



LXXXII. Explanation of a remarkable paradox in the calculus of functions noticed by Mr. Babbage

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As the iodate of ammonia is not noticed in any work with which I am acquainted, I think it right to observe here that it is a highly crystalline granular powder, possessed of but little solubility: it may be prepared by saturating the solution of the muriatic and iodic acids with pure ammonia, when it will fall down, the muriate remaining dissolved. I find that iodic acid is decomposed by sulphocyanic acid and the sulphocyanates of potash and soda; and also that saliva, in consequence probably of the sulpho-cyanate of potash it contains, decomposes iodic acid, and produces with it and starch a blue precipitate not to be distinguished from that produced under similar circumstances by morphia. The importance of this discovery in a medico-legal point of view is considerable, since iodic acid is now very much relied upon as a test for morphia.

I am, Gentlemen, yours, &c.

Roebuck Place, Great Dover Road,
Southwark.

LEWIS THOMPSON,
M.R.C.S.

LXXXII. *Explanation of a remarkable Paradox in the Calculus of Functions, noticed by Mr. Babbage.* By JOHN T. GRAVES, Esq., M.A., of the Inner Temple.

[Continued from p. 341, and concluded.]

HAVING thus proved that

$$e_0 \sqrt{-1} \frac{z}{\sqrt{z^2}} \cos^{-1} \frac{y}{\sqrt{y^2 + z^2}} = y + \sqrt{-1} z, \quad (15.)$$

we have seen in what manner it follows that

$$l \sqrt{y^2 + x^2} + \sqrt{-1} \frac{z}{\sqrt{z^2}} \cos^{-1} \frac{y}{\sqrt{y^2 + z^2}}$$

is an e -log of x . Q. E. D.

Let $l \sqrt{y^2 + z^2} + \sqrt{-1} \frac{z}{\sqrt{z^2}} \cos^{-1} \frac{y}{\sqrt{y^2 + z^2}}$ (which

I call the 0th e -log of x of the 0th order) be denoted by $l(y + \sqrt{-1} z)$ or lx . It is plain that when x is real and positive, lx resolves itself in point of quantity (as it ought to do, if our notation be consistent) into the arithmetical e -log of x . Following the same notation,

$$l \frac{1}{\sqrt{y^2 + z^2}} - \sqrt{-1} \frac{z}{\sqrt{z^2}} \cos^{-1} \frac{y}{\sqrt{y^2 + z^2}} \quad (16.)$$

may be denoted by $l \frac{1}{x}$, since it may be similarly proved to be an ϵ -log of $\frac{1}{x}$, and since it is the same individual function of $\frac{1}{x}$ that $l x$ is of x . That it is the *same individual function* (so far as such a phrase can be considered applicable), or in other words, that it has a better right than any other logarithm of $\frac{1}{x}$ to be considered the logarithm of $\frac{1}{x}$ corresponding to $l x$, will appear on substituting in $l x$ the respective constituents of $\frac{1}{x}$ for those of x , that is to say $\frac{y}{y^2 + z^2}$ for y and $\frac{-z}{y^2 + z^2}$ for z , for by such substitution the expression which I denote by $l \frac{1}{x}$ will be obtained.

Observing that in $l \frac{1}{x}$ the constituent (see (16.))

$$l \frac{1}{\sqrt{y^2 + z^2}} = -l \sqrt{y^2 + z^2}, \text{ we find}$$

$$l \frac{1}{x} = -l x. \quad (17.)$$

Let $l \sqrt{y^2 + z^2} = y'$ and $\frac{z}{\sqrt{z^2}} \cos_{0+}^{-1} \frac{y}{\sqrt{y^2 + z^2}} = z'$, and let $l l \theta$ be denoted by the notation $l^2 \theta$, then, by what precedes, we have

$$l (y' + \sqrt{-1} z') \text{ or } l^2 x = l \sqrt{y'^2 + z'^2} + \sqrt{-1} \frac{z'}{\sqrt{z'^2}} \cos_{0+}^{-1} \frac{y'}{\sqrt{y'^2 + z'^2}} \quad (18.)$$

$$\begin{aligned} \text{we have also } l (-y' - \sqrt{-1} z') \text{ or } l^2 \frac{1}{x} \\ = l \sqrt{y'^2 + z'^2} - \sqrt{-1} \frac{z'}{\sqrt{z'^2}} \cos_{0+}^{-1} \frac{y'}{\sqrt{y'^2 + z'^2}} \end{aligned} \quad (19.)$$

but it is easy to see that

$$\cos_{0+}^{-1} \frac{-y'}{\sqrt{y'^2 + z'^2}} = \pi - \cos_{0+}^{-1} \frac{y'}{\sqrt{y'^2 + z'^2}} \quad (20.)$$

$$\text{Hence } l^2 x = l^2 \frac{1}{x} + \sqrt{-1} \frac{z}{\sqrt{z^2}} \pi \quad (21.)$$

since $\frac{z^1}{\sqrt{z^2}}$ evidently = $\frac{z}{\sqrt{z^2}}$. But $\sqrt{-1} \pi$ or $-\sqrt{-1} \pi$ (one or other of which values $\sqrt{-1} \frac{z}{\sqrt{z^2}} \pi$ must possess at any time) is an e -log of -1 , for

$$e_0^{\pm \sqrt{-1} \pi} = \cos(\pm \pi) + \sqrt{-1} \sin(\pm \pi) = -1. \quad (22.)$$

Hence, if z be *positive*, or, x being real, if we choose or have reason to consider the infinitesimal or zero z *positive*, and if

$\psi x = c^{\frac{l^2 x}{\sqrt{-1} \pi}}$, I say that we shall have $c \psi \frac{1}{x} = \psi x$, whatever be c ; it being understood that corresponding powers are

to be compared in the expressions $c^{\frac{l^2 x}{\sqrt{-1} \pi}}$ and $c \cdot c^{\frac{l^2 \frac{1}{x}}{\sqrt{-1} \pi}}$.

$$\text{For } c \psi \frac{1}{x} = c \cdot c^{\frac{l^2 \frac{1}{x}}{\sqrt{-1} \pi}} = c^{1 + \frac{l^2 \frac{1}{x}}{\sqrt{-1} \pi}} = c^{\frac{l^2 \frac{1}{x} + \sqrt{-1} \pi}{\sqrt{-1} \pi}}. \quad (23.)$$

but by equation (21.) since z is now supposed positive,

$$c^{\frac{l^2 \frac{1}{x} + \sqrt{-1} \pi}{\sqrt{-1} \pi}} = c^{\frac{l^2 x}{\sqrt{-1} \pi}} = \psi x. \quad (24.)$$

On the other hand, if z be *negative*, or so considered, and

$\psi x = c^{\frac{l^2 x}{\sqrt{-1} \pi}}$, we shall no longer have $\psi x = c \psi \frac{1}{x}$, but if,

z being negative or so considered, $\psi x = c^{\frac{-l^2 x}{\sqrt{-1} \pi}}$, we may prove in similar manner that equation (1.) will subsist whatever be c .

We have thus therefore obtained two correlated and mutually complementary examples, both possessing, to a certain extent, the property of satisfying (1.), and both of them included

in the generic form $c^{\frac{-\log \log x}{\log(-1)}}$, mentioned by Mr. Babbage as derived from the process of Laplace, but in neither case does equation (1.) hold good for all values of z , positive or negative, nor even for all *real* values of x , *without an annexed supposition relating to those real values*. Thus we see matters so arranged, with most curious delicacy,

that we are never at liberty to suppose $\psi \frac{1}{x} = c \psi x$ (a

supposition which it is necessary to make if we would prove' $c^2 = 1$), without making ψ itself change the form it had when

ψx was equal to $c \psi \frac{1}{x}$; in other words, (1.) and (2.) are es-

entially non-simultaneous equations in the illustrative instances before us, for the z of x , of whatever sign it be or be considered to be, though such z be infinitesimal or zero, as in the case of x real, is always or must always be considered to

be of a different sign from the z of $\frac{1}{x}$. When $\psi x = c \sqrt{\frac{l^2 x}{-1}}$

equation (1.) will not be satisfied for all real quantitative values of x , unless zero be considered positive, nor again when

$\psi x = c \sqrt{\frac{l^2 x}{-1}}$, unless zero be considered negative. *Vice versâ* with respect to equation (2.). One supposition excludes the other: zero may be considered either positive or negative, but not both together. Hence, even in the case of x real, where the solutions would appear on first view to be concurrent, they are, in truth, alternative. We are bound to consider x the same in *state* as well as *quantity* on both sides of the equations (1.) and (2.), and here obscurity arises from the symbols of algebra not expressing to the eye a difference of state between reals having the same quantity. Such difference of *state* in things denoted by algebraic symbols is in most cases immaterial, unless no *quantity* remain in either of their constituents; but we know that it is of importance in the case of vanishing fractions, and we perceive that it may become so in certain other fine circumstances, such as those which we have just discussed.

We have shown therefore by a particular example (or rather by two correlated examples) that the paradox noticed by Mr. Babbage is only a remarkably subtle instance of the following general proposition which is not *à priori* improbable. Though we may prove it to be impossible to find *one fixed form* ψ , such that the equation $\psi x = F \psi \alpha x$ (F and α being given functions) shall hold good simultaneously in different cases where particular values of x are assumed (the term "value" including state as well as quantity), we are not therefore to despair of finding *distinct forms* of ψ , absolute or alternative, which for certain values of x , within appropriate limits, shall severally satisfy the equation $\psi x = F \psi \alpha x$. Such a partial form of ψx and the corresponding partial form of $F \psi \alpha x$ taken with it may be likened to two curves which co-

incide for a certain continuous space and divaricate in the rest of their course.

Those parts of this paper in which *infinitesimals* have been spoken of in the more popular language of mathematics, may advantageously be translated into the more rigid phraseology of *limits*. Various distinct continuities may terminate in the same quantity as a limit, as, for example, a line may be looked upon as having moved through any of the infinite number of planes of which it may be the boundary, and it is easy to conceive that there are properties of such a line, which (all things else remaining the same) vary with the plane in which its motion is deemed to have taken place; but it is, I believe, a novelty in algebra, to present an instance of a given individual function of a positive or negative quantity, which varies accordingly as the functional subject is regarded as the limit of this or that kind of imaginary quantity.

Professor De Morgan, in the place before cited, (p. 335.) mentions $\left(\frac{1-x}{1+x}\right)^{\frac{\log c}{\log(-1)}}$ as a form of ψx not obviously discontinuous that appears to satisfy equation (1.) independently of c . We may assume that always on the opposite sides of that equation $\frac{\log c}{\log(-1)}$ is intended to denote the same quantity, and that in the

expressions $\left(\frac{1-x}{1+x}\right)^{\frac{\log c}{\log(-1)}}$ and $c \left(\frac{1-\frac{1}{x}}{1+\frac{1}{x}}\right)^{\frac{\log c}{\log(-1)}}$ corre-

sponding powers are to be compared. With respect to this instance I shall only add that it would not be difficult to show by reasoning similar to that which I have already employed in this paper, that no definite case included in the indeterminate

expression $\left(\frac{1-x}{1+x}\right)^{\frac{\log c}{\log(-1)}}$ can be other than a partial or alternative solution for ψx , unless $c^2 = 1$; for let $\frac{1-x}{1+x} = y + \sqrt{-1}z$, then it may be proved by my exponential theorems that the equation

$$\left(\frac{1-x}{1+x}\right)^{\frac{\log c}{\log(-1)}} = c \left(\frac{1-\frac{1}{x}}{1+\frac{1}{x}}\right)^{\frac{\log c}{\log(-1)}},$$

if, being individualized, it hold good for z of one sign posi-

tive or negative, cannot hold good for z of an opposite sign, unless $c = \frac{1}{c}$.

Inner Temple, July 1836.

P.S. It is not out of place to mention, that I am gratified with the view which Professor De Morgan has taken in the last Number of this Magazine (October 1836, vol. ix. p. 252,) of my researches on logarithms, and that I agree with him in considering my results rather upon the whole as extensions or (as I should say) completions, than as corrections of what had before been accomplished. He has also properly noticed an oversight I committed in not observing the distinction he drew in his *Calculus of Functions*, with reference to the possibility of obtaining the most general solution, between functional equations where there are and where there are not independent variables. I may be permitted, however, to assent to his remarks on some other points with some qualifications, which may seem over-nice and pedantic, but are required by the delicacy of the subject, and I wish to prefix some explanation relative to the actual progress or improvement which I consider this branch of science to have received from my researches. The deficiencies in the ordinary theory which I have endeavoured to supply are the following: first, I found no formula which assigned even one value of a^x , much less all of them, when a and x were imaginary; secondly, I considered that as -2 was a value of $4\frac{1}{2}$, $\frac{1}{2}$ would be admitted to be, or at least deserved to be reckoned a logarithm of -2 to the base 4, and yet in no formula which I had met with for the 4-logs of -2 was any such quantity as $\frac{1}{2}$ included; thirdly, I observed generally great laxity, not to say inaccuracy, in the use of ambiguous exponential expressions, and saw equations employed without apparent restriction where, perhaps, the two sides had but one value in common. For instance, the equation $e^{1+\sqrt{-1}2m\pi} = e$ is not correct without restricting the meaning of the left-hand side, for though every quantity included in the formula $1 + \sqrt{-1}2m\pi$ is an e -log of e , $e^{1+\sqrt{-1}2m\pi}$ has an infinite number of real values besides e for any given m , except $m = 0$. Hence, I scruple to call $e^{1+\sqrt{-1}2m\pi}$ merely a *general algebraic form* of e , and think it necessary to devise a notation to characterize that particular value of $e^{1+\sqrt{-1}2m\pi}$ which is equal to e .

The still more general form $e^{\frac{1+\sqrt{-1}2m\pi}{1+\sqrt{-1}2n\pi}}$ has, equally

with $e^{1+\sqrt{-1}2m\pi}$, one of its values equal to e , and is *the most general exponential form* that possesses this property; a property which seems to me, even according to ordinary acceptance, to confer on every quantity included in the formula

$\frac{1+\sqrt{-1}2m\pi}{1+\sqrt{-1}2n\pi}$ a right to the appellation *e-log* of e . Here

let me remark that I cannot conceive how any difference in results can be obtained from operating correctly on two *strictly equivalent algebraic forms* such as $\cos \theta$ and $\cos (2i\pi \pm \theta)$. It is true that one may suggest what we have to *recollect* with respect to the other, and it is true that in the treatment of such forms there are many specious fallacies to be guarded against. Thus, it would not be correct to reason as though $\cos \{c(2i\pi \pm \theta)\}$ were the same function of $\cos (2i\pi \pm \theta)$ that $\cos (c\theta)$ is of $\cos \theta$, if θ denote a particular individual value of $\cos^{-1} \cos \theta$. On account of the preceding considerations, among others, instead of obtaining my formula for x , the general logarithm of y , by the method stated by Professor De Morgan as substantially the same as mine, viz. by setting

out at once with the unelementary definition $e^{(1+\sqrt{-1}2m\pi)x} = e^{ly+\sqrt{-1}2n\pi}$, I should prefer building, as I have done, on received principles of analogy, which, I think, would naturally entitle ψy to the name of an *e-log* of y , if any value of $a^{\psi y}$ were equal to y , especially if we found that ψy possessed the property $\psi y + \psi y' = \psi yy'$. I do not meet in books any *explicit exclusion* of $\frac{1}{2}$ from the name of logarithm of -10 to the base 100 on the ground that -10 is not what is called

the arithmetical value of $100^{\frac{1}{2}}$, and, in legal phrase, I submit that the *onus* of making out their case lies on those who advocate such an exclusion. It would refuse the name of logarithm to any function whatever of y , where y and the base a were not real and positive, or would require some definition of what is meant by the arithmetical value of a^x for all values of a and x , as well imaginary as real. Now, for some values of a , it may be matter of arbitrary decision to determine which one of a certain pair of values of a^x is to be considered as *corresponding* to that which is the arithmetical value of a^x ,

when a and x are real and positive. Is $\sqrt{-1}$ or $\frac{1}{\sqrt{-1}}$ the

arithmetical value of $(-1)^{\frac{1}{2}}$? For some values of a and x , a^x has more than one real positive value, and for some, again, a^x has a single real positive value corresponding to

what would be in general an imaginary value of a^2 , if a and x were real and positive. But I have already occupied too much space, and need not labour these arguments, for Professor De Morgan does not materially differ from me here. He seems to regard the ordinary theory as an edifice complete in itself, but is content to receive my results as an extension which may prove useful, whereas I regard them rather as the erection of a wing, required for symmetry, if not for use.

LXXXIII. On the Conducting Power of Iodine for Electricity.

By JAMES INGLIS, M.D.*

[Addressed to the Chemical Section of the British Association.]

IT may not, perhaps, have escaped the notice of some of the members of this Section, that in extracts from a Prize Essay of mine, published some months ago, in the *Philosophical Magazine*, I stated that I had found iodine to be a conductor of electricity. Nor may the experiments of Mr. Solly tending to prove the contrary have passed by unobserved. Nevertheless, being satisfied in my own mind what I had published was correct, I determined at the earliest opportunity to resume the investigation, and instead of answering that gentleman directly through the medium of the *Philosophical Magazine*, I thought it might be better to lay before you the result, in as much as I shall by experiment prove my former statement, and then furnish you with that portion of iodine which you have seen conduct, that you may for yourselves judge of its purity.

In Mr. Solly's first paper, no mention is made of experiments performed with fused iodine; but his attention being drawn to the subject by a note of mine, he published a second, in which he throws a doubt on the purity of the iodine I had used, saying that it contained "most probably the iodide of iron, which is not unfrequently present in the iodine of the shops." (*Lond. and Edinb. Phil. Mag.*, No. 48. p. 401.)

The iodine I used was obtained from the manufactory of Mr. Whitelaw of Glasgow, where no iron vessel is ever employed, and in which, in its veriest impurity, no iron can be detected. Here, for instance, is one tube containing an aqueous solution of ioduret of iron; a second, an aqueous solution of the iodine to be tested; and a third having in it a solution of the ferrocyanate of potassa. Now, on adding a small portion of this last solution to the one containing iron, immediately the blue ferrocyanate of the peroxide of iron results. But no such effect is produced when the test is added

* Read before the Chemical Section of the British Association at Bristol, Aug. 26, 1836: and now communicated by the Author.