \cos\tfrac{1}{2}(n-m')\pi \,\frac{r^n}{r_1^{n+1}}$$
$$\times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) \quad (r<r_1),$$

$$V = \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{n=m'}^{\infty} 2^{m'} \frac{\Pi\{\frac{1}{2}(n+m'-1)\}}{\Pi\{\frac{1}{2}(n-m')\}\,\Pi(-\frac{1}{2})} \cos\tfrac{1}{2}(n-m')\pi \,\frac{r_1^n}{r^{n+1}}$$
$$\times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_n^{-m'}(\mu) \quad (r>r_1);$$

or

$$\left.\begin{aligned} V &= \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} 2^{m'} \frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})} \cos k\pi \,\frac{r^{m'+2k}}{r_1^{m'+2k+1}} \\ &\qquad \times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_{m'+2k}^{-m'}(\mu) \quad (r<r_1) \\ V &= \frac{4\pi q}{\phi_0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} 2^{m'} \frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})} \cos k\pi \,\frac{r_1^{m'+2k}}{r^{m'+2k+1}} \\ &\qquad \times \sin\frac{m\pi\phi}{\phi_0} \sin\frac{m\pi\phi_1}{\phi_0} P_{m'+2k}^{-m'}(\mu) \quad (r>r_1) \end{aligned}\right\} . \quad (12)$$

These expressions give V in terms of the harmonic functions appropriate to the space; the results can be obtained from them in the form previously given by the writer* as follows.

It is known that

$$P_{m'+2k}^{-m'}(\mu) = \frac{2^{m'}\Pi(m'-\frac{1}{2})\,\Pi(2k)}{\Pi(2m'+2k)\,\Pi(-\frac{1}{2})} (1-\mu^2)^{\frac{1}{2}m} C_{2k}^{m'+\frac{1}{2}}(\mu),$$

* *Proc. London Math. Soc.*, Vol. XXVI, p. 160, 1895.

where $C_{2k}^{m'+\frac{1}{2}}(\mu)$ is the coefficient of x^{2k} in the development of

$$(1-2\mu x+x^2)^{m'+\frac{1}{2}}$$

in a series of powers of x, hence

$$V=\frac{4q}{r_1\phi_0}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{r}{r_1}\right)^{m'+2k}(1-\mu^2)^{\frac{1}{2}m'}\cos k\pi\,\frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}$$

$$\times\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}C_{2k}^{m'+\frac{1}{2}}(\mu)\quad(r<r_1),$$

where r and r_1 are interchanged, when $r>r_1$. Writing

$$S_m=\sum_{k=0}^{\infty}\left(\frac{r}{r_1}\right)^{2k}\frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)}\cos k\pi\,.\,C_{2k}^{m'+\frac{1}{2}}(\mu),$$

it follows that

$$S_m=\sum_{k=0}^{\infty}\int_{-1}^{1}(1-u^2)^{m'-\frac{1}{2}}\,u^{2k}x^{2k}\cos k\pi\,.\,C_{2k}^{m'+\frac{1}{2}}(\mu)\,du,$$

where $x=r/r_1$, hence

$$S_m=\int_{-1}^{1}\frac{(1-u^2)^{m'-\frac{1}{2}}\,du}{(1-2\iota\mu xu-x^2u^2)^{m'+\frac{1}{2}}},$$

and, substituting

$$1-2\iota\mu xu-x^2u^2=(1-u^2)\,x\sin\theta\,.\,e^{-\iota\zeta},$$

this becomes

$$S_m=\frac{\iota}{2}(x\sin\theta)^{-m'}\int_{\iota\infty-\gamma}^{\iota\infty+\gamma}\frac{e^{m'\zeta\iota}\,d\zeta}{\sqrt{(2x\sin\theta\cos\zeta-1-x^2)}},$$

where the path of integration lies wholly in the upper part of the ζ-plane and passes between the origin and the branch point $\iota\zeta_0$, where

$$\cosh\zeta_0=(1+x^2)/2x\sin\theta$$

of the integrand which lies on the imaginary axis in the upper half, but none of the other branch points $\pm\iota\zeta_0\pm 2n\pi$. Now

$$V=\frac{4q}{r_1\phi_0}\sum_{m=0}^{\infty}\left(\frac{r}{r_1}\right)^{m'}S_m\sin^{m'}\theta\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi_1}{\phi_0}\quad(r<r_1),$$

therefore

$$V=\frac{2q\iota}{\phi_0}\sum_{m=0}^{\infty}\int_{\iota\infty-\gamma}^{\iota\infty+\gamma}\frac{e^{m\pi\zeta\iota/\phi_0}\,d\zeta}{\sqrt{(2rr_1\sin\theta\cos\zeta-r^2-r_1^2)}}\sin\frac{m\pi\phi}{\phi_0}\sin\frac{m\pi\phi}{\phi_1},$$

or $$V = \frac{q}{2\phi_0}\int_{\iota\infty-\gamma}^{\iota\infty+\gamma}\frac{d\zeta}{\sqrt{(2rr_1\sin\theta\cos\zeta - r^2 - r_1^2)}}$$

$$\times\left\{\frac{\sin\frac{\pi\zeta}{\phi_0}}{\cos\frac{\pi\zeta}{\phi_0}-\cos\frac{\pi}{\phi_0}(\phi-\phi_1)} - \frac{\sin\frac{\pi\zeta}{\phi_0}}{\cos\frac{\pi\zeta}{\phi_0}-\cos\frac{\pi}{\phi_0}(\phi+\phi_1)}\right\},$$

that is, writing for ζ, $\iota\zeta$ and $(r^2+r_1^2)/2rr_1\sin\theta = \cosh\eta$,

$$V = \frac{q}{\phi_0\sqrt{(2rr_1\sin\theta)}}\int_\eta^\infty \frac{d\zeta}{\sqrt{(\cosh\zeta-\cosh\eta)}}$$

$$\times\left\{\frac{\sinh\frac{\pi\zeta}{\phi_0}}{\cosh\frac{\pi\zeta}{\phi_0}-\cos\frac{\pi}{\phi_0}(\phi-\phi_1)} - \frac{\sinh\frac{\pi\zeta}{\phi_0}}{\cosh\frac{\pi\zeta}{\phi_0}-\cos\frac{\pi}{\phi_0}(\phi+\phi_1)}\right\}; \quad (13)$$

the result in its previous form.

The velocity potential of the motion of liquid, in the space bounded by the planes $\phi = 0$, $\phi = \phi_0$, due to a source of strength q at the point r_1, $\frac{1}{2}\pi$, ϕ_1, is immediately obtained in a similar way, and is given by

$$\frac{2\pi q}{r_1\phi_0}\sum_{k=0}^{\infty}\left(\frac{r}{r_1}\right)^{2k}\frac{\Pi(k-\frac{1}{2})}{\Pi(k)\Pi(-\frac{1}{2})}\cos k\pi\, P_{2k}(\mu) + \frac{4\pi q}{r_1\phi_0}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}2^{m'}$$

$$\times\cos k\pi\frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\Pi(-\frac{1}{2})}\left(\frac{r}{r_1}\right)^{m'+2k}P_{m'+2k}^{-m'}(\mu)\cos\frac{m\pi\phi}{\phi_0}\cos\frac{m\pi\phi_1}{\phi_0} \quad (r<r_1), \quad (14)$$

and the corresponding expression with r and r_1 interchanged, when $r > r_1$. These expressions can, as above, be transformed into

$$\frac{q}{\phi_0\sqrt{(2rr_1\sin\theta)}}\int_\eta^\infty\frac{d\zeta}{\sqrt{(\cosh\zeta-\cosh\eta)}}$$

$$\times\left\{\frac{\sinh\frac{\pi\zeta}{\phi_0}}{\cosh\frac{\pi\zeta}{\phi_0}-\cos\frac{\pi}{\phi_0}(\phi-\phi_1)} + \frac{\sinh\frac{\pi\zeta}{\phi_0}}{\cosh\frac{\pi\zeta}{\phi_0}-\cos\frac{\pi}{\phi_0}(\phi+\phi_1)}\right\},$$

where $$\cosh\eta = (r^2+r_1^2)/2rr_1\sin\theta.$$

The solution of the problem of the diffraction of waves of sound due to a source at the point r_1, $\frac{1}{2}\pi$, ϕ_1 is obtained as follows. The linear differential equation that involves κ, where $2\pi/\kappa$ is the wave length, is

$$\frac{d^2L}{dr^2}+\frac{2}{r}\frac{dL}{dr}+\left\{\kappa^2-\frac{n(n+1)}{r^2}\right\}L = 0,$$

and the solutions of this equation are required which tend to the value r^n/r_1^{n+1}, when $r < r_1$, and to the value r_1^n/r^{n+1}, when $r > r_1$. Since L is finite when $r = 0$, the solution, when $r < r_1$, is

$$L = Ar^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r),$$

and, since the waves are all travelling outwards at infinity, the solution, when $r > r_1$, is

$$L = Br^{-\frac{1}{2}}K_{n+\frac{1}{2}}(\iota\kappa r);$$

and, since the two solutions are identical, when $r = r_1$,

$$L = Cr^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r)\,K_{n+\frac{1}{2}}(\iota\kappa r_1) \quad (r < r_1),$$

$$L = Cr^{-\frac{1}{2}}J_{n+\frac{1}{2}}(\kappa r_1)K_{n+\frac{1}{2}}(\iota\kappa r) \quad (r > r_1),$$

Now, when $\kappa = 0$,

$$L = \frac{C}{2n+1}\,\frac{r^n}{r_1^{n+\frac{1}{2}}}\,e^{-(n+\frac{1}{2})\frac{1}{2}\pi\iota} \quad (r < r_1),$$

therefore
$$\frac{C}{2n+1}\,\frac{r^n}{r_1^{n+\frac{1}{2}}}\,e^{-(n+\frac{1}{2})\frac{1}{2}\pi\iota} = \frac{r^n}{r_1^{n+1}},$$

whence
$$C = (2n+1)\,e^{(n+\frac{1}{2})\frac{1}{2}\pi\iota}\,r_1^{-\frac{1}{2}},$$

and, if the velocity potential due to the source alone is $e^{\iota\kappa(at-R)}/R$ at a distance R from it, where a is the velocity of sound, the velocity potential of the wave motion is $Ue^{\iota\kappa at}$, where

$$U = \frac{2\pi}{\phi_0}\sum_{k=0}^{\infty}(4k+1)e^{(2k+\frac{1}{2})\frac{1}{2}\pi\iota}r^{-\frac{1}{2}}r_1^{-\frac{1}{2}}J_{2k+\frac{1}{2}}(\kappa r)K_{2k+\frac{1}{2}}(\iota\kappa r_1)\cos k\pi\frac{\Pi(k-\frac{1}{2})}{\Pi(k)\Pi(-\frac{1}{2})}P_{2k}(\mu)$$

$$+\frac{4\pi}{\phi_0}\sum_{m'=1}^{\infty}\sum_{k=0}^{\infty}2^{m'}(2m'+4k+1)\,e^{(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota}r^{-\frac{1}{2}}r_1^{-\frac{1}{2}}J_{m'+2k+\frac{1}{2}}(\kappa r)\,K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1)$$

$$\times\cos k\pi\frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\Pi(-\frac{1}{2})}\,P_{m'+2k}^{-m'}(\mu)\cos\frac{m\pi\phi}{\phi_0}\cos\frac{m\pi\phi_1}{\phi_0} \quad (\text{when } r < r_1), \quad (15)$$

and the corresponding expression with r and r_1 interchanged, when $r > r_1$. Now*

$$J_{m'+2k+\frac{1}{2}}(\kappa r)\,K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1)$$
$$= -\tfrac{1}{2}e^{-(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota}\int_{c-\infty\iota}^{0} e^{\frac{1}{2}s-\kappa^2(r^2+r_1^2)/2s}\,I_{m'+2k+\frac{1}{2}}\left(\frac{\kappa^2 rr_1}{s}\right)\frac{ds}{s},$$

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* *Proc. London Math. Soc.*, Vol. XXXII, p. 155, 1900.

and* $$I_{m'+2k+\frac{1}{2}}\left(\frac{\kappa^2 rr_1}{s}\right) = -\frac{1}{2\pi}\int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\kappa^2 rr_1/s\cdot\cos\zeta+(m'+2k+\frac{1}{2})\zeta\iota}\,d\zeta,$$

where $$2\pi > \gamma > \pi, \qquad 0 > \gamma' > -\pi,$$

hence, writing

$$S_m = 2^{m'} \sum_{k=0} (2m'+4k+1)\, e^{(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota} \times \cos k\pi\, J_{m'+2k+\frac{1}{2}}(\kappa r) K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1) \frac{\Pi(m'+k-\frac{1}{2})}{\Pi(k)\,\Pi(-\frac{1}{2})} P_{m'+2k}^{-m'}(\mu),$$

that is,

$$S_m = \frac{1}{\pi}\sin^{m'}\theta \sum_{k=0}^{\infty} (2m'+4k+1)\, e^{(m'+2k+\frac{1}{2})\frac{1}{2}\pi\iota} \times \cos k\pi\, J_{m'+2k+\frac{1}{2}}(\kappa r)\, K_{m'+2k+\frac{1}{2}}(\iota\kappa r_1) \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)} C_{2k}^{m'+\frac{1}{2}}(\mu),$$

it follows that

$$S_m = \frac{1}{4\pi^2}\sin^{m'}\theta \sum_{k=0}^{\infty} (2m'+4k+1) \times \cos k\pi \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)} \int_{c-\infty\iota}^{0}\int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\cos\zeta)+(m'+2k+\frac{1}{2})\zeta\iota} \times s^{-1}\,d\zeta\,ds\; C_{2k}^{m'+\frac{1}{2}}(\mu),$$

or, integrating by parts with respect to ζ,

$$S_m = \frac{1}{2\pi^2\iota}\sin^{m'}\theta \sum_{k=0}^{\infty} \cos k\pi \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)} C_{2k}^{m'+\frac{1}{2}}(\mu)\,\kappa^2 rr_1 \int_{\iota-\infty\iota}^{0}\int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} \times e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\cos\zeta)+(m'+2k+\frac{1}{2})\zeta\iota}\sin\zeta\,d\zeta\,\frac{ds}{s^2}.$$

Now, by the above,

$$\sum_{k=0}^{\infty}\cos k\pi \frac{\Pi(m'-\frac{1}{2})\,\Pi(k-\frac{1}{2})}{\Pi(m'+k)} e^{2k\zeta\iota} C_{2k}^{m'+\frac{1}{2}}(\mu) = \tfrac{1}{2}\iota\,(e^{\zeta\iota}\sin\theta)^{-m'} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} \frac{e^{m'\zeta_1\iota}\,d\zeta_1}{(2e^{\zeta\iota}\sin\theta\cos\zeta_1 - 1 - e^{2\zeta\iota})^{\frac{1}{2}}},$$

where the extremities of the path of integration lie in the upper half of the plane, and the path encloses the branch point of the integrand that is continuous with the branch point of the integrand on the positive

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* *Electric Waves*, p. 191, 1902.

imaginary axis when ζ is a pure imaginary, therefore

$$S_m = \frac{\kappa^2 r r_1}{4\pi^2} \int_{c-\infty\iota}^{0} \int_{\infty\iota+\gamma}^{\infty\iota+\gamma'} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\cos\zeta)+m'\zeta_1\iota}$$

$$\times\, 2^{-\frac{1}{2}}(\sin\theta\cos\zeta_1-\cos\zeta)^{-\frac{1}{2}}\sin\zeta\, d\zeta_1\, d\zeta\, \frac{ds}{s^2},$$

whence, changing the order of integration, and integrating with respect to ζ,

$$S_m = -(2\pi)^{-\frac{3}{2}}\,\kappa(rr_1)^{\frac{1}{2}} \int_{c-\infty\iota}^{0} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)+m'\zeta_1\iota}\, s^{-\frac{3}{2}}\, d\zeta_1\, ds,$$

where $$2\pi > \gamma_1' > \pi, \qquad \pi > \gamma_1 > 0;$$

or, replacing the integral with respect to s by its known value,

$$S_m = \frac{1}{2\pi}(rr_1)^{\frac{1}{2}} \int_{\infty\iota-\gamma_1}^{\infty\iota+\gamma_1'} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)}+m'\zeta_1\iota}(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)^{-\frac{1}{2}}d\zeta_1,$$

and, substituting this result in the expression for U, it becomes

$$U = -\frac{1}{\phi_0}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)}}\,(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)^{-\frac{1}{2}}$$

$$\times\left(1+2\sum_1^\infty e^{m\pi\zeta_1\iota/\phi_0}\cos\frac{m\pi\phi}{\phi_0}\cos\frac{m\pi\phi_1}{\phi_0}\right)d\zeta_1,$$

that is,

$$U = \frac{\iota}{2\phi_0}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)}}\,(r^2+r_1^2-2rr_1\sin\theta\cos\zeta_1)^{-\frac{1}{2}}$$

$$\times\left\{\frac{\sin\dfrac{\pi\zeta_1}{\phi_0}}{\cos\dfrac{\pi\zeta_1}{\phi_0}-\cos\dfrac{\pi}{\phi_0}(\phi-\phi_1)}+\frac{\sin\dfrac{\pi\zeta_1}{\phi_0}}{\cos\dfrac{\pi\zeta_1}{\phi_0}-\cos\dfrac{\pi}{\phi_0}(\phi+\phi_1)}\right\}.\text{*}$$

When $\phi_0 = \pi/n$, where n is an integer, the result agrees with that obtained by the method of images, for

$$\frac{n\sin n\zeta}{\cos n\zeta-\cos n(\phi\pm\phi_1)} = \sum_{k=0}^{n-1}\frac{\sin\zeta}{\cos\zeta-\cos(\phi\pm\phi_1-2k\pi/n)},$$

* In the particular case when $\phi_0 = 2\pi$, this result is equivalent to that given by Carslaw *Proc. London Math. Soc.*, Vol. xxx, p. 139, 1898.

and each term of the sum is immediately integrable, giving

$$U = \sum_{k=0}^{n-1} \left[e^{-\iota\kappa\sqrt{\{r^2+r_1^2-2rr_1 \sin\theta \cos(\phi-\phi_1-2k\pi/n)\}}} \times \{r^2+r_1^2-2rr_1 \sin\theta \cos(\phi-\phi_1-2k\pi/n)\}^{-\frac{1}{2}}\right]$$

$$+ e^{-\iota\kappa\sqrt{\{r^2+r_1^2-2rr_1 \sin\theta \cos(\phi+\phi_1-2k\pi/n)\}}} \times \{r^2+r_1^2-2rr_1 \sin\theta \cos(\phi+\phi_1-2k\pi/n)\}^{-\frac{1}{2}}.$$

For the case of the straight edge or half plane $\phi_0 = 2\pi$, and

$$U = \frac{\iota}{4\pi}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)}} (r^2+r_1^2-2rr_1 \sin\theta \cos\zeta)^{-\frac{1}{2}} \times \left\{\frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)} + \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi+\phi_1)}\right\} d\zeta,$$

which can be expressed in terms of integrals between real limits in a form suitable for approximation as follows. Writing

$$I = \frac{\iota}{4\pi}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)}} \times (r^2+r_1^2-2rr_1\sin\theta\cos\zeta)^{-\frac{1}{2}} \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)}\, d\zeta,$$

it follows that

$$I = -\tfrac{1}{2}\iota\kappa(2\pi)^{-\frac{1}{2}}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1}\int_{c-\infty\iota}^{0} e^{\frac{1}{2}s-\kappa^2/2s\,(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)} \frac{\sin\frac{1}{2}\zeta\, d\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)}\frac{ds}{s^{\frac{1}{2}}},$$

and writing

$$W = \int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{\kappa^2 rr_1/s\,\sin\theta\cos\zeta} \frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)}\, d\zeta,$$

this becomes, on substituting

$$\cos\tfrac{1}{2}\zeta = t\cos\tfrac{1}{2}(\phi-\phi_1),$$

$$W = 2\int_{t_0}^{t_1} e^{\kappa^2 rr_1/s\,\sin\theta\,[2t^2\cos^2\frac{1}{2}(\phi-\phi_1)-1]}\frac{dt}{t-1},$$

where, when $\cos\frac{1}{2}(\phi-\phi_1)$ is positive,

$$t_0 = c'+\infty\iota, \qquad t_1 = -c'-\infty\iota,$$

and the path of integration crosses the positive real axis to the right of the point $t = 1$, since, when ζ is a pure imaginary t is greater than unity;

when $\cos\frac{1}{2}(\phi-\phi_1)$ is negative,

$$t_0 = -c' - \infty\iota, \qquad t_1 = c' + \infty\iota,$$

and the path of integration cuts the real axis to the left of the origin since t is negative when ζ is a pure imaginary. Hence, when $\cos\frac{1}{2}(\phi-\phi_1)$ is positive,

$$W = 2e^{-\kappa^2 rr_1/s \sin\theta} \int_{c'+\infty\iota}^{-c'-\infty\iota} e^{2\kappa^2 rr_1/s\, t^2 \sin\theta \cos^2\frac{1}{2}(\phi-\phi_1)} \frac{dt}{t-1},$$

that is, deforming the path so as to make it coincide with the imaginary axis, since it passes over the singularity $t=1$,

$$W = -4\pi\iota e^{-\kappa^2 rr_1/s \sin\theta \cos(\phi-\phi_1)} + 2e^{-\kappa^2 rr_1/s \sin\theta} \int_{\infty\iota}^{-\infty\iota} e^{2\kappa^2 rr_1/s\, t^2 \sin\theta \cos^2\frac{1}{2}(\phi-\phi_1)} \frac{dt}{t-1};$$

now writing

$$W' = \int_{\infty\iota}^{-\infty\iota} e^{ht^2} \frac{dt}{t-1},$$

it follows that

$$W' = -2\int_0^{\infty\iota} \frac{e^{ht^2}}{t^2-1}\, dt = 2\iota \int_0^{\infty} \frac{e^{-ht^2} dt}{t^2+1},$$

or

$$W' = 2\iota \int_0^{\infty}\int_0^{\infty} e^{-(h+u)t^2-u}\, dt\, du = \iota\sqrt{\pi}\int_0^{\infty} e^{-u}(h+u)^{-\frac{1}{2}}\, du,$$

that is, writing

$$h+u = w^2/2s,$$

since the real part of s is always positive,

$$W' = \iota(2\pi)^{\frac{1}{2}} \int_{w_0}^{\infty} e^{-w^2/2s} s^{-\frac{1}{2}}\, dw,$$

where

$$w_0 = (2sh)^{\frac{1}{2}};$$

therefore

$$W = -4\pi\iota e^{-\kappa^2 rr_1/s \sin\theta\cos(\phi-\phi_1)} + 2\iota\,(2\pi)^{\frac{1}{2}} s^{-\frac{1}{2}} e^{\kappa^2 rr_1/s \sin\theta \cos(\phi-\phi_1)} \int_{w_0}^{\infty} e^{-w^2/2s}\, dw.$$

where

$$w_0 = 2\kappa(rr_1 \sin\theta)^{\frac{1}{2}} \cos\tfrac{1}{2}(\phi-\phi_1).$$

Hence

$$I = -(2\pi)^{-\frac{1}{2}}\kappa \int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}s-\kappa^2/2s\,\{r^2+r_1^2-2rr_1\sin\theta\cos(\phi-\phi_1)\}} s^{-\frac{3}{2}}\, ds$$

$$+(2\pi)^{-1}\kappa \int_{c-\infty\iota}^{0}\int_{w_0}^{\infty} e^{\frac{1}{2}s-\kappa^2/2s\,\{r^2+r_1^2-2rr_1\sin\theta\cos(\phi-\phi_1)\}-w^2/2s} s^{-2}\, ds\, dw,$$

and writing $$R^2 = r^2+r_1^2-2rr_1\sin\theta\cos(\phi-\phi_1),$$

where R is the distance of the point (r, θ, ϕ) from the source, this becomes

$$I = \frac{e^{-\iota\kappa R}}{R} - \frac{\iota\kappa}{\pi}\int_{w_0}^{\infty} \frac{K_1\{\iota\sqrt{(\kappa^2R^2+w^2)}\}}{\sqrt{(\kappa^2R^2+w^2)}}\,dw,$$

that is $$I = \frac{e^{-\iota\kappa R}}{R} - \frac{\iota\kappa}{\pi}\int_{\xi_0}^{\infty} K_1(\iota\kappa R\cosh\xi)\,d\xi,$$

where $$\sinh\xi_0 = 2R^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1)\,;$$

now $$\frac{\iota\kappa}{\pi}\int_{-\infty}^{\infty} K_1(\iota\kappa R\cosh\xi)\,d\xi = \frac{e^{-\iota\kappa R}}{R},$$

therefore the result can be written in the form

$$I = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi.$$

Again, when $\cos\frac{1}{2}(\phi-\phi_1)$ is negative,

$$W = 2e^{-\kappa^2rr_1/s\sin\theta}\int_{-c'-\infty\iota}^{c'+\infty\iota} e^{2\kappa^2rr_1/s\sin\theta\cos^2\frac{1}{2}(\phi-\phi_1)\,t^2}\frac{dt}{t-1},$$

and, deforming the path of integration so as to make it coincide with the imaginary axis, since it does not pass over the pole of the integrand

$$W = 2\int_{-\infty\iota}^{\infty\iota} e^{2\kappa^2rr_1/s\sin\theta\cos^2\frac{1}{2}(\phi-\phi_1)\,t^2}\frac{dt}{t-1},$$

whence, by the above,

$$W = -2\iota(2\pi)^{\frac{1}{2}}s^{-\frac{1}{2}}e^{\kappa^2rr_1/s\sin\theta\cos(\phi-\phi_1)}\int_{w_0}^{\infty} e^{-w^2/2s}\,dw,$$

where $$w_0 = -2\kappa(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1)\,;$$

and therefore $$I = \frac{\iota\kappa}{\pi}\int_{\xi_0'}^{\infty} K_1(\iota\kappa R\cosh\xi)\,d\xi,$$

where $$\sinh\xi_0' = -2R^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1),$$

and this can be written

$$I = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi,$$

where $$\sinh\xi_0 = 2R^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1).$$

Similarly, writing

$$I_1 = \frac{\iota}{4\pi}\int_{\infty\iota+\gamma_1'}^{\infty\iota-\gamma_1} e^{-\iota\kappa\sqrt{(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)}}(r^2+r_1^2-2rr_1\sin\theta\cos\zeta)^{-\frac{1}{2}}$$
$$\times\frac{\sin\frac{1}{2}\zeta}{\cos\frac{1}{2}\zeta-\cos\frac{1}{2}(\phi-\phi_1)}\,d\zeta,$$

it follows that $$I_1 = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_1} K_1(\iota\kappa R'\cosh\xi)\,d\xi,$$

where $$R'^2 = r^2+r_1^2-2rr_1\sin\theta\cos(\phi+\phi_1),$$
$$\sinh\xi_1 = 2R'^{-1}(rr_1\sin\theta)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi+\phi_1),$$

and R' is the distance of the point (r, θ, ϕ) from the image of the source in the plane $\phi = 0$. Hence

$$U = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi + \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_1} K_1(\iota\kappa R'\cosh\xi)\,d\xi.$$

The corresponding expression for the velocity potential of the liquid motion in the space due to a source of unit strength is obtained from this by substituting $\kappa = 0$, and gives

$$\frac{1}{\pi R}\int_{-\infty}^{\xi_0}\frac{d\xi}{\cosh\xi} + \frac{1}{\pi R'}\int_{-\infty}^{\xi_1}\frac{d\xi}{\cosh\xi},$$

that is $\frac{1}{R}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\frac{1}{2}\xi_0)\right\} + \frac{1}{R'}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\frac{1}{2}\xi_1)\right\}$,

and the corresponding expression for Green's function is

$$\frac{1}{R}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\tfrac{1}{2}\xi_0)\right\} - \frac{1}{R'}\left\{\frac{1}{2}+\frac{2}{\pi}\tan^{-1}(\tanh\tfrac{1}{2}\xi_1)\right\},$$

a known result.

The expression given above for U is in a form suitable for approximation. When R is small, that is, when the point (r, θ, ϕ) is near to the source, ξ_0 is very large and tends to infinity as R tends to zero, hence the value of the first integral tends to $e^{-\iota\kappa R}/R$. When the point (r, θ, ϕ) is not near to the source, R is finite and $R\cosh\xi$ is not small within the range of integration, hence $K_1(\iota\kappa R\cosh\xi)$ can be replaced by its asymptotic expansion, and the principal part of the first integral is given by

$$I = \frac{\iota\kappa}{\pi}\int_{-\infty}^{\xi_0} K_1(\iota\kappa R\cosh\xi)\,d\xi = \kappa^{\frac{1}{2}}(2\pi R)^{-\frac{1}{2}}e^{\frac{1}{4}\pi\iota}\int_{-\infty}^{\xi_0} e^{-\iota\kappa R\cosh\xi}(\cosh\xi)^{-\frac{1}{2}}\,d\xi,$$

that is, writing $$(2\kappa R)^{\frac{1}{2}}\sinh\tfrac{1}{2}\xi = u,$$

by $$\pi^{-\frac{1}{2}}R^{-1}e^{\frac{1}{4}\pi\iota-\iota\kappa R}\int_{-\infty}^{u_0}\left(1+\frac{u^2}{\kappa R}\right)^{-\frac{1}{2}}\left(1+\frac{u^2}{2\kappa R}\right)^{-\frac{1}{2}}e^{-\iota u^2}\,du,$$

where $$u_0 = \pm\sqrt{\{(r^2+r_1^2+2rr_1\sin\theta)^{\frac{1}{2}}-R\}}\,\kappa^{\frac{1}{2}},$$

and the upper or lower sign of the radical is taken according as $\cos\frac{1}{2}(\phi-\phi_1)$ is positive or negative, hence the principal value of I is $\bar{I}$, where

$$\bar{I} = \pi^{-\frac{1}{2}}R^{-1}e^{\frac{1}{4}\pi\iota-\iota\kappa R}\int_{-\infty}^{u_0}e^{-\iota u^2}\,du.$$

In the second integral the Bessel's function can always be replaced by its asymptotic expansion, since as R' tends to zero, $R'\cosh\xi$ is always finite within the range of integration, being always not less than

$$\sqrt{(r^2+r_1^2+2rr_1\sin\theta)},$$

its value at the upper limit ξ_1; hence the principal part of I_1 is

$$\pi^{-\frac{1}{2}}\kappa e^{\frac{1}{4}\pi\iota-\iota\kappa R'}\int_{-\infty}^{u_1}(u^2+\kappa R')^{-\frac{1}{2}}(\tfrac{1}{2}u^2+\kappa R')^{-\frac{1}{2}}e^{-\iota u^2}\,du,$$

where $$u_1 = \pm\sqrt{\{(r^2+r_1^2+2rr_1\sin\theta)^{\frac{1}{2}}-R'\}}\,\kappa^{\frac{1}{2}},$$

and the upper or lower sign of the radical is taken according as $\cos\frac{1}{2}(\phi-\phi_1)$ is positive or negative, therefore the value of the principal part of I_1 is $\bar{I}_1$, where

$$\bar{I}_1 = \pi^{-\frac{1}{2}}R'^{-1}e^{\frac{1}{4}\pi\iota-\iota\kappa R'}\int_{-\infty}^{u_1}e^{-\iota u^2}\,du,$$

when R' is not small, and $\bar{I}_1$ tends to the value

$$2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\kappa e^{\frac{1}{4}\pi\iota}\int_{-\infty}^{-u_1'}u^{-2}e^{-\iota u^2}\,du,$$

where $$u_1' = \sqrt{(2\kappa r_1)},$$

when R' tends to zero. When r_1 is very large, u_0 and u_1 tend to the values

$$\sqrt{(2\kappa r\sin\theta)}\cos\tfrac{1}{2}(\phi-\phi_1),\qquad \sqrt{(2\kappa r\sin\theta)}\cos\tfrac{1}{2}(\phi+\phi_1),$$

and the result becomes that for plane waves agreeing with the known result.

Similar forms of the solution can be obtained for the case of a line source parallel to the edge of the half plane at a finite distance from it; the method of reduction of the integrals in the known solution* is identical

* *Electric Waves*, pp. 192, 195.

with the preceding investigation, and the results are, when the electric force due to the line source in the line (r_1, ϕ_1), is

$$K_0\{\iota\kappa\sqrt{[r^2+r_1^2-2rr_1\cos(\phi-\phi_1)]}\}\,e^{\iota\kappa Vt},$$

the resulting electric force is given by $Ze^{\iota\kappa Vt}$, where

$$Z = \tfrac{1}{2}\int_{-\infty}^{\xi_0} e^{-\iota\kappa R\cosh\xi}\,d\xi - \tfrac{1}{2}\int_{-\infty}^{\xi_1} e^{-\iota\kappa R'\cosh\xi}\,d\xi_1,$$

where

$$R\sinh\xi_0 = 2(rr_1)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi-\phi_1), \qquad R'\sinh\xi_1 = 2(rr_1)^{\frac{1}{2}}\cos\tfrac{1}{2}(\phi+\phi_1),$$

$$R^2 = r^2+r_1^2-2rr_1\cos(\phi-\phi_1), \qquad R'^2 = r^2+r_1^2-2rr_1\cos(\phi+\phi_1);$$

when the magnetic force due to the line source on the line (r_1, ϕ_1) is

$$K_0\{\iota\kappa\sqrt{[r^2+r_1^2-2rr_1\cos(\phi-\phi_1)]}\}\,e^{\iota\kappa Vt},$$

the resulting magnetic force is given by $\gamma e^{\iota\kappa Vt}$, where

$$\gamma = \tfrac{1}{2}\int_{-\infty}^{\xi_0} e^{-\iota\kappa R\cosh\xi}\,d\xi + \tfrac{1}{2}\int_{-\infty}^{\xi_1} e^{-\iota\kappa R'\cosh\xi}\,d\xi.$$