ALTERNANTS AND CONTINUOUS GROUPS

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THE present paper is concerned with the proof of a fundamental proposition of non-commutative algebra, and with shewing that this leads at once to various results fundamental for the theory of continuous groups which are usually developed only with much more detailed considerations than are here employed.

One of the most noticeable peculiarities of the algebra of two non-commutative quantities A, B is the occurrence of the alternant

$$(A, B) = AB-BA ;$$

from this we may form the alternants $(A, (A, B))$, $(B, (A, B))$, and so on, each new quantity being the alternant of a previous quantity with either A or B; and, if these be called simple alternants, we may proceed to form alternants of simple alternants; it is, however, easy to see that these are expressible by simple alternants with numerical coefficients. Now it is an obvious suggestion of Lie's theory of groups that the product

$$e^A e^B = \left(1+A+\frac{A^2}{2!}+\dots\right)\left(1+B+\frac{B^2}{2!}+\dots\right)$$

is of the form e^C, where C is a series of alternants of A and B. For, if, in the ordinary notation,

$$X = e_1 X_1+\dots+e_r X_r, \qquad X' = e_1' X_1+\dots+e_r' X_r, \qquad X'' = e_1'' X_1+\dots+e_r'' X_r,$$

we have

$$e^X e^{X'} x_i = e^X f_i(x, e') = f_i[f(x, e), e'] = f_i(x, e'') = e^{X''} x_i,$$

and the ground for this simplification can only be the alternant relations

$$(X_\rho, X_\sigma) = \Sigma c_{\rho\sigma\tau} X_\tau.$$

The fact that the product $e^A e^B$ is capable of the form e^C has been insisted on by Mr. Campbell in his recent interesting book on groups, and he has given an elementary proof of it (*Proc. London Math. Soc.*, Vol. XXIX., 1898, p. 14), which cannot, however, I think, be regarded as final. It follows from this that the terms of any the same dimension in A and B in the expansion of

$$(e^A e^B-1)-\tfrac{1}{2}(e^A e^B-1)^2+\tfrac{1}{3}(e^A e^B-1)^3-\ldots$$

must be simple alternants or sums of such with numerical coefficients; and to the fourth dimension we, in fact, find for this expansion

$$A+B+\tfrac{1}{2}(A,\ B)+\tfrac{1}{12}(A-B,\ (A,\ B))-\tfrac{1}{24}(A,\ (B,\ (A,\ B))).$$

On the other hand, starting with Schur's formulæ for the first parameter group, and using the notation of the theory of matrices, I have, in several papers in these *Proceedings*, deduced from the theory of groups various propositions primarily for the parameter group; and, in particular, if $e''_\sigma = \psi_\sigma(e,\ e')$, which we shall write $e'' = \psi(e,\ e')$, be the equations of that group, with canonical variables e, e'' and canonical parameters e', if E be the matrix whose general element is

$$E_{\rho\sigma} = \sum_{\tau} c_{\sigma\tau\rho} e_\tau \quad (\rho,\ \sigma,\ \tau = 1,\ 2,\ \ldots,\ r),$$

and $\Delta = e^E$, while E', Δ' and E'', Δ'' depend respectively on e' and e'', as E, Δ depend on e, it has been shown (*Proc. London Math. Soc.*, Vol. XXXIV., 1901, p. 98) that

$$\Delta'' = \Delta'\Delta,$$

leading (*ib.*, Vol. XXXV., p. 333) to

$$E'' = (\Delta'\Delta-1)-\tfrac{1}{2}(\Delta'\Delta-1)^2+\ldots.$$

The present paper professes to show that these investigations for the matrix E, so long as they refer to the general group of unassigned order, have a much wider bearing, and that, in fact, they furnish a proof of the general theorem $e^A e^B = e^C$, of a simple character, and so also of the fundamental group property which arises when, as in Mr. Campbell's work, A and B are linear differential operators. But there is more than this: we have proved (*ib.*, Vol. XXXIV., 1902, p. 352) that the E matrix of the row Ee' is the alternant $(E,\ E')$; hence, if $E_1,\ \ldots,\ E_m$ be each E matrices, either of e, or e', or of some other row, the E matrix of the row $E_1 E_2 \ldots E_m E e'$ is the alternant $(E_1,\ (E_2,\ \ldots,\ (E_m,\ (E,\ E'))\ldots))$; thus every alternant of E matrices is an E matrix. From this it follows that the equations of the first parameter group are of the form

$$e'' = e+e'+(H_0+H_1+\ldots)\,Ee',$$

where H_n is a homogeneous polynomial of dimension n in the matrices E, E'—in itself a remarkable result, since this is then the parameter group of every possible continuous group when the appropriate matrix E

is employed—but it also follows that every term in this expression gives rise to one of the terms in the expansion of

$$e^A e^B - 1 - \tfrac{1}{2}(e^A e^B - 1)^2 + \tfrac{1}{3}(e^A e^B - 1)^3 - \ldots,$$

and that there are no terms in this latter but such as arise in this way. The series for $e^{\prime\prime}$ is manifestly simpler to deal with than the series of alternants: in this paper it is obtained without the assumption of any of the propositions of the theory of groups, or the use of the theory of matrices; and a law of recurrence for its terms, whereby any term is deducible from the preceding, is obtained. The result can then conversely be used to establish the existence of general functions possessing the group property, and in particular it gives incidentally the origin of Schur's formulæ. The method employed is of the kind usually called symbolical, the quantities being defined by their laws of operation and not belonging to the ordered aggregate of natural number; if it be not conceded that sufficient explanation is given to shew that these laws are in this case self-consistent, the work retains its validity when interpreted in terms of the theory of matrices; but it is claimed that the paper is a contribution to the calculus of alternants of any non-commutative quantities.

1. *Alternants and their Bases.*

The capitals A, B, C, ... denote any quantities which can be added and subtracted associatively and commutatively, and can be multiplied associatively and distributively, but not commutatively. We do not recognise the existence of solutions of the equations $xA = 1$ or $Ax = 1$. With each capital, A, we associate a quantity, a, which we call its *base*, and we call A the *derivative* of a, either determining the other uniquely. By the base of a sum or difference of derivatives we understand the sum or difference of their bases; thus the base of $-A$ is $-a$, and the base of zero is zero, and conversely; and bases can be added associatively and commutatively. Further, with every alternant of capitals we associate a definite base of which it is the derivative; and a derivative is either a capital or an alternant of capitals. As we have not as yet introduced products of capitals and bases, it is legitimate to denote the base of (A, B) by Ab; the identity $(A, B)+(B, A) = 0$ will then require $Ab+Ba = 0$; in particular $Aa = 0$. The base of an alternant $(A, (B, C))$ will then naturally be denoted by Ad, where d is Bc, the base of (B, C), and so may be denoted by $A(Bc)$; we shall, however, denote this by ABc; it being observed that the notation Ab has as yet been introduced only for the case when A is a single capital, and b a base, there is no confusion possible of ABc with the symbol $AB.c$ regarded as the base of (AB, C) or $ABC-CAB$. In general we assign similarly, to an alternant

$$(A_1, (A_2, (\ldots, A_n, B) \ldots))),$$

a base denoted by $A_1A_2 \ldots A_n b$; if $c = A_h A_{h+1} \ldots A_n b$, the alternant takes the form $(A_1, (A_2, \ldots (A_{h-1}, C) \ldots))$, and its base has the notation $A_1 \ldots A_{h-1} c$; so that in the product $A_1 \ldots A_n b$ the symbols are associative.

We shew next that this notation for the base of a simple alternant may conveniently be extended to the base of a compound alternant; as a compound alternant is expressible by a sum of simple alternants, its base is a sum of bases of the form considered; we shew, however, that its expression in a succinct form is equivalent to allowing the symbols in the base of a compound alternant to obey a distributive law. First, the alternant $((A, B), C)$ is easily verified to be the sum of $(A, (B, C))$ and $-(B, (A, C))$, of which the bases are respectively ABc and $-BAc$; if then the base $(A, B)c$ be assigned to $((A, B), C)$, we have

$$(A, B)c = ABc-BAc \quad \text{or} \quad (AB-BA)c = ABc-BAc,$$

as is most natural. In passing we notice that the base of

$$((A, B), C) = -(C, (A, B))$$

is also denoted by $-CAb$; so that we have

$$ABc+BCa+CAb = 0.$$

In general, if $A = (A_1, (A_2, \ldots (A_{n-1}, A_n) \ldots))$ be any alternant, and B be a single capital, we denote the base of (A, B) by Ab, and we examine now the implications of this notation. Suppose, in expanded form,

$$A = \Sigma \pm K_1K_2 \ldots K_n,$$

where $K_1K_2 \ldots K_n$ is a permutation of $A_1A_2 \ldots A_n$, so that

$$Ab = (\Sigma \pm K_1K_2 \ldots K_n)\, b.$$

Any term $K_1K_2 \ldots K_n b$ is the base of an alternant $(K_1, (K_2, \ldots (K_n, B) \ldots))$; we show that

$$(A, B) = \Sigma \pm (K_1, (K_2, \ldots (K_n, B) \ldots));$$

when this is proved we can write

$$(\Sigma \pm K_1K_2 \ldots K_n)b = \Sigma \pm (K_1K_2 \ldots K_n b).$$

Let P_{n-1} denote $(A_2, (A_3, \ldots (A_{n-1}, A_n) \ldots))$; then (A, B), $= ((A_1, P_{n-1}), B)$, is, as just remarked, equal to $(A_1, (P_{n-1}, B))-(P_{n-1}, (A_1, B))$; denote

(A_1, B) by C, and assume the law of expansion just stated, which we wish to prove true for an alternant (A, B) of $(n+1)$ dimensions, to be true for alternants (P_{n-1}, B), (P_{n-1}, C), of n dimensions; we have remarked that it holds for $n = 3$. Supposing

$$P_{n-1} = \Sigma \pm H_1 \dots H_{n-1},$$

we have thence $(P_{n-1}, B) = \Sigma \pm (H_1, (\dots (H_{n-1}, B) \dots))$,

$$(P_{n-1}, C) = \Sigma \pm (H_1, (\dots (H_{n-1}, C) \dots)),$$

and thus

$$(A, B) = (A_1, (P_{n-1}, B)) - (P_{n-1}, C)$$

$$= \Sigma \pm (A_1, (H_1, (\dots (H_{n-1}, B) \dots))) - \Sigma \pm (H_1, (\dots (H_{n-1}, (A_1, B)) \dots)),$$

while $A = (A_1, P_{n-1}) = \Sigma \pm A_1 H_1 \dots H_{n-1} - \Sigma \pm H_1 \dots H_{n-1} A_1$;

so that the result is established.

More generally, let $B = (B_1, (B_2, \dots (B_{m-1}, B_m) \dots))$ be any alternant, of which the base is $B_1 B_2 \dots B_{m-1} b_m$. When A was a single capital, we agreed to denote the base of (A, B) by $AB_1 \dots B_{m-1} b_m$; it is natural to seek to extend this to the case when, as above,

$$A = (A_1, (A_2, \dots (A_{n-1}, A_n) \dots)),$$

and to denote the base of (A, B) by $(\Sigma \pm K_1 K_2 \dots K_n) B_1 B_2 \dots B_{m-1} b_m$; in fact, the proof given above for the expansion of (A, B) when B is a single capital holds equally here, and we have

$$\Sigma \pm (K_1 K_2 \dots K_n B_1 \dots B_{m-1} b_m) = (\Sigma \pm K_1 K_2 \dots K_n) B_1 B_2 \dots B_{m-1} b_m.$$

Again, if $C = (C_1, (C_2 \dots (C_{h-1}, C_h) \dots))$, the equation

$$\Sigma \pm C_1 C_2 \dots C_h K_1 K_2 \dots K_n B_1 \dots B_{m-1} b_m$$

$$= C_1 C_2 \dots C_h (\Sigma \pm K_1 \dots K_n) B_1 \dots B_{m-1} b_m$$

$$= C_1 C_2 \dots C_h \Sigma \pm (K_1 \dots K_n B_1 \dots B_{m-1} b_m)$$

requires the obvious identity

$$\Sigma \pm (C_1, (C_2 \dots (C_h, (K_1 \dots (K_n, B) \dots)) \dots))$$

$$= (C_1, (C_2 \dots [C_h, (\Sigma \pm (K_1, (\dots (K_n, B) \dots)))] \dots)).$$

Thus in a product or sum of products of the form $A_1 \dots A_{n-1} a_n$, where $A_1, \dots, A_n$ are single capitals, when the sum is of the form

$$A_1 \dots A_h P A_k \dots A_{n-1} a_n,$$

where P is an alternant, the symbols obey the associative and distributive laws. The same is true when $A_1, \dots, A_n$ are themselves alternants, as may be similarly shewn. For instance, returning to the theorem of expansion proved above, the base of (A, B) or $((A_1, (A_2 \dots (A_{n-1}, A_n) \dots)), B)$ may equally be denoted by $-BA_1 \dots A_{n-1} a_n$; let $p_{n-1} = A_2 A_3 \dots A_{n-1} a_n$, so that the derivative P_{n-1} is as above; then

$$-BA_1 \dots A_{n-1} a_n = -BA_1 p_{n-1};$$

the alternants shew, as in the work above, that this is the same as

$$A_1 P_{n-1} b + P_{n-1} B a_1 = A_1 P_{n-1} b - P_{n-1} A_1 b;$$

but the base of $(A, B), = ((A_1, P_{n-1}), B)$, is $(A_1, P_{n-1})\, b$; thus we have $(A_1 P_{n-1} - P_{n-1} A_1)\, b$ equal to $A_1 P_{n-1} b - P_{n-1} A_1 b$. And so on. And by the way we notice the identities such as

$$BA_1 \dots A_{n-1} a_n + \Sigma \pm K_1 \dots K_n b = 0,$$

the general form of which is $Ab + Ba = 0$, including $Aa = 0$, where A, B are any two derivatives, that is, either single capitals or alternants of capitals.

It is manifest that the notation of bases, once established, as here, furnishes a compendious way of expressing the relations among alternants, and may be of great use in expressing a compound alternant by simple alternants.

Many identities arise from the relation $Ab + Ba = 0$, by supposing a to be a base of the form $A_1 A_2 \dots A_m a_{m+1}$, and b a base of the form $B_1 B_2 \dots B_n b_{n+1}$, where $A_1, \dots, A_{m+1}, B_1, \dots, B_{n+1}$ are any derivatives. Restricting ourselves, as sufficient for purposes of illustration and as giving results to be utilised below, to the cases when each of $A_1, \dots, A_{m+1}$, $B_1, \dots, B_{n+1}$ is either A or B, we find on examination that the following are all the identities so obtainable up to those of the sixth dimension.

(1) From $Aa = 0$, putting Ab for a, we have $(AB - BA)\, Ab = 0$. This corresponds to the alternant relation $((A, B), (A, B)) = 0$, which is the same as $(A, (B, (A, B))) = (B, (A, (A, B)))$.

(2) Also from $Aa = 0$, by putting $A^2 b$ for a, we have

$$(A, (A, B))\, A^2 b = 0,$$

that is $$(A^2 B - 2ABA + BA^2)\, A^2 b = 0,$$

or $$(A^3 B - 2ABA^2 + BA^3)\, Ab = 0.$$

This corresponds to the identity

$$((A, (A, B)), (A, (A, B))) = 0,$$

which is equivalent to

$$(A, (A, (B, (A, (A, B)))))-2(A, (B, (A, (A, (A, B)))))$$
$$+(B, (A, (A, (A, (A, B))))) = 0,$$

an identity which we may agree to denote by

$$[1^2 21^2 2]-2[121^3 2]+[21^4 2] = 0.$$

By interchange of A and B we derive

$$(AB^3-2BAB^2+B^3A)Ab = 0.$$

(3) From $Ab+Ba = 0$, putting A^2b for a, BAb for b, we derive

$$(A, (A, B))BAb+(B, (A, B))A^2b = 0,$$

which is the same as

$$(A^2B^2-B^2A^2+2BABA+BA^2B-2ABAB-AB^2A)Ab = 0,$$

or, in virtue of $(AB-BA)Ab = 0$, the same as

$$(A^2B^2-B^2A^2+3BA^2B-3AB^2A)Ab = 0.$$

This corresponds to the identity

$$((A, (A, B)), (B, (A, B)))+((B, (A, B)), (A, (A, B))) = 0,$$

which expanded is the same as

$$[1^2 2^2 12]-[2^2 1^3 2]+3[21^2 212]-3[12^2 1^2 2] = 0.$$

The identities

$$S(A, B)Ab = (A^3B+BA^3-2ABA^2)Ab = 0,$$
$$T(A, B)Ab = (A^2B^2-B^2A^2+3BA^2B-3AB^2A)Ab = 0$$

are not really independent; for we find

$$S(A+B, B)Ab = [S(A, B)-S(B, A)+T(A, B)]Ab,$$
$$T(A+B, B)Ab = [T(A, B)-2S(B, A)]Ab,$$

while $$(A+B)b = Ab.$$

In what has preceded we have considered only bases of the form Ab, wherein A and B are alternants (or simple capitals); and, if

$$A = \Sigma \pm K_1K_2 \dots K_n,$$

it has been shown that $Ab = \Sigma \pm (K_1K_2 \dots K_n b)$.

If now P_1, P_2, ... be simple capitals or alternants, it is natural to regard $(P_1P_2 \dots P_h+P_{h+1}P_{h+2} \dots P_k+\dots)b$ as an appropriate notation for the base which is the sum of the bases $P_1 P_2 \dots P_h b$, $P_{h+1} P_{h+2} \dots P_k b$, ... ; so that its derivative will be

$$(P_1, (P_2, \dots (P_h, B) \dots)) + (P_{h+1}, (P_{h+2}, \dots (P_k, B) \dots)) + \dots .$$

When $P_1 P_2 \dots P_h + P_{h+1} P_{h+2} \dots P_k + \dots$ is an alternant we have shewn this to be legitimate; we extend it to the general case, for which there is no corresponding condition for alternants satisfied, the derivative not being equal to

$$((P_1 \dots P_h + P_{h+1} \dots P_k + \dots), B);$$

this expression, in fact, not being an alternant of alternants, is not under consideration.

A particular case of this definition is

$$c = (\lambda_0 + \lambda_1 A + \lambda_2 A^2 + \dots) b,$$

where A is a derivative; this we regard as the base of

$$C = \lambda_0 B + \lambda_1 (A, B) + \lambda_2 (A, (A, B)) + \dots .$$

If $\mu_0, \mu_1, \dots$ be such functions of $\lambda_0, \lambda_1, \dots$ that

$$(\mu_0 + \mu_1 A + \mu_2 A^2 + \dots)(\lambda_0 + \lambda_1 A + \lambda_2 A^2 + \dots) = 1,$$

we have

$$b = (\mu_0 + \mu_1 A + \mu_2 A^2 + \dots) c,$$

and the substitution in the series above given for C of the series

$$B = \mu_0 C + \mu_1 (A, C) + \mu_2 (A, (A, C)) + \dots$$

reduces that series to C.

2. *Of the Substitutional Operation.*

We introduce a symbol to express the operation of replacing a base a by a base b, and write

$$\left(b \frac{\partial}{\partial a}\right) a = b;$$

the operation of replacing the capital A by the capital B is also considered, and may be denoted by the same symbol, so that

$$\left(b \frac{\partial}{\partial a}\right) A = B;$$

and, if P be a product of capitals, some of which are A, by the effect of the operator upon P is meant a sum of terms each differing from P by the substitution of B for A in one factor without change of order; for instance.

$$\left(b \frac{\partial}{\partial a}\right) ABA = B^2 A + AB^2, \qquad \left(b \frac{\partial}{\partial a}\right) A^2 Ca = BACa + ABCa + A^2 Cb.$$

Denoting $\left(b\frac{\partial}{\partial a}\right)$ by δ, and the operation of forming the derivative A of a by ϵ, so that $\epsilon a = A$, we have

$$\delta\epsilon a = \delta A = B = \epsilon b = \epsilon\delta a.$$

Let the alternant $(A, (A, \ldots (A, C) \ldots))$, where A occurs m times, be, for brevity, denoted by $(A_m C)$; then

$$\delta\epsilon(Ac) = \delta(AC-CA) = BC-CB = \epsilon(Bc) = \epsilon\delta(Ac),$$

$$\delta\epsilon(A^m c) = \delta(A_m C) = (B, (A_{m-1}C))+(A, (B, (A_{m-2}C)))+\ldots+(A_{m-1}(B, C))$$
$$= \epsilon(BA^{m-1}c+ABA^{m-2}c+\ldots+A^{m-1}Bc) = \epsilon\delta(A^m c),$$

$$\delta\epsilon(A^n DA^m c) = \delta(A_n(D(A_m C))) = \delta_1(A_n(D(A_m C)))+\delta_2(A_n(D(A_m C))),$$

where δ_1 operates on A occurring under A_n, and δ_2 on A occurring under A_m,

$$= \epsilon\delta_1(A^n DA^m c)+\epsilon\delta_2(A^n DA^m c) = \epsilon\delta(A^n DA^m c);$$

and so on. Thus, if $F(A)$ be a symbol for the sum of a finite or an infinite number of products of capitals, of which each product involves A, and the other capitals in the product are unaffected by δ, we infer that

$$\delta\epsilon F(A)c = \epsilon\delta F(A)c.$$

We shall presently be concerned with a base b which is a particular case of the form $F(A)c$, namely, is

$$b = (\lambda_0+\lambda_1 A+\lambda_2 A^2+\ldots)c;$$

then
$$B = \lambda_0 C+\lambda_1(A_1 C)+\lambda_2(A_2 C)+\ldots,$$

and
$$\delta^2 a = \left(b\frac{\partial}{\partial a}\right)^2 a = \delta b = [\lambda_1 B+\lambda_2(BA+AB)+\ldots]c$$

is also, clearly, on substitution for B, of the form $F(A)c$, and the same is similarly true of every expression $\delta^m a$; thus

$$\epsilon\delta^2 a = \epsilon\delta(\delta a) = \epsilon\delta b = \delta\epsilon(\delta a) = \delta(\epsilon\delta a) = \delta(\delta\epsilon a) = \delta^2\epsilon a,$$

$$\epsilon\delta^3 a = \epsilon\delta(\delta^2 a) = \delta\epsilon(\delta^2 a) = \delta(\epsilon\delta^2 a) = \delta^3\epsilon a,$$

and so on; and in general $\epsilon\delta^m a = \delta^m\epsilon a$.

Now, with the same b, let t be an ordinary number, and put

$$f = \left[1+t\left(b\frac{\partial}{\partial a}\right)+\frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]a,$$

so that

$$F = \epsilon a+t\epsilon\delta a+\frac{t^2}{2!}\epsilon\delta^2 a+\ldots = \left[1+t\left(b\frac{\partial}{\partial a}\right)+\frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]A.$$

Put, further,
$$\Upsilon = \mu_0+\mu_1 F+\mu_2 F^2+\ldots,$$

so that, after substitution for F and arrangement according to powers of t,

$$\nabla = \nabla^{t=0}+t\left(\frac{d\nabla}{dt}\right)_{t=0}+\frac{t^2}{2!}\left(\frac{d^2\nabla}{dt^2}\right)_{t=0}+\ldots.$$

We have $\frac{dF}{dt} = [\delta+t\delta^2+\ldots]A = \delta[1+t\delta+\ldots]A = \delta F,$

and thus

$$\frac{d\nabla}{dt} = \mu_1\frac{dF}{dt}+\mu_2\left(\frac{dF}{dt}F+F\frac{dF}{dt}\right)+\mu_3\left(\frac{dF}{dt}F^2+F\frac{dF}{dt}F+F^2\frac{dF}{dt}\right)+\ldots$$

$$= \mu_1\delta F+\mu_2(\delta F.F+F.\delta F)+\mu_3(\delta F.F^2+F\delta F.F+F^2\delta F)+\ldots = \delta\nabla,$$

from which $$\left(\frac{d^m\nabla}{dt^m}\right)_{t=0} = (\delta^m\nabla)_{t=0} = \delta^m\nabla_{t=0}.$$

Therefore, the substitution

$$F = \left[1+t\left(b\frac{\partial}{\partial a}\right)+\frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]A$$

in $$\nabla = \mu_0+\mu_1F+\mu_2F^2+\ldots$$

gives $$\nabla = \left[1+t\left(b\frac{\partial}{\partial a}\right)+\frac{t^2}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]\nabla_0$$

where $$\nabla_0 = \mu_0+\mu_1A+\mu_2A^2+\ldots.$$

Here b is any series of the form

$$b = (\lambda_0+\lambda_1A+\lambda_2A^2+\ldots)c.$$

We shall presently utilise this result. It is convenient to give here another particular result. If b reduce to Ac, and $F(A, C)$, or simply F, denote any sum of terms each of which is a product of powers of A and C, with a numerical coefficient, we have, as we shall now prove,

$$\left(b\frac{\partial}{\partial a}\right)F(A, C) = FC-CF = (F, C).$$

In particular, if $F(A, C)$ reduce to A,

$$\left(b\frac{\partial}{\partial a}\right)A = B = (A, C).$$

To prove the result, notice

$$\left(b\frac{\partial}{\partial a}\right)A^2 = (A, C)A+A(A, C) = A^2C-CA^2 = (A^2, C),$$

and, if $$\left(b\frac{\partial}{\partial a}\right)H = (H, C), \qquad \left(b\frac{\partial}{\partial a}\right)K = (K, C),$$

then $\left(b\frac{\partial}{\partial a}\right)C^mHC^n = C^m(H, C)C^n = C^mHC^{n+1}-C^{m+1}HC^n = (C^mHC^n, C)$

and $\left(b\frac{\partial}{\partial a}\right)HK = (H, C)K+H(K, C) = HKC-CHK = (HK; C)$.

Another result to be utilised follows hence. We have

$$(A_2C) = (A,(A, C)) = A^2C-2ACA+CA^2,$$

and in general, easily, by induction,

$$(A_rC) = \sum_{s=0}^{r}(-1)^s\binom{r}{s}A^{r-s}CA^s,$$

and so

$$(A_1C)\binom{m}{1}A^{m-1}+(A_2C)\binom{m}{2}A^{m-2}+(A_3C)\binom{m}{3}A^{m-3}+\ldots+(A_mC)$$

$$= (AC-CA)\binom{m}{1}A^{m-1}+(A^2C-2ACA+CA^2)\binom{m}{2}A^{m-2}$$

$$+(A^3C-3A^2CA+3ACA^2-CA^3)\binom{m}{3}A^{m-3}+\ldots$$

$$+\left[A^mC-\binom{m}{1}A^{m-1}CA+\ldots+(-1)^{m-1}\binom{m}{1}ACA^{m-1}+(-1)^mCA^m\right]$$

$$= CA^m[(1-1)^m-1]+\binom{m}{1}ACA^{m-1}[(1-1)^{m-1}]$$

$$+\binom{m}{2}A^2CA^{m-2}[(1-1)^{m-2}]+\ldots+A^mC$$

$$= (A^m, C) = \left(b\frac{\partial}{\partial a}\right)A^m,$$

and thus, if $\phi(A) = \nu_0+\nu_1A+\nu_2A^2+\ldots$, $b = Ac$,

$$\left(b\frac{\partial}{\partial a}\right)\phi(A) = (A, C)\frac{\phi'(A)}{1!}+(A_2C)\frac{\phi''(A)}{2!}+(A_3C)\frac{\phi'''(A)}{3!}+\ldots,$$

where $\phi'(A) = \nu_1+2\nu_2A+3\nu_3A^2+\ldots$, $\phi''(A) = 2\nu_2+3.2.\nu_3A+\ldots$.

This is the same as

$$\left(b\frac{\partial}{\partial a}\right)\phi(A) = B\frac{\phi'(A)}{1!}+(A_1B)\frac{\phi''(A)}{2!}+(A_2B)\frac{\phi'''(A)}{3!}+\ldots,$$

and this result, though proved for $B = (A_1C)$, can easily be shown to hold for arbitrary B. A particular case, given by Mr. Campbell, *Proc. London Math. Soc.*, Vol. XXIX., 1897, p. 16, is

$$BA^{m-1}+ABA^{m-2}+\ldots+A^{m-1}B = \binom{m}{1}BA^{m-1}+\binom{m}{2}(A_1B)A^{m-2}$$

$$+\binom{m}{3}(A_2B)A^{m-3}+\ldots+(A_{m-1}B).$$

3. *The Exponential Theorem.*

Applying the theorem just obtained for arbitrary A and B,

$$\left(b\frac{\partial}{\partial a}\right)\phi(A) = B\frac{\phi'(A)}{1!}+(A_1B)\frac{\phi''(A)}{2!}+(A_2B)\frac{\phi'''(A)}{3!}+\ldots,$$

to the case where for $\phi(A)$ we have

$$\Delta = 1+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots,$$

so that each of $\phi'(A)$, $\phi''(A)$, ... is also Δ, we obtain

$$\left(b\frac{\partial}{\partial a}\right)\Delta = \left(B+\frac{(A, B)}{2!}+\frac{(A, (A, B))}{3!}+\ldots\right)\Delta.$$

Now put, with arbitrary a',

$$b = \left(1+\frac{A}{2!}+\frac{A^2}{3!}+\ldots\right)^{-1}a' = \left(1-\frac{A}{2}+\frac{\varpi_1}{2!}A^2+\frac{\varpi_2}{4!}A^4+\ldots\right)a',$$

where $\varpi_1 = \frac{1}{6}$, $\varpi_2 = -\frac{1}{30}$, ... are Bernoulli's numbers with associated signs; then, as

$$B+\frac{(A, B)}{2!}+\frac{(A, (A, B))}{3!}+\ldots$$

is the derivative of $\left(1+\frac{A}{2!}+\frac{A^2}{3!}+\ldots\right)b$

or a', we shall obtain

$$B+\frac{(A, B)}{2!}+\frac{(A, (A, B))}{3!}+\ldots = A'.$$

Thus, with the particular b so defined from A and A', we have

$$\left(b\frac{\partial}{\partial a}\right)\Delta = A'\Delta,$$

and hence $\left(b\frac{\partial}{\partial a}\right)^m\Delta = A'^m\Delta.$

Next, put $a'' = \left[1+\left(b\frac{\partial}{\partial a}\right)+\frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]a,$

and $\Delta'' = 1+A''+\frac{A''^2}{2!}+\ldots;$

then, as follows from a result proved in the previous section (p. 33),

$$\Delta'' = \left[1+\left(b\frac{\partial}{\partial a}\right)+\frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]\Delta = \left(1+A'+\frac{A'^2}{2!}+\ldots\right)\Delta,$$

which we appropriately write $\Delta'' = \Delta'\Delta$;

this is then the consequence of

$$a'' = \left[1+\left(b\frac{\partial}{\partial a}\right)+\frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\ldots\right]a,$$

where, a and a' being arbitrary,

$$b = \left(1+\frac{A}{2!}+\frac{A^2}{3!}+\ldots\right)^{-1}a'.$$

This, however, gives

$$B = A'-\tfrac{1}{2}(A, A')+\frac{\varpi_1}{2!}(A_2A')+\frac{\varpi_2}{4!}(A_4A')+\ldots,$$

$$\left(b\frac{\partial}{\partial a}\right)b = \left\{-\frac{B}{2}+\frac{\varpi_1}{2}[BA]+\frac{\varpi_2}{4!}[BA^3]+\ldots\right\}a',$$

where $[BA^{m-1}] = BA^{m-1}+ABA^{m-2}+\ldots+A^{m-1}B,$

so that $\left(b\frac{\partial}{\partial a}\right)b$ is of the form $F(A, A')a'$, where $F(A, A')$ is a series of polynomials in A and A' with numerical coefficients. So therefore is every term $\left(b\frac{\partial}{\partial a}\right)^m a$. Hence, as $A'a' = 0$, we have an equation

$$a'' = a+a'+(H_0+H_1+H_2+H_3+\ldots)Aa',$$

where H_r is a homogeneous polynomial in A and A', of dimensions r. As A'', the derivative of this, is a series of alternants, we infer therefore, with arbitrary non-commutative symbols A, A', if

$$\Delta = 1+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\ldots, \qquad \Delta' = 1+A'+\frac{A'^2}{2!}+\frac{A'^3}{3!}+\ldots,$$

and the expansion of

$$(\Delta'\Delta-1)-\tfrac{1}{2}(\Delta'\Delta-1)^2+\tfrac{1}{3}(\Delta'\Delta-1)^3-\ldots$$

be arranged according to homogeneous polynomials in A and A' of ascending dimension, that each polynomial is an aggregate of alternants of A and A', any one such aggregate being the derivative of the terms of the same dimension in

$$a'' = a+a'+(H_0+H_1+H_2+\ldots)Aa'.$$

For instance, as will be presently seen,

$$H_0 = -\tfrac{1}{2}, \quad H_1 = \tfrac{1}{12}(A-A'), \quad H_2 = \tfrac{1}{24}AA',$$

and we find on trial, as was remarked in the introduction, that to the fourth dimension the expansion in question is

$$A+A'-\tfrac{1}{2}(A, A')+\tfrac{1}{12}(A-A', (A, A'))+\tfrac{1}{24}(A, (A', (A, A'))).$$

The theorem has a converse. When A, A' are perfectly general there exists no other derivative A_{11} such that

$$\Delta_{11} = 1+A_{11}+\frac{A_{11}^2}{2!}+\dots$$

is equal to $\Delta'\Delta$.

For otherwise $\Delta_{11} = \Delta''$, $\Delta_{11}^{-1}\Delta'' = 1$, and so, if we put

$$\psi(a, a') = a+a'+(H_0+H_1+H_2+\dots)Aa',$$

and denote by A_0 the derivative of $\psi(a'', -a_{11})$, we shall have $\Delta_0 = 1$, namely,

$$\left(1+\frac{A_0}{2!}+\frac{A_0^2}{3!}+\dots\right)A_0 = 0;$$

or, on multiplying by $1-\tfrac{1}{2}A_0+\frac{\varpi_1}{2!}A_0^2+\dots$, we shall have $A_0 = 0$, namely, $\psi(a'', -a_{11}) = 0$. The only solution of this which holds for all values of a'' must be one for which the terms of any order are separately zero; in particular, from the terms of the first order $a''-a_{11} = 0$; while, conversely, if this be so, $\Delta'' = \Delta_{11}$.

4. *On certain Properties of $\psi(a, a')$, and the Determination of its Coefficients.*

The actual form of $\psi(a, a')$ can be determined from its definition

$$\left[1+\left(b\frac{\partial}{\partial a}\right)+\frac{1}{2!}\left(b\frac{\partial}{\partial a}\right)^2+\dots\right]a;$$

it is at once seen, however, that beyond the first few terms this is an intricate process, and, moreover, the terms then appear arranged according to ascending dimension in a' only, and not, as will appear more proper, according to ascending dimension in a and a' jointly.

Some information appears on the face of the series. First we have

$$\psi(-a', -a) = -\psi(a, a').$$

For $\Delta^{-1}\Delta'^{-1}\Delta'\Delta = 1$, or $\Delta''^{-1} = \Delta^{-1}\Delta'^{-1}$, namely, as

$$\Delta''^{-1} = 1 - A'' + \frac{A''^2}{2!} - \dots,$$

we have $\quad -A'' = (\Delta^{-1}\Delta'^{-1} - 1) - \tfrac{1}{2}(\Delta^{-1}\Delta'^{-1} - 1)^2 + \dots.$

Thus in the series

$$\psi(a, a') = a + a' + (H_0 + H_1 + H_2 + \dots) Aa'$$

H_{2n} is symmetrical in A and A', while H_{2n-1} is changed in sign by interchanging A and A'. Next, the terms of the first dimension in a' in $\psi(a, a')$ are

$$\left(b\frac{\partial}{\partial a}\right) a = \left(1 - \frac{A}{2} + \frac{\varpi_1}{2!}A^2 + \frac{\varpi_2}{4!}A^4 + \dots\right) a';$$

thus $H_0 = -\frac{1}{2}$, while H_{2n} has no terms in A^{2n} or A'^{2n}, and H_{2n-1} contains the terms $\frac{\varpi_n}{(2n)!}(A^{2n-1} - A'^{2n-1})$. This gives $H_1 = \frac{1}{12}(A - A')$, and H_2 = numerical multiple of $AA' + A'A$; we have seen, however, that $(AA' - A'A)Aa' = 0$, so that H_2 is a numerical multiple of AA'.

To obtain further information we prove first two lemmas: (α) if a, a' be arbitrary and $c = \Delta a'$, then $C = \Delta A'\Delta^{-1}$, and hence

$$\Delta_c = 1 + C + \frac{C^2}{2!} + \dots = \Delta\Delta'\Delta^{-1};$$

(β) we have $\psi(a', a) = \Delta\psi(a, a') = \psi(a, \Delta a')$, which are both included in

$$\psi(a, \Delta^\lambda a') = \Delta^{\lambda-1}\psi(a', a).$$

The result (α) is obvious on forming the expression

$$\Delta A'\Delta^{-1} = \left(1 + A + \frac{A^2}{2!} + \dots\right) A' \left(1 - A + \frac{A^2}{2!} - \dots\right),$$

in which the terms of dimension $(n+1)$ are

$$\frac{A^n}{n!}A' - \frac{A^{n-1}}{(n-1)!}A'\frac{A}{1!} + \frac{A^{n-2}}{(n-2)!}A'\frac{A^2}{2!} - \dots + (-1)^n A'\frac{A^n}{n!},$$

which is $\frac{1}{n!}(A_n A')$, the derivative of $\frac{A^n a'}{n!}$. Or it follows thus: From $c = \Delta a'$, we have

$$C\Delta = \left(A' + \frac{(A, A')}{1!} + \frac{(A_2 A')}{2!} + \dots\right)\Delta$$

$$= A'\Delta + \left(B + \frac{(A, B)}{2!} + \frac{(A_2 B)}{3!} + \dots\right)\Delta,$$

where $b = Aa'$; by previous theorems (§ 2), this is

$$A'\Delta + \left(b\frac{\partial}{\partial a}\right)\Delta = A'\Delta + (\Delta, A') = \Delta A',$$

so that $C = \Delta A'\Delta^{-1}$. To prove the result $\psi(a', a) = \Delta\psi(a, a')$ of lemma (β) we have only to notice that, if $a_{11} = \Delta a''$, then $\Delta_{11} = \Delta\Delta''\Delta^{-1}$, and so, if $a'' = \psi(a, a')$, and therefore $a_{11} = \Delta\psi(a, a')$, we have

$$\Delta_{11} = \Delta\Delta'\Delta\Delta^{-1} = \Delta\Delta',$$

which establishes $a_{11} = \psi(a', a)$. The same proof stated differently gives, from $a_{11} = \Delta\psi(a, a')$, the series

$$\begin{aligned} A_{11} = \Delta A''\Delta^{-1} &= \Delta(\Delta'\Delta - 1)\Delta^{-1} - \tfrac{1}{2}\Delta(\Delta'\Delta - 1)^2\Delta^{-1} + \ldots \\ &= (\Delta\Delta' - 1) - \tfrac{1}{2}(\Delta\Delta' - 1)^2 + \ldots, \end{aligned}$$

which shows that $a_{11} = \psi(a', a)$. The result $\psi(a, \Delta a') = \Delta\psi(a, a')$ of lemma (β) is obvious from the form

$$\psi(a, a') = a + a' + (H_0 + H_1 + H_2 + \ldots)Aa'.$$

For consider any term $P = A^{\lambda}A'^{\lambda'}A^{\mu}A'^{\mu'} \ldots Aa'$;

if herein we replace a' by $\Delta a'$, and therefore A' by $\Delta A'\Delta^{-1}$, it becomes

$$A^{\lambda}\Delta A'^{\lambda'}\Delta^{-1}A^{\mu}\Delta A'^{\mu'}\Delta^{-1} \ldots A\Delta a',$$

or, since Δ and A are commutable,

$$\Delta A^{\lambda}A'^{\lambda'}A^{\mu}A'^{\mu'} \ldots \Delta^{-1}\Delta Aa',$$

or ΔP: while $a + a'$ becomes $\Delta a + \Delta a'$, it being noticed that in virtue of $Aa = 0$ we have $\Delta a = a$; thus $\psi(a, \Delta a') = \Delta\psi(a, a')$.

Consider now the equation

$$-\psi(-a, -a') = \Delta\psi(a, a');$$

it gives

$$\begin{aligned} &a + a' - (H_0 - H_1 + H_2 - \ldots)Aa' \\ &\qquad = \left\{1 + A + \frac{A^2}{2!} + \ldots\right\}\{a + a' + (H_0 + H_1 + H_2 + \ldots)Aa'\}; \end{aligned}$$

comparing here terms of second dimension, we have

$$-H_0Aa' = H_0Aa' + A(a + a') = (H_0 + 1)Aa',$$

so that $H_0 = -\tfrac{1}{2}$; comparing terms of third dimension,

$$H_1Aa' = H_1Aa' + (H_0 + \tfrac{1}{2})A^2a',$$

giving also $H_0 = -\frac{1}{2}$; comparing terms of $(n+2)$-th dimension, we have

$$(-1)^{n-1}H_n Aa' = \left(H_n + AH_{n-1} + \frac{A^2}{2!}H_{n-2} + \ldots + \frac{A^n}{n!}H_0\right)Aa' + \frac{A^{n+1}a'}{(n+1)!};$$

thus when n is even, $= 2m$, say, we obtain

$$\left[2H_{2m} + AH_{2m-1} + \frac{A^2}{2!}H_{2m-2} + \ldots + \frac{A^{2m}}{(2m)!}\left(H_0 + \frac{1}{2m+1}\right)\right]Aa' = 0,$$

from which $H_{2m}Aa'$ can at once be calculated when all the preceding terms are known; for $n = 2m+1$ we obtain a less convenient equation in which the term of highest dimension is again the term involving H_{2m}; if, however, in the equation just obtained we interchange a and a', and subtract the result from this equation, we obtain

$$\left[(A+A')H_{2m-1} + \frac{A^2 - A'^2}{2!}H_{2m-2} + \ldots + \frac{A^{2m} - A'^{2m}}{(2m)!}\left(H_0 + \frac{1}{2m+1}\right)\right]Aa' = 0;$$

we proceed to show that this equation determines $H_{2m-1}Aa'$ when all preceding terms are known. To explain this, let K_n denote what H_n becomes when $A - A'$ is put for A; if K_n were determined, H_n could then be found by putting $A + A'$ for A; we have thus, since $(A-A')a' = Aa'$,

$$\left[AK_{2m-1} + \frac{(A-A')^2 - A'^2}{2!}K_{2m-2} + \ldots + \frac{(A-A')^{2m} - A'^{2m}}{(2m)!}\left(H_0 + \frac{1}{2m+1}\right)\right]$$
$$\times Aa' = 0;$$

this gives $AK_{2m-1}Aa'$ when preceding terms are known; we say two solutions for K_{2m-1} are not then possible; for, if so, and P were their difference, there would exist an equation $APAa' = 0$, in which P is an integral polynomial in A and A'; we have remarked the existence of identities $QAa' = 0$, in which Q is such an integral polynomial; we say that Q is incapable of the form AP; for let $PAa' = f$, so that $APAa' = 0$ is equivalent to $Af = 0$ or $AF = FA$; let $F = A^h G$, where G is incapable of a form AG_1, in which G_1 is an integral polynomial in A and A'; then $A^{h+1}G = A^h GA$, clearly an impossible result, since the highest powers of A occurring as left-side factors are different for these two quantities.

As illustrations of these formulæ we proceed to calculate H_1, H_2, H_3, H_4. For H_1 we have

$$\left[(A+A')H_1 + \frac{A^2 - A'^2}{2}\left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right]Aa' = 0,$$

or, in the notation above,

$$\left(AK_1 - \frac{A^2 - AA' - A'A}{12}\right) Aa' = 0;$$

in virtue of $(AA' - A'A)Aa' = 0$, this gives $H_1 Aa' = \frac{1}{12}(A - A')Aa'$. For H_2 we then have

$$\left[2H_2 + A\,\frac{A - A'}{12} + \frac{A^2}{2}\,(-\tfrac{1}{2} + \tfrac{1}{3})\right] Aa' = 0,$$

and hence at once $H_2 Aa' = \frac{1}{24} AA'.Aa'$. Hence we have for K_3

$$\left[AK_3 + \frac{(A-A')^2 - A'^2}{2}\,\frac{(A-A')A'}{24} + \frac{(A-A')^3 + A'^3}{6}\,\frac{A - 2A'}{12}\right.$$
$$\left. + \frac{(A-A')^4 - A'^4}{24}\,(-\tfrac{1}{2} + \tfrac{1}{5})\right] Aa' = 0.$$

Taking account of $(AA' - A'A)Aa' = 0$, this is found to be

$$[720AK_3 + A^4 + 3A^3A' - 4A^2A'^2 + 4AA'^3 - 3AA'^2A - AA'A^2 + Q]Aa' = 0,$$

where $\quad QAa' = [4A'AA'^2 - 3A'A^2A' - A'A^3 + A'^2A^2 - 2A'^3A]Aa',$

which must then be capable of the form $APAa'$; in fact the previously given identities (§ 1) lead to

$$(A'^3A - 2A'AA'^2)Aa' = -AA'^3Aa',$$
$$(A'^2A^2 - 3A'A^2A')Aa' = (A^2A'^2 - 3AA'^2A)Aa',$$
$$A'A^3Aa' = (-A^3A' + 2AA'A^2)Aa',$$

and QAa' then takes the form in question; putting then $A + A'$ for A, we find

$$H_3Aa' = -\tfrac{1}{720}[A^3 - A'^3 + 6(A^2A' - A'^2A) + 2(AA'^2 - A'A^2)]Aa',$$

in which H_3 is changed in sign when A, A' are interchanged, and has $\varpi_2/4!$ for coefficient of A^3. Lastly, for H_4 we obtain, on substituting for H_1, H_2, H_3,

$$[-1440H_4 + A^3A' + 2A^2A'^2 - 6AA'^2A - 2AA'A^2 - AA'^3]Aa' = 0;$$

putting here, from the identities in question,

$$(A^3A' - 2AA'A^2)Aa' = -A'A^3Aa',$$

and $(2A^2A'^2 - 6AA'^2A)Aa' = (A^2A'^2 - 3AA'^2A + A'^2A^2 - 3A'A^2A')Aa',$

we find

$$H_4Aa' = \tfrac{1}{1440}[A^2A'^2+A'^2A^2-3AA'^2A-3A'A^2A'-AA'^3-A'A^3]Aa',$$

wherein H_4 is symmetrical in form. We have therefore, to the terms found,

$$\begin{aligned} a'' = a+a'+\{-\tfrac{1}{2}+\tfrac{1}{12}(A-A')+\tfrac{1}{24}AA' \\ -\tfrac{1}{720}(A^3-A'^3+6A^2A'-6A'^2A+2AA'^2-2A'A^2) \\ -\tfrac{1}{1440}(A'A^3+AA'^3+3A'A^2A'+3AA'^2A-A^2A'^2-A'^2A^2)\}Aa'. \end{aligned}$$

By taking the derivative of this we have therefore the expansion by simple alternants of $H(a, a') = (\Delta'\Delta-1)-\tfrac{1}{2}(\Delta'\Delta-1)^2+\ldots$, up to terms of the sixth dimension in A and A'. And the law of succession for the calculation of the terms of $\psi(a, a')$ corresponds of course to a law for the terms of $H(a, a')$. In fact, $H(a', a) = -H(-a, -a') = \Delta H(a, a')\Delta^{-1}$, say

$$C_1-C_2+C_3-C_4+\ldots = \left(1+A+\frac{A^2}{2!}+\ldots\right)(C_1+C_2+\ldots)\left(1-A+\frac{A^2}{2!}-\ldots\right),$$

giving, if we notice $(A_m C) = \sum_{s=0} (-1)^s \binom{m}{s} A^{m-s}CA^s$, on equating terms of dimension r,

$$(-1)^{r-1}C_r = C_r+(A, C_{r-1})+\frac{1}{2!}(A, (A, C_{r-2}))+\ldots+\frac{1}{(r-1)!}(A_{r-1}C_1).$$

It is manifest, however, that the application of this law is more laborious than the rule we have followed for the bases of the alternants.

In what has preceded the derivatives A, A' have been perfectly general. But the fact that the series $H(a, a')$ is obtainable by forming the derivative of the series $\psi(a, a')$ remains true when A, A' are less general. One of the most natural ways in which we may allow restrictions is to suppose that the powers and products of A and A' are also derivatives. We carry out a particular example of this suggestion, of a very simple kind, which is yet general enough to give a great deal of information in regard to the expansion of $\psi(a, a')$. We suppose A^2 to be a derivative, in fact equal to μA, where μ is a number; and similarly $A'^2 = \mu' A'$. Then $A^2a' = \mu Aa'$, and therefore

$$(A, (A, A')) = \mu(A, A'), \quad \text{or} \quad A^2A'-2AA'A+A'A^2 = \mu(AA'-A'A),$$

which gives $\mu A'A = AA'A$; and similarly $\mu' AA' = A'AA'$. It is consistent with these to suppose $AA' = \mu A'$ and $A'A = \mu' A$. These four equations give, if $B = pA + qA'$, whatever numbers p and q may be, $B^2 = (p\mu + q\mu')B$, and conversely. It is then easily seen that

$$H_m(A, A')Aa' = H_m(\mu, \mu')Aa',$$

and we have

$$\psi(a, a') = a + a' + [H_0 + H_1(\mu, \mu') + H_2(\mu, \mu') + \ldots]Aa' = a + a' + \lambda Aa',$$

say, where λ is a number, giving $A'' = (1 - \lambda\mu')A + (1 + \lambda\mu)A'$. On the other hand, from $A^m = \mu^{m-1}A$, we have $\Delta = 1 + \dfrac{1}{\sigma(\mu)}A$, where

$$\sigma(\mu) = \left(1 + \frac{\mu}{2!} + \frac{\mu^2}{3!} + \ldots\right)^{-1} = 1 - \frac{\mu}{2} + \frac{\varpi_1}{2!}\mu^2 + \frac{\varpi_2}{4!}\mu^4 + \ldots = \frac{\mu}{e^\mu - 1};$$

thus $\Delta'\Delta = \left(1 + \dfrac{A'}{\sigma(\mu')}\right)\left(1 + \dfrac{A}{\sigma(\mu)}\right) = 1 + \dfrac{A'}{\sigma(\mu')} + \dfrac{1}{\sigma(\mu)}\left(1 + \dfrac{\mu'}{\sigma(\mu')}\right)A;$

also, from the form $A'' = (1 - \lambda\mu')A + (1 + \lambda\mu)A'$, we have

$$A''^2 = (\mu + \mu')A'',$$

and

$$\Delta'' = 1 + \frac{A''}{\sigma(\mu + \mu')};$$

putting this equal to $\Delta'\Delta$, we have

$$1 - \lambda\mu' = \frac{\sigma(\mu + \mu')}{\sigma(\mu)}\left[1 + \frac{\mu'}{\sigma(\mu')}\right], \qquad 1 + \lambda\mu = \frac{\sigma(\mu + \mu')}{\sigma(\mu')},$$

which are consistent in virtue of

$$\frac{\mu + \mu'}{\sigma(\mu + \mu')} = \frac{\mu}{\sigma(\mu)} + \frac{\mu'}{\sigma(\mu')}\left(1 + \frac{\mu}{\sigma(\mu)}\right),$$

or

$$e^{\mu + \mu'} - 1 = e^\mu - 1 + (e^{\mu'} - 1)e^\mu,$$

and give λ; namely, in this case $H_m(\mu, \mu')$ consists of the terms of order m in the expansion of $[\sigma(\mu')]^{-1}\dfrac{\sigma(\mu + \mu') - \sigma(\mu')}{\mu}$. We find

$$\frac{1}{\sigma(\mu')}\,\frac{\sigma(\mu + \mu') - \sigma(\mu')}{\mu} = -\tfrac{1}{2} + \frac{\mu - \mu'}{12} + \frac{\mu\mu'}{24} - \tfrac{1}{720}(\mu^3 - \mu'^3 + 4\mu^2\mu' - 4\mu'^2\mu)$$
$$- \tfrac{1}{1440}(\mu^3\mu' + \mu'^3\mu + 4\mu^2\mu'^2) + \ldots;$$

thus we have at once the values of H_0, H_1, H_2 and infer, putting down for $H_3(A, A')$, $H_4(A, A')$ the most general possible respectively unsymmetrical and symmetrical polynomials of dimensions three and four certain relations among the numerical coefficients in these.

An important Property of the Series $\psi(a, a')$.

Denoting $1+A+A^2/2!+\dots$ by Δ_a, we manifestly have

$$\Delta_{a''}\Delta_{\psi(a,a')} = \Delta_{a''}\Delta_{a'}\Delta_a = \Delta_{\psi(a',a'')}\Delta_a,$$

leading to $$\psi[\psi(a, a'), a''] = \psi[a, \psi(a', a'')],$$

which can be verified directly from the series $\psi(a, a')$.

5. *Some Applications to the Theory of Groups.*

The applications we shall notice are of two kinds: the first, that which was in view in Mr. Campbell's paper referred to in the introduction to this paper, when the capitals $A, B, \dots$ are linear operators of the form

$$X = \xi_1(x_1 \dots x_n)\frac{\partial}{\partial x_1} + \dots + \xi_n(x_1 \dots x_n)\frac{\partial}{\partial x_n};$$

the second when the capitals are certain matrices. As regards the first, it is easy to show that, if

$$x_i' = \left(1+tX+\frac{t^2}{2!}X^2+\dots\right)x_i = e^{tX}x_i,$$

where t is a number, then, at least in a neighbourhood where the analytic function $F(x_1 \dots x_n)$ is regular, a similar condition being assumed for $\xi_1(x_1 \dots x_n) \dots \xi_n(x_1 \dots x_n)$,

$$F(x_1' \dots x_n') = \left(1+tX+\frac{t^2}{2!}X^2+\dots\right)F(x_1 \dots x_n).$$

If then we put $$F_1(x) = F(e^{tX_1}x) = e^{tX_1}F(x),$$

where X_1 is one linear operator, and define similarly F_2 from another operator X_2, the exponential theorem proved above gives

$$e^{tX_1}e^{tX_2} = e^{tX_3},$$

where $$tX_3 = (e^{tX_1}e^{tX_2}-1)-\tfrac{1}{2}(e^{tX_1}e^{tX_2}-1)^2+\dots$$

is a series of alternants and therefore also a linear operator, and therefore

$$F_3(x) = e^{tX_1}e^{tX_2}F(x) = e^{tX_1}F_2(x) = F_2(e^{tX_1}x).$$

In particular, putting $F(x) = x_i$, and, with linear operators $P_1, \dots, P_r$,

$$X_1 = e_1P_1+\dots+e_rP_r, \qquad X_2 = e_1'P_1+\dots+e_r'P_r,$$

where $e_1, \ldots, e_r, e_1', \ldots, e_r'$ are numerical constants, we have, if

$$e^{X_1} x_i = f_i(x, e), \qquad e^{X_2} x_i = f_i(x, e'),$$

the equation

$$e^{X_3} x_i = f_i[f(x, e), e'].$$

If, further, every alternant of two of the operators $P_1, \ldots, P_r$ is expressible linearly by $P_1 \ldots P_r$, say in the form

$$(P_\rho, P_\sigma) = c_{\rho\sigma 1} P_1 + \ldots + c_{\rho\sigma r} P_r \quad (\rho, \sigma = 1, 2, \ldots, r),$$

then X_3 will be also so expressible, say in the form $e_1'' P_1 + \ldots + e_r'' P_r$, and we shall have

$$f_i[f(x, e), e'] = f_i(x, e''),$$

the fundamental equation of Lie's group theory.

Now we have seen (§ 3) that

$$X_3 = X_1 + X_2 + \tfrac{1}{2}(X_1, X_2) + \tfrac{1}{12}(X_1 - X_2, (X_1, X_2)) - \tfrac{1}{24}(X_1, (X_2, (X_1, X_2))) + \ldots;$$

and we have

$$(X_1, X_2) = (e_1 P_1 + \ldots + e_r P_r,\ e_1' P_1 + \ldots + e_r' P_r) = \sum_\tau \sum_\rho \sum_\sigma c_{\rho\sigma\tau} e_\rho e_\sigma' P_\tau$$
$$(\rho, \sigma, \tau = 1, 2, \ldots, r).$$

We introduce now a matrix of r rows and columns whose general element is

$$E_{\rho\sigma} = \sum_\tau c_{\sigma\tau\rho} e_\tau \quad (\tau = 1, \ldots, r),$$

so that each element is a linear function of $e_1, \ldots, e_r$ with coefficients chosen from the constants $c_{\rho\sigma\tau}$; then we have, since from $(P_\rho, P_\sigma) = -(P_\sigma, P_\rho)$ follows $c_{\rho\sigma\tau} = -c_{\sigma\rho\tau}$,

$$(X_1, X_2) = -\sum_\tau \sum_\sigma E_{\tau\sigma} e_\sigma' P_\tau = -\sum_\tau (Ee')_\tau P_\tau,$$

where Ee' is a symbol for the set of r quantities

$$(Ee')_\tau = E_{\tau_1} e_1' + E_{\tau_2} e_2' + \ldots + E_{\tau_r} e_r',$$

and clearly $Ee' = -E'e$, if E' be the same function of $e_1, \ldots, e_r'$ as is E of $e_1 \ldots e_r$. Hence it follows, if $(Ee')_\tau$ be momentarily denoted by f_τ,

$$(X_1, (X_1, X_2)) = -(e_1 P_1 + \ldots + e_r P_r,\ f_1 P_1 + \ldots + f_r P_r) = \sum_\tau (Ef)_\tau P_\tau,$$

$$(X_2, (X_1, X_2)) = -(e_1' P_1 + \ldots + e_r' P_r,\ f_1 P_1 + \ldots + f_r P_r) = \sum_\tau (E'f)_\tau P_\tau,$$

and therefore again, if $(E'f)$ be momentarily denoted by g_τ,

$$(X_1, (X_2, (X_1, X_2))) = (e_1 P_1 + \ldots + e_r P_r,\ g_1 P_1 + \ldots + g_r P_r) = -\sum_\tau (Eg)_\tau P_\tau;$$

but $Ef = E^2e'$, $E'f = E'Ee'$, $Eg = EE'f = EE'Ee'$; thus we have

$$X_3 = \sum_{\tau=1}^{r} (e_\tau + e'_\tau - \tfrac{1}{2}(Ee')_\tau + \tfrac{1}{12}(E^2e')_\tau - \tfrac{1}{12}(E'Ee')_\tau + \tfrac{1}{24}(EE'Ee')_\tau + \ldots)P_\tau;$$

or, since $X_3 = \sum_{\tau=1}^{r} e''_\tau P_\tau$, we have r equations

$$e''_\tau = e_\tau + e'_\tau - \tfrac{1}{2}(Ee')_\tau + \tfrac{1}{12}(E^2e')_\tau - \tfrac{1}{12}(E'Ee')_\tau + \tfrac{1}{24}(EE'Ee')_\tau + \ldots,$$

all of which are represented by

$$e'' = e + e' + [-\tfrac{1}{2} + \tfrac{1}{12}(E - E') + \tfrac{1}{24}EE' + \ldots]Ee',$$

and this is the series we have previously denoted by $\psi(e, e')$, there being r functions $\psi_1(e, e'), \ldots, \psi_r(e, e')$. Thus it appears that the fundamental equations of Lie's group theory are

$$f_i[f(x, e), e'] = f_i[x, \psi(e, e')],$$

and we have proved, to employ the ordinary nomenclature, not only that the equations of the parameter group have the form given by $e'' = \psi(e, e')$ expressible by the matrices E, E', but that there is a one to one correspondence between the terms of this, and the terms, which are alternants, in the expansion of

$$e^A e^B - 1 - \tfrac{1}{2}(e^A e^B - 1)^2 + \ldots .$$

And this brings us to the second application of our preceding general theory to the theory of groups. It is, in fact, easily recognised that the laws fundamental for derivatives and bases which were set out at the beginning of this paper hold for matrices E, such as have been here introduced, and for the sets $e_1, \ldots, e_r$. To see this it is only necessary to notice (1) the fact proved above that $Ee' + E'e = 0$, (2) the fact that the E matrix of the set Ee' is $EE' - E'E$. This last follows from the equation

$$EE'e'' + E'E''e + E''Ee' = 0,$$

which expresses that relation among the constants of structure which follow from the identity

$$(X_1, (X_2, X_3)) + (X_2, (X_3, X_1)) + (X_3, (X_1, X_2)) = 0;$$

for this equation is $\quad (EE' - E'E)e'' + E''Ee' = 0.$

It must be remarked, however, that it is only for groups of general form that the equation $E = 0$ involves $e_1 = 0, \ldots, e_r = 0$, or $e = 0$. The group must have no special transformations.

Conversely that intricate part of the algebra of the theory of groups which is concerned with the theory of structure may conveniently be stated in the general terms of the earlier part of this paper. As the theory is there set out, the primary set of capitals may be of infinite number, and in any case the alternants formed from them of successive dimensions form an indefinitely extended system from which there is no return to the original capitals, save perhaps by reverting an infinite series

$$\lambda_0 B + \lambda_1 (A, B) + \lambda_2 (A, (A, B)) + \dots .$$

If we wish to have a finite system of elements, we may then naturally suppose, to speak in terms of the bases of the alternants, that there is but a finite number of bases, say $a_1, a_2, \dots, a_r$, in terms of which every other base is expressible linearly with numerical coefficients. We shall thus have, for instance, equations such as

$$A_\rho a_\sigma = -(c_{\rho\sigma 1} a_1 + \dots + c_{\rho\sigma r} a_r).$$

It is at once evident that this leads to equations $F(A_\rho) a_\sigma = 0$, where $F(A_\rho)$ is an integral polynomial in A_ρ with numerical coefficients; and so on. It is interesting to consider from this point of view such propositions, for instance, as one, due to Cartan, relating to the necessary and sufficient condition that a group be integrable; with our proposition that the E matrix of Ee' is (E, E'), this condition is that the constants of structure should be such that for some positive integral λ we should have, for every e and e', an equation $(EE' - E'E)^\lambda = 0$.