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XXXII.—*On the Evaluation of Certain Combinations of the Ber, Bei and Allied Functions.* By S. BUTTERWORTH, M.Sc., Assistant Lecturer in Physics, School of Technology, Manchester.

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1. It has been pointed out by Russell* that, in most formulæ, the ber and bei functions nearly always occur in one or other of the following combinations :—

$$X(x) = \text{ber}^2 x + \text{bei}^2 x,$$

$$V(x) = \text{ber}'^2 x + \text{bei}'^2 x,$$

$$Z(x) = \text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x,$$

$$W(x) = \text{ber } x \text{ bei}' x - \text{bei } x \text{ ber}' x,$$

while in most cases only the combinations Z/X , W/X , Z/V , W/V occur.

Russell has given formulæ for evaluating these combinations, suitable either for high or low values of the argument. These formulæ have been extended by Savidge† and he has published a four-figure table of the functions. A five-figure table of the combinations Z/V and W/V for smaller intervals of the argument has also been prepared by Rosa and Grover.‡

In the present Paper formulæ are established for functions φ_n and ψ_n defined by the relation

$$\varphi_n + i\psi_n = \frac{J_{n+1}(\sqrt{-ix})}{J_{n-1}(\sqrt{-ix})}.$$

From the definition of the ber and bei functions it may readily be shown that

$$\left. \begin{aligned} W/X &= x(1 + \varphi_1)/2, & W/V &= 2/x - x\psi_2/4 \\ Z/X &= -x\psi_1/2, & Z/V &= x(1 + \varphi_2)/4 \end{aligned} \right\}, \quad \dots \quad (1)$$

so that Russell's formulæ are particular cases of the formulæ of this Paper.

The functions φ_n and ψ_n also occur in the problem of a spherical conductor in an alternating magnetic field, n in this case being of the form $(2m+1)/2$, where m is an integer.

* Russell, "Phil. Mag.," 6, 17, p. 524, 1909.

† Savidge, "Phil. Mag.," 6, 19, p. 49, 1910.

‡ Rosa and Grover, "Bulletin" Bureau of Standards, p. 226, Jan., 1912.

2. *Formulae for Low Values of x .*—By writing down the series for $J_+(x)$ and $J_{n-1}(x)$, and dividing, afterwards putting $\sqrt{-ix}$ for x , we get

$$\left. \begin{aligned} \varphi_n &= -\frac{2x^4}{(2n)^2(2n+2)(2n+4)} + \frac{(28n-48)x^8}{(2n)^4(2n+2)^2(2n+4)(2n+6)(2n+8)} \\ \psi_n &= -\frac{x^2}{2n(2n+2)} + \frac{(10n+12)x^6}{(2n)^3(2n+2)^2(2n+4)(2n+6)} \end{aligned} \right\} \text{(A)}$$

These formulae are useful only for very small values of x .

3. *Formulae for High Values of x .*—By making use of the relation

$$J_n(ix) = i^n I_n(x),$$

and using the semi-convergent expansion

$$I_n(x) = \frac{1}{\sqrt{2\pi x}} e_x \left\{ 1 - \frac{4n^2-1}{8x} + \frac{(4n^2-1)(4n^2-9)}{2(8x)^2} - \dots \right\},$$

we get

$$\left. \begin{aligned} \varphi_n &= -1 + \frac{2n}{\sqrt{2} \cdot x} - \frac{2n(2n-1)(2n-3)}{8\sqrt{2}x^3} \\ &\quad - \frac{2n(2n-1)(2n-3)}{8x^4} + \frac{2n(2n-1)(4n^2-9)(2n-7)}{\sqrt{2} \cdot 128x^5} \\ \psi_n &= -\frac{2n}{\sqrt{2}x} + \frac{2n(2n-1)}{2x^2} - \frac{2n(2n-1)(2n-3)}{8\sqrt{2}x^3} \\ &\quad - \frac{2n(2n-1)(4n^2-9)(2n-7)}{\sqrt{2} \cdot 128x^5} \end{aligned} \right\} \text{(B)}$$

Putting $n=1$ or 2 , and making use of relation (1), Russell's formulae for W/X , W/V , Z/X , Z/V are readily obtained.

4. *Formulae for Intermediate Values of x .*—It is found on application that formulae (A) and (B) fail for a considerable range of the argument.

If, however, the roots of $J_{n-1}(x)=0$ are known, this gap may be filled up.

Let $\alpha_1, \alpha_2, \alpha_3 \dots$ be the positive roots of $J_{n-1}(x)=0$.

Then we may write

$$J_{n-1}(x) = a_n(x^2 - \alpha_1^2)(x^2 - \alpha_2^2)(x^2 - \alpha_3^2) \dots$$

Since the series for $J_{n+1}(x)$ is more convergent than that for

$J_{n-1}(x)$, $J_{n+1}(x)/J_{n-1}(x)$ may be separated into partial fractions by the usual method, and we find

$$\frac{J_{n+1}}{J_{n-1}} = 4nx^2 \sum_{r=1}^{\infty} \frac{1}{\alpha_r^2(\alpha_r^2 - x^2)} \quad \dots \quad (2)$$

Putting $\sqrt{-ix}$ for x ,

$$\left. \begin{aligned} \phi_n &= -4nx^4 \sum_{r=1}^{\infty} \frac{1}{\alpha_r^2(\alpha_r^4 + x^4)} \\ \psi_n &= -4nx^2 \sum_{r=1}^{\infty} \frac{1}{\alpha_r^4 + x^4} \end{aligned} \right\} \dots \quad (3)$$

These formulæ are always convergent, but the convergence is very slow.

However, by expanding the later terms in powers of x , practical formulæ may be obtained.

For values of x up to $x = \alpha_2$, I find the most convenient form to use is

$$\left. \begin{aligned} \phi_n &= -4nx^4 \left\{ \frac{1}{\alpha_1^2(\alpha_1^4 + x^4)} + \frac{1}{\alpha_2^4(\alpha_2^4 + x^4)} + C_3 \right. \\ &\quad \left. - C_5 x^4 + C_7 x^8 - \dots \right\} \\ \psi_n &= -4nx^2 \left\{ \frac{1}{\alpha_1^4 + x^4} + \frac{1}{\alpha_2^4 + x^4} + C_2 - C_4 x^4 + C_6 x^8 - \dots \right\} \end{aligned} \right\}, \quad (C)$$

where

$$C_s = \sum_{r=3}^{\infty} \frac{1}{\alpha_r^2},$$

When $x > \alpha_2$ formulæ (B) are suitable.

The coefficients C_s are easily determined by summation, except in the case of C_2 . However, by comparing the coefficients of x^2 in formulæ (A) and (C) we find

$$C_2 + \frac{1}{\alpha_1^4} + \frac{1}{\alpha_2^4} = \frac{1}{8n^2(2n+2)},$$

and this relation is sufficiently accurate to determine C_2 .

5. *Numerical Values from Formulæ (C).*—The zeroes of $J_0(x)$ and $J_1(x)$ are given to a high degree of accuracy in Gray and Mathews' Bessel Function (pp. 244 and 280). From these values I obtain the values of the coefficients given in Table I. In Table II. are given the values of W/X , Z/X (α) as calculated by formulæ (C), using relation (1), (b) as calculated by Savidge.

The values of the argument chosen are those for which the calculation by the usual formulæ is most tedious.

In Table III. are the corresponding values of W/V and Z/V as compared with Rosa and Grover's values.

Numerical values from formulæ (B) are unnecessary, as these reduce to Russell's formulæ when $n=1$ or 2.

TABLE I.

s .	C_s (for $n=1$).	C_s (for $n=2$).
2	2.7338×10^{-4}	1.56456×10^{-4}
3	2.8933×10^{-6}	1.16285×10^{-6}
4	3.5003×10^{-8}	9.9744×10^{-9}
5	4.4599×10^{-10}	9.0709×10^{-11}
6	5.8173×10^{-12}	8.4836×10^{-13}
7	7.6745×10^{-14}	8.0504×10^{-15}
8	1.0183×10^{-15}	7.7129×10^{-1}
9	1.3552×10^{-1}	7.3960×10^{-19}
10	1.8065×10^{-19}	7.1222×10^{-21}
a_1	2.4048256	3.8317060
a_2	5.5200781	7.0155867

TABLE II.

x	W/X .		Z/X .	
	Savidge.	Formula (C).	Savidge.	Formula (C).
3	0.7485	0.748531	0.5399	0.539948
4	0.7141	0.714078	0.5842	0.584194
5	0.7101	0.710145	0.6040	0.603998
6	0.7101	0.710139	0.6211	0.621090

TABLE III.

x	W/V .		Z/V .	
	Rosa and Grover.	Formula (C).	Rosa and Grover.	Formula (C).
5	0.81709	0.817090	0.69496	0.694960
6	0.79786	0.797862	0.69781	0.697809
7	0.78377	0.783768	0.70037	0.700372
8	0.77361	0.773610	0.70214	0.702130

ABSTRACT.

Formulæ are obtained for the real and imaginary parts of J_{n+1}/J_{n-1} , when the argument is of the form $\sqrt{-1}x$. These formulæ are useful for evaluating the functions W/X , Z/X , W/V , Z/V defined by Russell, and have other applications.

The formulæ are of two types :—

(a) Those suitable for low or moderate values of the argument. These are particularly useful in those cases where computation by other methods is most laborious.

(b) Those suitable for high values of the argument. Russell's formulæ are particular cases of these formulæ.