351. Some Interesting Formulae in Arithmetical Progression

Author(s): E. N. Barisien

Source: *The Mathematical Gazette*, Vol. 6, No. 94 (Oct., 1911), pp. 154-155

Published by: [The Mathematical Association](#)

Stable URL: <http://www.jstor.org/stable/3603256>

Accessed: 15/01/2015 04:52

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

<http://www.jstor.org>

Hence, if P be any point on the P.C.,

$$PA^2 = 2 \cdot AH \cdot PM;$$

and if Q be any point whatever,

$$QA^2 - \text{Power of } Q \text{ for P.C.} = 2 \cdot AH \cdot QN;$$

$$PM, QC \text{ being } \perp^{\text{rs}} \text{ on } B'C'.$$

2. I have shown elsewhere that, if Q be the centre of an in-conic, with foci S, S' , and axes $2p, 2q$,

$$\text{then } 4R \cdot QN = SA \cdot S'A.$$

$$\begin{aligned} \therefore 2 \cdot AH \cdot QN &= SA \cdot S'A \cdot \cos A \\ &= \frac{1}{2}(SA^2 + S'A^2 - 4p^2), \text{ by a well-known theorem,} \\ &= (Q \cdot A^2 - p^2 + q^2). \end{aligned}$$

$$\therefore \text{Power of } Q \text{ for P.C.} = p^2 + q^2,$$

or the P.C. and Director Circle cut orthogonally.

[My proof (analytical) that $4R \cdot QN = SA \cdot S'A$ is somewhat long and cumbrous. I should be glad to meet with a short proof by Pure Geometry.]

W. GALLATLY.

350. [Vol. v. p. 396.] I think that Mr. Milne has made a slip in stating that Apollonius' "Inclinationes" was restored by Anderson in 1612. In *The Ladies' Diary*, 1840, Appendix, pp. 49-61, is given an "Abstract of the Principal Geometrical Writings of Alexander Anderson, of Aberdeen." The title of the first is given as follows:

Alexandri Andersoni, Aberdonensis Supplementum, Apollonii Redivivi (of Ghetaldus). *Sive Analysis Problematis hactenus desiderati ad Apollonii Pergaei doctrinam περι νέσεων*, à Marino Ghetaldo Patritio Ragusino hujusque, non ita pridem restitutum. In qua exhibetur mechanice aequalitatum tertii gradus sive solidarum, in quibus magnitudo omnino data, aequatur homogeneae sub altero tantum coefficiente ignoto. Huic subnexa est variorum problematum practice eodem authore. Paris, 1612. In 4to. Pp. 36.

In this book there is no treatment of the problem of inclinations.

R. C. ARCHIBALD.

Dr. Archibald is quite correct. I was misled by a reference in Hutton's *Mathematical Dictionary* (1796), vol. i., under the heading Apollonius, where he says, "Alex. Anderson, Supplem. Apol. Redivivi. Incl. Paris, 1612, 4to." Not having the book, I was unable to observe the golden rule: "Verify your references."

J. J. MILNE.

[Barlow's *New Mathematical and Philosophical Dictionary* (1814) gives the identical reference, with the exception of "Red." for "Redivivi." W. J. G.]

351. [A. 1. a.] *Some Interesting Formulae in Arithmetical Progression.*

With the usual notation, $S_1 = \frac{n}{2}(2a + n - 1)b$.

Let $M = \frac{a+b}{2}$, and let S_2, S_3 be the sums of the squares and cubes respectively of the terms of S_1 .

$$\text{Then } S_1 = n(a-b) + \frac{n(n+1)}{2}b,$$

$$S_2 = n(a-b)^2 + n(n+1)b(a-b) + \frac{n(n+1)(2n+1)}{6}b^2,$$

$$S_3 = \left[n(a-b) + \frac{n(n+1)}{2}b \right] \left[(a-b)^2 + (n+1)b(a-b) + \frac{n(n+1)}{2}b^2 \right]$$

$$= S_1 [bS_1 + a(a-b)],$$

$$\text{and } = [bn^2 + (2a-b)n][(bn^2 + 2a-bn)b + 2a(a-b)]/4.$$

$$S_1 = nM; \quad S_2 = n \left[M^2 + \frac{b^2(n-1)}{2} \right]; \quad S_3 = nM \left[M^2 + \frac{b^2(n^2-1)}{4} \right].$$

Hence the well-known relations :

$$n^2 S_3 - 3n S_1 S_2 + S_1^3 = 0, \dots\dots\dots(1)$$

$$n S_2 - S_1^2 = b^2 n^2 (n^2 - 1) / 12, \dots\dots\dots(2)$$

$$b = 2\sqrt{3} (n S_2 - S_1^2) / (n\sqrt{n^2 - 1}), \dots\dots\dots(3)$$

$$a = S_1 / n - b(n-1)/2. \dots\dots\dots(4)$$

From (1) we see that if the progression is in positive integers, we have $9S_2^2 - 8S_1 S_3$ is always a perfect square.

These formulae are useful in the solution of otherwise not very easy problems :

(1) Find an A.P., given S_1, S_2, S_3 .

Here (1) gives n ; (3) gives b when n is known ; and a follows by (4).

(2) Find an A.P., given S_1, S_2, n ; S_1, S_3, n ; or S_2, S_3, n . E. N. BARISIEN.

352. [v. 1. a.] We all owe Dr. Nunn a great debt for the fascinating article on Wallis and his marvellously ingenious evaluation of π . The thinking of $\sqrt{r^2 - a^2}$ as $[\frac{1}{2}\sqrt{r} - \frac{1}{2}\sqrt{a}]^{\frac{1}{2}}$ is in itself very wonderful ; and then to get the unknown σ by the artifice of imprisoning a long multiple of σ between two known sets of factors, and so getting limits as close as we like between which σ must lie, merely from consideration of a table, was an inspiration.

I notice that Wallis' table was formed apparently by clever guess work. But, as a matter of fact, it is easy to see by induction from any special case that the characteristic fraction for $[\frac{1}{2}\sqrt{r} - \frac{1}{2}\sqrt{a}]^p$ is $\frac{R|P}{R+P}$; (I append the sort of work involved—quite elementary and simple, and rather pretty). Hence the figures in the table are $\frac{R+P}{R|P}$. These are easily evaluated unless *both* R and P are fractional ; in that case we get σ and its multiples, σ being $\frac{1}{\frac{1}{2}|\frac{1}{2}}$. So that the whole investigation is closely connected with the interesting extension of the factorial notation to fractions with $\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$.

We see at once that the multipliers in each row are the successive values of $\frac{R+P}{P}$, or, to avoid complex fractions, $\frac{2R+2P}{2P}$.

Thus, if $R = \frac{1}{2}$, $P = \frac{1}{2}$ we get σ , and for $P = 1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, etc. we get $\sigma \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times \dots$, and for $R = \frac{3}{2}$, $P = 0$ we get 1, and for $P = 1, 2, 3, \dots$ we get $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \dots$. In fact, we get all Dr. Nunn's multipliers, *with the reasons* for them.

This investigation seems to explain why Euler's first integral should have been one with a solution of the factorial type like $\frac{R+P}{R|P}$, and his second integral the Γ function itself, if he was led by Wallis' researches to hunt for a general method of evaluating fractional factorials.

$$\left(\frac{1}{r^p} - \alpha^{\frac{1}{p}}\right)^4 = r^{\frac{4}{p}} - 4r^{\frac{3}{p}} \cdot \alpha^{\frac{1}{p}} + 6r^{\frac{2}{p}} \cdot \alpha^{\frac{2}{p}} - 4r^{\frac{1}{p}} \cdot \alpha^{\frac{3}{p}} + \alpha^{\frac{4}{p}}.$$

$$\text{Characteristic fraction} = 1 - \frac{4}{1 + \frac{1}{p}} + \frac{6}{1 + \frac{2}{p}} - \frac{4}{1 + \frac{3}{p}} + \frac{1}{1 + \frac{4}{p}}$$