THE DIOPHANTINE EQUATION $x^n - Ny^n = z$

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1. The Diophantine equality

$$\lambda x = \mu y + z$$

is equivalent to the Diophantine inequality

$$\lambda x \geq \mu y.$$

It has been shewn* that the arithmetically independent solutions of this inequality are obtained by forming the descending intermediate series of convergents to the continued fraction

$$\frac{\mu}{\lambda}.$$

If this finite series be

$$\frac{1}{0}, \frac{a_0}{\beta_0}, \frac{a_1}{\beta_1}, \dots, \frac{a_t}{\beta_t}, \frac{a_{t+1}}{\beta_{t+1}}, \dots, \frac{a_n}{\beta_n},$$

the general solution is

$$x = (A+1)a_t + Ba_{t+1},$$

$$y = (A+1)\beta_t + B\beta_{t+1},$$

where A, B are arbitrary positive (including zero) integers.

The fundamental solutions are given by

$$x = a_t,$$

$$y = \beta_t,$$

and it was shewn long ago by Sylvester that, if

$$\frac{p_0}{q_0}, \frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}$$

* "The Diophantine Inequality $\lambda x \geq \mu y$," *Camb. Phil. Trans.*, Vol. XIX., Part I., 1900.

be the principal convergents to $\frac{\mu}{\lambda}$,

and $\frac{p'_0}{q'_0}, \frac{p'_1}{q'_1}, \dots, \frac{p'_n}{q'_n}$

the principal convergents to the fraction whose partial quotients are the same as those of

$$\frac{\mu}{\lambda}.$$

but taken in reverse order,

$$p_n q_s - p_s q_n = (-)^{s+1} p_{n-s-1},$$

$$p_{n-1} q_s - p_s q_{n-1} = (-)^{s+1} q'_{n-s-1}.$$

These important results involve the complete solution of the problem of finding the arithmetically independent solutions of the Diophantine equality

$$\lambda x = \mu y + z;$$

for, taking the descending intermediate series of convergents to

$$\frac{p_n}{q_n} \quad \text{or} \quad \frac{\mu}{\lambda},$$

$$\begin{aligned} p_n \beta_t - a_t q_n &= p_n (q_{2m} + s q_{2m+1}) - (p_{2m} + s p_{2m+1}) q_n \\ &= -p'_{n-2m-1} + s p'_{n-2m-2}; \end{aligned}$$

hence $\lambda a_t - \mu \beta_t = p'_{n-2m-3} + (a_{2m+1} - s) p'_{n-2m-2},$

where a_{2m+2} is the partial quotient which is the maximum value of s .

Now $p'_{n-2m-3} + (a_{2m+1} - s) p'_{n-2m-2}$

is the numerator of one of the descending intermediate series of convergents to the fraction which is the reverse of

$$\frac{\mu}{\lambda}.$$

Writing this series $\frac{a'_0}{\beta'_0}, \frac{a'_1}{\beta'_1}, \dots, \frac{a'_{n'}}{\beta'_{n'}}$,

$$\lambda a_t - \mu \beta_t = a'_{n'-t-1}$$

where, when $t = n'$, we take a'_{-1} to be zero.

We have therefore the following theorem:—

“The fundamental arithmetically independent solutions of the Diophantine equality

$$\lambda x = \mu y + z$$

are formed by constructing

“(i.) the descending intermediate series of convergents to the fraction $\frac{\mu}{\lambda}$, viz.,

$$\frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_{n'}}{\beta_{n'}}.$$

“(ii.) The descending intermediate series to the fraction which is the reverse of $\frac{\mu}{\lambda}$, viz.,

$$\frac{\alpha'_0}{\beta'_0}, \frac{\alpha'_1}{\beta'_1}, \dots, \frac{\alpha'_{n'}}{\beta'_{n'}}.$$

“The solutions are then $n' + 1$ in number, viz.,

$$x = \alpha_t, \quad y = \beta_t, \quad z = \alpha'_{n'-t-1},$$

where α'_{-1} is to be taken equal to zero.”

As an example, take $31x = 222y + z$,

where $\frac{222}{31} = 7 + \frac{1}{6} + \frac{1}{5}, \quad \frac{222}{43} = 5 + \frac{1}{6} + \frac{1}{7}:$

the descending series to $\frac{222}{31}$

is $\frac{1}{0}, \frac{8}{1}, \frac{15}{2}, \frac{22}{3}, \frac{29}{4}, \frac{36}{5}, \frac{43}{6}, \frac{222}{31}.$

and to $\frac{222}{43}$

$$\frac{1}{0}, \frac{6}{1}, \frac{11}{2}, \frac{16}{3}, \frac{21}{4}, \frac{26}{5}, \frac{31}{6}, \frac{222}{43},$$

and we derive the fundamental solutions

x	y	z
1	0	31
8	1	26
15	2	21
22	3	16
29	4	11
36	5	6
43	6	1
222	31	0

and every other solution is a linear function of these eight.

2. It is a corollary from the foregoing theorem that the fundamental (that is arithmetically independent) solutions of the Diophantine inequality

$$x \geq N^{1/m} y,$$

N not being a perfect m -th power, are obtained by forming the infinite descending intermediate series of convergents to

$$N^{1/m}.$$

If this series be $\frac{1}{0}, \frac{a_1}{\beta_1}, \dots, \frac{a_t}{\beta_t}, \frac{a_{t+1}}{\beta_{t+1}}, \dots$, ad inf.,

the general solution is $x = (A+1)a_t + Ba_{t+1}$,

$$y = (A+1)\beta_t + B\beta_{t+1}.$$

Hence this is also the solution of the Diophantine inequality

$$x^m \geq Ny^m;$$

and thence we say, in regard to the Diophantine equality

$$x^m = Ny^m + z,$$

that the fundamental values of x and y are as above, and it remains to determine the properties of z .

My first aim is to connect z with the continued fraction which denotes

$$N^{1/m}.$$

Putting, in a usual notation,

$$\begin{aligned} N^{1/m} &= x_1 = a_1 + \frac{1}{x_2}, \\ x_2 &= a_2 + \frac{1}{x_3}, \\ &\dots \dots \dots, \\ x_k &= a_k + \frac{1}{x_{k+1}}, \end{aligned}$$

let the principal convergents be

$$\frac{p_{-1}}{q_{-1}}, \frac{p_0}{q_0}, \frac{p_1}{q_1}, \dots$$

or

$$\frac{0}{1}, \frac{1}{0}, \frac{a_1}{1}, \dots;$$

we have

$$\begin{aligned} x_{n+2} &= \frac{q_n N^{1/m} - p_n}{p_{n+1} - q_{n+1} N^{1/m}} \\ &= \frac{(-)^m (p_n p_{n+1}^{m-1} - N q_n q_{n+1}^{m-1}) + p_{n+1}^{m-2} N^{1/m} + p_{n+1}^{m-3} q_{n+1} N^{2/m} + \dots + q_{n+1}^{m-2} N^{(m-1)/m}}{(-)^{n+1} (p_{n+1}^m - N q_{n+1}^m)}. \end{aligned}$$

The expressions $(-)^n (p_n p_{n+1}^{m-1} - Nq_n q_{n+1}^{m-1}) = L_{n+1}$,

$$(-)^{n+1} (p_{n+1}^m - Nq_{n+1}^m) = M_{n+1}$$

are essentially positive; L_{n+1} has been termed a "rational dividend," and M_{n+1} a "divisor."

Substituting this value of x_{n+2} in the relation

$$x_{n+1} x_{n+2} = a_{n+1} x_{n+2} + 1$$

and comparing coefficients, we find, putting

$$(-)^n (p_n^{m-1} p_{n+1} - Nq_n^{m-1} q_{n+1}) = K_{n+1},$$

the two fundamental relations

$$L_n L_{n+1} + (-)^{n+1} N (p_{n+1}^{m-1} q_n^{m-1} - p_n^{m-1} q_{n+1}^{m-1}) = a_{n+1} L_{n+1} M_n + M_n M_{n+1}, \quad (1)$$

$$L_n + K_{n+1} = a_{n+1} M_n, \quad (2)$$

where, comparing the coefficients of $N^{s/m}$, the first relation corresponds to $s = 0$, and the second relation is given by all values of s from 1 to $m-1$, and no other relation is obtainable.

Now, the intermediate convergent of the descending series appertaining to

$$N^{1/m}$$

is, in general,

$$\frac{\alpha_t}{\beta_t},$$

where

$$\alpha_t = p_{2n} + sp_{2n+1},$$

$$\beta_t = q_{2n} + sq_{2n+1};$$

so that the general expression of z is

$$\begin{aligned} z &= [(A+1)(p_{2n} + sp_{2n+1}) + B \{ p_{2n} + (s+1)p_{2n+1} \}]^m \\ &\quad - N[(A+1)(q_{2n} + sq_{2n+1}) + B \{ q_{2n} + (s+1)q_{2n+1} \}]^m \\ &= [(A+B+1)p_{2n} + \{s(A+B+1) + B\} p_{2n+1}]^m \\ &\quad - N[(A+B+1)q_{2n} + \{s(A+B+1) + B\} q_{2n+1}]^m; \end{aligned}$$

or, writing

$$A+B+1 = u,$$

$$s(A+B+1) + B = u_s,$$

we have

$$z = (p_{2n}^m - Nq_{2n}^m, p_{2n}^{m-1} p_{2n+1} - Nq_{2n}^{m-1} q_{2n+1}, \dots, p_{2n+1}^m - Nq_{2n+1}^m) \propto (u, u_s)^m.$$

There are $m+1$ coefficients of which

$$\text{the first } p_{2n}^m - Nq_{2n}^m = M_{2n},$$

$$\text{the second } p_{2n}^{m-1} p_{2n+1} - Nq_{2n}^{m-1} q_{2n+1} = K_{2n+1} = a_{2n+1} M_{2n} - L_{2n},$$

$$\text{the } m\text{-th } p_{2n} p_{2n+1}^{m-1} - Nq_{2n} q_{2n+1}^{m-1} = L_{2n+1},$$

$$\text{the } m+1\text{-th } p_{2n+1}^m - Nq_{2n+1}^m = -M_{2n+1}.$$

We have to express the remaining coefficients from the 3rd to the $(m-1)$ -th inclusive in terms of the numbers L , M and the partial quotients a . Postponing the general consideration of this matter, I observe that the expression for z , viz.,

$$(up_{2n} + u_s p_{2n+1})^m - N(uq_{2n} + u_s q_{2n+1})^m,$$

is a linear transformation of $x^m - Ny^m$

to new variables u, u_s ;

the modulus of transformation being

$$p_{2n} \quad p_{2n+1} = 1.$$

$$q_{2n} \quad q_{2n+1} =$$

Hence we may employ known results in the theory of invariants of binary forms in the investigation.

It is convenient to consider in particular and in detail the cases of $m = 2, 3, 4, \dots$ in order to lead up to the theory of the general case.

$$3. \text{ The case } m = 2. \quad x^2 = Ny^2 + z.$$

The general value of z is

$$(M_{2n}, L_{2n+1}, -M_{2n+1}) \chi(u, u_s)^2,$$

where

$$u = A + B + 1,$$

$$u_s = su + B,$$

A, B being arbitrary positive integers, and s a positive integer not exceeding a_{2n+2} .

Since here $K_{2n+1} = L_{2n+1}$, we have the relations

$$L_{2n} L_{2n+1} + N = a_{2n+1} L_{2n+1} M_{2n} + M_{2n} M_{2n+1},$$

$$L_{2n} + L_{2n+1} = a_{2n+1} M_{2n},$$

and thence

$$M_{2n} M_{2n+1} + L_{2n+1}^2 = N,$$

where $M_{2n} M_{2n+1}$ is an essentially positive number. This relation at once establishes the periodicity of the numbers M , the numbers L , and the numbers α . Also, for given values of A , B , it is clear that z has a finite number of numerical values.

The relation
$$M_{2n} M_{2n+1} + L_{2n+1}^2 = N$$

is obtained at once by equating the invariants of

$$x^2 - Ny^2$$

and

$$(M_{2n}, L_{2n+1}, -M_{2n+1} \chi(u, u_s)).$$

It leads to a relation between three values of z , if such values depend upon the same three numbers

$$M_{2m}, L_{2m+1}, L_{2m+1};$$

for write $z_{A, B, s} = (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(u_1, u_2))^2 = \alpha_u^2 = \beta_u^2 = \dots,$

$$z_{C, D, t} = (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(v_1, v_2))^2 = \alpha_v^2 = \beta_v^2 = \dots,$$

$$z_{E, F, r} = (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(w_1, w_2))^2 = \alpha_w^2 = \beta_w^2 = \dots$$

We may clearly eliminate

$$M_{2m}, L_{2m+1}, -M_{2m+1}$$

between these three equations and the invariant relation

$$M_{2n} M_{2n+1} + L_{2n+1}^2 = N;$$

for, squaring the symbolic identity

$$\alpha_u \beta_v - \alpha_v \beta_u = (\alpha \beta)(uv),$$

we obtain

$$\alpha_u^2 \beta_v^2 + \alpha_v^2 \beta_u^2 - (\alpha \beta)^2 (uv)^2 = 2\alpha_u \alpha_v \beta_u \beta_v$$

and two similar identities which lead to the relations

$$z_{C, D, t} z_{E, F, r} + N (vw)^2 = \{ (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(v_1, v_2) \chi(w_1, w_2) \}^2,$$

$$z_{E, F, r} z_{A, B, s} + N (wu)^2 = \{ (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(w_1, w_2) \chi(u_1, u_2) \}^2,$$

$$z_{A, B, s} z_{C, D, t} + N (uv)^2 = \{ (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(u_1, u_2) \chi(v_1, v_2) \}^2,$$

and, since $(\alpha\beta)^2 = -2N$,

$$\alpha_v \alpha_w = (M_{2m}, L_{2m+1}, -M_{2m+1}) (v_1, v_2) (w_1, w_2).$$

Moreover $(vw) \alpha_u + (wu) \alpha_v + (uv) \alpha_w = 0$;

whence $(vw)^2 \alpha_u^2 = (vu)^2 \alpha_v^2 + (uv)^2 \alpha_w^2 + 2(vu)(uv) \alpha_v \alpha_w$,

or

$$(vw)^2 z_{A, B, s} = (vu)^2 z_{C, D, t} + (uv)^2 z_{E, F, r} + 2(vu)(uv) \sqrt{\{z_{C, D, t} z_{E, F, r} + N(vw)^2\}}$$

and rationalising,

$$\begin{aligned} (vw)^4 z_{A, B, s}^2 + (vu)^4 z_{C, D, t}^2 + (uv)^4 z_{E, F, r}^2 - 2(wu)^2 (uv)^2 z_{C, D, t} z_{E, F, r} \\ - 2(uv)^2 (vw)^2 z_{E, F, r} z_{A, B, s} - 2(vw)^2 (vu)^2 z_{A, B, s} z_{C, D, t} \\ = 4(uv)^2 (vw)^2 (wu)^2 N. \end{aligned}$$

This relation can also be obtained as follows:—

The four quadratics $\alpha_u^2, \alpha_v^2, \alpha_w^2, \alpha_x^2$

are connected by the relation

$$\alpha_x^2 = \frac{(xv)(xw)}{(uv)(uw)} \alpha_u^2 + \frac{(xw)(xu)}{(vw)(vu)} \alpha_v^2 + \frac{(xu)(xv)}{(wu)(wv)} \alpha_w^2,$$

since the right-hand side is a homogeneous quadratic function of x_1, x_2 , and has the values

$$\alpha_u^2, \alpha_v^2, \alpha_w^2$$

for $x = u, v, w$ respectively; if we compare the coefficients of

$$x_1^2, x_1 x_2, x_2^2$$

on the two sides of the identity, we obtain the expressions of

$$M_{2m}, L_{2m+1}, M_{2m+1}$$

as linear functions of $\alpha_u^2, \alpha_v^2, \alpha_w^2$.

Moreover, equating the discriminants of the two quadratic functions of x_1, x_2 , we obtain the sought quadratic relation between

$$\alpha_u^2, \alpha_v^2, \alpha_w^2.$$

The expressions $(vw), (wu), (uv)$

involve arbitrary positive integers

$$A, B, C, D, E, F,$$

which can be determined so that

$$(vw)^2 = (wu)^2 = (uv)^2 = 1;$$

and then

$$z_{A, B, s}^2 + z_{C, D, t}^2 + z_{E, F, r}^2 - 2z_{C, D, t} z_{E, F, r} - 2z_{E, F, r} z_{A, B, s} - 2z_{A, B, s} z_{C, D, t} = 4N,$$

a representation of $4N$ as a ternary quadratic form.

Also

$$z_{C, D, t} z_{E, F, r} + N,$$

$$z_{E, F, r} z_{A, B, s} + N,$$

$$z_{A, B, s} z_{C, D, t} + N$$

are, each of them, perfect squares.

Now, it is clear that (vw) , (wu) , (uv) cannot all be of the same sign.

If we choose the constants so that

$$(vw) = -(vu) = -(uv) = +1,$$

$$z_{A, B, s} = z_{C, D, t} + z_{E, F, r} + 2\sqrt{(z_{C, D, t} z_{E, F, r} + N)},$$

$$z_{C, D, t} = z_{E, F, r} + z_{A, B, s} - 2\sqrt{(z_{E, F, r} z_{A, B, s} + N)},$$

$$z_{E, F, r} = z_{A, B, s} + z_{C, D, t} - 2\sqrt{(z_{A, B, s} z_{C, D, t} + N)}.$$

In every case two of the signs attached to the radical will be negative and one positive.

Mr. G. B. Mathews has pointed out to me that, if

$$a = (vw)^2 a_u^2, \quad b = (wu)^2 a_v^2, \quad c = (uv)^2 a_w^2,$$

$$f = (wu)(uv) a_o a_v, \quad g = (uv)(vw) a_m a_u, \quad h = (vw)(vu) a_u a_v,$$

the ternary quadratic form

$$(a, b, c, f, g, h)(x, y, z)^2 = \{ (vw) a_u x + (wu) a_v y + (uv) a_w z \}^2,$$

$$bc - f^2 = ca - g^2 = ab - h^2 = -(vw)^2 (wu)^2 (uv)^2 N,$$

$$gh - af = hf - bg = fg - ch = -(vw)^2 (wu)^2 (uv)^2 N,$$

$$a - b = b - g = c - h,$$

$$a - b - c - 2f = b - c - a - 2g = c - a - b - 2h = 0,$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

and

$$af + bg + ch = + 2N(vw)^2(wu)^2(uv)^2,$$

$$gh + hf + fg = - N(vw)^2(wu)^2(uv)^2.$$

4. The case $m = 3$.— $x^3 = Ny^3 + z$.

The general value of z is

$$(M_{2n}, a_{2n+1}M_{2n} - L_{2n}, L_{2n+1}^2, -M_{2n+1}\chi(u, u_s)^3;$$

and we have the relation

$$L_{2n}L_{2n+1} + N(p_{2n+1}q_{2n} + p_{2n}q_{2n+1}) = a_{2n+1}L_{2n+1}M_{2n} + M_{2n}M_{2n+1}.$$

Since $(p_{2n+1}q_{2n} + p_{2n}q_{2n+1})^2 = 1 + 4p_{2n}q_{2n}p_{2n+1}q_{2n+1}$

and $(p_{2n+1}q_{2n} + p_{2n}q_{2n+1}) - (p_{2n}q_{2n-1} + p_{2n-1}q_{2n}) = 2a_{2n+1}p_{2n}q_{2n},$

there is no difficulty in eliminating p and q and obtaining a relation between the quantities L , M , a , and N .

The following is a better method of procedure.

Equating the invariants of $x^3 - Ny^3$

and $(M_{2n}, a_{2n+1}M_{2n} - L_{2n}, L_{2n+1}, -M_{2n+1}\chi(u, u_s)^3,$

we obtain at once

$$M_{2n}^2M_{2n+1}^2 + 6M_{2n}M_{2n+1}L_{2n+1}(a_{2n+1}M_{2n} - L_{2n}) - 4(a_{2n+1}M_{2n} - L_{2n})^3M_{2n+1} \\ + 4M_{2n}L_{2n+1}^3 - 3(a_{2n+1}M_{2n} - L_{2n})^2L_{2n+1}^2 = N^2.$$

This invariant relation between

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, a_{2n+1}, \text{ and } N$$

is the fundamental property of the continued fraction to

$$N^{\frac{1}{2}}.$$

It may be written

$$M_{2n}^2M_{2n+1}^2 - 6M_{2n}M_{2n+1}L_{2n}L_{2n+1} + 4M_{2n+1}L_{2n}^3 + 4M_{2n}L_{2n+1}^3 - 3L_{2n}^2L_{2n+1} \\ + 6(M_{2n}M_{2n+1}L_{2n+1} - 2M_{2n+1}L_{2n}^2 + L_{2n}L_{2n+1}^2)M_{2n}a_{2n+1} \\ + 3(4M_{2n+1}L_{2n} - L_{2n+1}^2)M_{2n}^2a_{2n+1} \\ - 4M_{2n+1}M_{2n}^3a_{2n+1}^3 \\ = N^2.$$

From the Hessian of z , we obtain

$$\begin{aligned} & [M_{2n}L_{2n+1} - (a_{2n+1}M_{2n} - L_{2n})^2, \frac{1}{2} \{-M_{2n}M_{2n+1} - L_{2n+1}(a_{2n+1}M_{2n} - L_{2n})\}, \\ & \quad -M_{2n+1}(a_{2n+1}M_{2n} - L_{2n}) - L_{2n+1}^2] (u+u_s)^2 \\ & = -Nxy = -N(up_{2n} + u_s p_{2n+1})(uq_{2n} + u_s q_{2n+1}), \end{aligned}$$

yielding by comparison of coefficients

$$\begin{aligned} M_{2n}L_{2n+1} - (a_{2n+1}M_{2n} - L_{2n})^2 &= -Np_{2n}q_{2n}, \\ -M_{2n}M_{2n+1} - L_{2n+1}(a_{2n+1}M_{2n} - L_{2n}) &= -N(p_{2n}q_{2n+1} + p_{2n+1}q_{2n}), \\ -M_{2n+1}(a_{2n+1}M_{2n} - L_{2n}) - L_{2n+1}^2 &= -Np_{2n+1}q_{2n+1}. \end{aligned}$$

Of these the second relation has been previously found as a fundamental equation derivable from the continued fraction. Similarly from the cubic covariant

$$\begin{aligned} & -M_{2n}^2M_{2n+1} - 3M_{2n}(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} + 2(a_{2n+1}M_{2n} - L_{2n})^3 \\ & = -N(p_{2n}^3 + Nq_{2n}^3), \end{aligned}$$

and three other relations.

By the solution of a cubic equation it is obvious that z can be expressed in terms of

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, N.$$

Let a value of z be

$$(M_{2n}, K_{2n+1}, L_{2n+1}, -M_{2n+1}) \chi(u_1, u_2)^3 = \alpha_u^3,$$

where

$$K_{2n+1} = a_{2n+1}L_{2n} - M_{2n},$$

and let other values of z , depending upon the same consecutive pair of principal convergents

$$\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}},$$

be

$$\alpha_r^3, \alpha_v^3, \alpha_x^3, \alpha_y^3;$$

then we have the obvious identity

$$\alpha_u^3 = \frac{(yv)(yw)(yx)}{(uv)(uv)(ux)} \alpha_u^3 + \frac{(yw)(yx)(yu)}{(vv)(vx)(vu)} \alpha_v^3 + \frac{(yx)(yu)(yv)}{(vx)(vx)(vv)} \alpha_x^3 + \frac{(yu)(yv)(yw)}{(xu)(xv)(xv)} \alpha_y^3,$$

which is symmetrical in u, v, w, x, y , in that an identity remains whatever substitution be impressed upon these letters. Each side of this identity is a homogeneous cubic function of

$$y_1, y_2,$$

and, equating the discriminants, we obtain a homogeneous quartic function of

$$\alpha_u^3, \alpha_v^3, \alpha_w^3, \alpha_x^3$$

equal to a constant. This constant is a function of

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, a_{2n+1},$$

homogeneous and of degree 4, which, as shewn above, is equal to

$$N^2.$$

Hence, by giving special values to

$$u_1, u_2, v_1, v_2, w_1, w_2, x_1, x_2,$$

we get a representation of N^2 by means of a quaternary quartic form.

5. *The case $n = 4$.*— $x^4 = Ny^4 + z$.

We require an expression for

$$p_{2n}^2 p_{2n+1}^2 - N q_{2n}^2 q_{2n+1}^2 = X_{2n+1}.$$

Since

$$p_{2n+2} = a_{2n+2} p_{2n+1} + p_{2n},$$

we have

$$\begin{aligned} M_{2n+2} &= M_{2n} + 4(a_{2n+1} M_{2n} - L_{2n}) a_{2n+2} + 6X_{2n+1} a_{2n+2}^2 + 4L_{2n+1} a_{2n+2}^3 \\ &\quad - M_{2n+1} a_{2n+2}^4, \\ -L_{2n+2} &= a_{2n+1} M_{2n} - L_{2n} + 3X_{2n+1} a_{2n+2} + 3L_{2n+1} a_{2n+2}^2 - M_{2n+1} a_{2n+2}^3, \\ X_{2n+2} &= X_{2n+1} + 2L_{2n+1} a_{2n+2} - M_{2n+1} a_{2n+2}^2. \end{aligned}$$

The second of these relations enables the required expression.

From the first two relations, eliminating X_{2n+1} ,

$$\begin{aligned} &M_{2n+2} + 2L_{2n+2} a_{2n+2} \\ &= M_{2n} + 2(a_{2n+1} M_{2n} - L_{2n}) a_{2n+2} - 2L_{2n+1} a_{2n+2}^3 + M_{2n+1} a_{2n+2}^4. \end{aligned}$$

We now investigate the relations obtained from invariant theory. The expressions for z are

$$(up_{2n} + u_s p_{2n+1})^4 - N(uq_{2n} + u_s q_{2n+1})^4$$

and $(M_{2n}, a_{2n+1} M_{2n} - L_{2n}, X_{2n+1}, L_{2n+1}, -M_{2n+1} \chi(u, u_s)^4$

Equating the two invariants of the two forms

$$\begin{aligned} -M_{2n}M_{2n+1} - 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} + 3X_{2n+1}^2 &= -N, \\ -M_{2n}X_{2n+1}M_{2n+1} + 2(a_{2n+1}M_{2n} - L_{2n})X_{2n+1}L_{2n+1} - M_{2n}L_{2n+1}^2 \\ &+ (a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} - X_{2n+1}^3 = 0, \end{aligned}$$

whence

$$\begin{aligned} \{ -M_{2n}M_{2n+1} - 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1}, X_{2n+1} - 3M_{2n}M_{2n+1}X_{2n+1} \\ + 6(a_{2n+1}M_{2n} - L_{2n})L_{2n+1}X_{2n+1} - 3M_{2n}L_{2n+1}^2 \\ + 3(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} + NX_{2n+1} = 0; \end{aligned}$$

therefore

$$\begin{aligned} X_{2n+1} &= \frac{3M_{2n}L_{2n+1}^2 - 3(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1}}{-M_{2n}M_{2n+1} - 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - 3M_{2n}M_{2n+1} \\ &\quad + 6(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} + N} \\ &= 3 \frac{(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} - M_{2n}L_{2n+1}^2}{4M_{2n}M_{2n+1} - 2(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - N}, \end{aligned}$$

the required expression of X_{2n+1} .

Thence, from the invariant relations,

$$\begin{aligned} M_{2n}M_{2n+1} + 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - N \\ = 27 \left\{ \frac{(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} - M_{2n}L_{2n+1}^2}{4M_{2n}M_{2n+1} - 2(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - N} \right\}^2, \end{aligned}$$

the invariant relation between

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, a_{2n+1} \text{ and } N,$$

which is the fundamental property of the continued fraction to

$$N^{\frac{1}{3}}.$$

Let the value of z

$$(u_1p_{2n} + u_2p_{2n+1})^4 - N(u_1q_{2n} + u_2q_{2n+1})^4$$

be denoted by

$$\alpha_u^4;$$

then we have the identity

$$\begin{aligned} \alpha_z^4 &= \frac{(zv)(zw)(zx)(zy)}{(uv)(uw)(ux)(uy)} \alpha_u^4 + \frac{(zw)(zx)(zy)(zu)}{(vw)(vx)(vy)(vu)} \alpha_v^4 + \frac{(zx)(zy)(zu)(zv)}{(wx)(wy)(wu)(wv)} \alpha_w^4 \\ &\quad + \frac{(zy)(zu)(zv)(zw)}{(xy)(xu)(xv)(xw)} \alpha_x^4 + \frac{(zu)(zv)(zw)(zx)}{(yu)(yv)(yw)(yx)} \alpha_y^4; \end{aligned}$$

and, equating the invariants of degrees 2 and 3, we find

(i.) a homogeneous quadratic function of

$$a_u^4, a_v^4, a_w^4, a_x^4, a_y^4$$

equal to N ,

(ii.) a homogeneous cubic function of the same values of z equal to zero.

Results in integers are obtained by giving special values to

$$u_1, u_2, v_1, v_2, w_1, w_2, x_1, x_2, y_1, y_2$$

consistent with the definition of these quantities.

We obtain representation of N

by means of a quinary quadratic form.

In general the special properties of the continued fraction to

$$N^{1/m}$$

are obtainable from the invariant theory of the binary m -ic.