On the Condition that, in the Transformation of any Figure by Curvilinear Coordinates in Three Dimensions, every Angle in the new Figure shall be equal to the corresponding Angle in the original Figure. By J. CLERK MAXWELL, F.R.S.

[Read May 9th, 1872.]

In the corresponding problem in two dimensions, the only condition is

$$x + \sqrt{-1}y = f(\xi + \sqrt{-1}\eta) \dots\dots\dots (1),$$

where x, y are the coordinates in one system, and ξ, η in the other.

In three dimensions the solution is more restricted.

Let x, y, z be functions of ξ, η, ζ ; then the point in the system x, y, z , for which ξ is constant, will be in a certain surface, and by giving ξ a series of values we obtain a series of such surfaces. There will be two other series of surfaces, corresponding to η and ζ respectively. If in the second system ξ, η, ζ are rectangular coordinates, the surfaces corresponding to ξ, η, ζ in the first system will intersect at right angles.

The condition of this is $\frac{dx}{d\eta} \frac{dx}{d\zeta} + \frac{dy}{d\eta} \frac{dy}{d\zeta} + \frac{dz}{d\eta} \frac{dz}{d\zeta} = 0 \dots\dots\dots (2),$

with two other equations in ζ and ξ , and in ξ and η .

$$\left. \begin{aligned} \text{If we now write } \alpha^2 &= \left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2 + \left(\frac{dz}{d\xi}\right)^2 \\ \beta^2 &= \left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2 + \left(\frac{dz}{d\eta}\right)^2 \\ \gamma^2 &= \left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2 + \left(\frac{dz}{d\zeta}\right)^2 \end{aligned} \right\} \dots\dots\dots (3);$$

and $d\xi$ is the intercept of the line ($\eta=\text{const.}, \zeta=\text{const.}$) cut off between the surfaces ξ and $\xi+d\xi$.

The angle e , at which a line whose coordinates are functions of p cuts a line whose coordinates are functions of q , is found from the equation

$$\begin{aligned} & \frac{dx}{dp} \frac{dx}{dq} + \frac{dy}{dp} \frac{dy}{dq} + \frac{dz}{dp} \frac{dz}{dq} \\ &= \cos e \left\{ \left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 \right\}^{\frac{1}{2}} \left\{ \left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2 \right\}^{\frac{1}{2}} \\ & \dots\dots\dots (4). \end{aligned}$$

Expressing this in terms of ξ, η , and ζ , it becomes

$$\begin{aligned} & \alpha^2 \frac{d\xi}{dp} \frac{d\xi}{dq} + \beta^2 \frac{d\eta}{dp} \frac{d\eta}{dq} + \gamma^2 \frac{d\zeta}{dp} \frac{d\zeta}{dq} \\ &= \cos e \left\{ \alpha^2 \left(\frac{d\xi}{dp}\right)^2 + \beta^2 \left(\frac{d\eta}{dp}\right)^2 + \gamma^2 \left(\frac{d\zeta}{dp}\right)^2 \right\}^{\frac{1}{2}} \\ & \quad \times \left\{ \alpha^2 \left(\frac{d\xi}{dq}\right)^2 + \beta^2 \left(\frac{d\eta}{dq}\right)^2 + \gamma^2 \left(\frac{d\zeta}{dq}\right)^2 \right\}^{\frac{1}{2}} \dots\dots\dots (5). \end{aligned}$$

In order that the angle e , at which these lines intersect in the system (x, y, z) , should be always equal to the angle ϵ , at which the corresponding lines in the system (ξ, η, ζ) intersect, it is necessary and sufficient that

$$\alpha = \beta = \gamma \dots \dots \dots (6),$$

for these are the conditions that equations (4) and (5) should be of the same form.

Now consider the quadrilateral ABCD, cut off from the surface $(\zeta = \text{const.})$ by the surfaces ξ , $\xi + d\xi$, η , and $\eta + d\eta$.

$$AB = \alpha d\xi,$$

$$AD = \beta d\eta,$$

$$CD = \left(\alpha + \frac{d\alpha}{d\eta} d\eta \right) d\xi,$$

$$BC = \left(\beta + \frac{d\beta}{d\xi} d\xi \right) d\eta.$$

Since the three sets of surfaces are orthogonal, their intersections are lines of curvature, by Dupin's Theorem. Hence AB is a line of curvature of the surface η , drawn parallel to α . Hence the normals at A and B will intersect at some point O. Let us call OA, the radius of curvature of η in the plane of α , $R_{\eta\alpha}$, then

$$AB : DC :: R_{\eta\alpha} : R_{\eta\alpha} + AD.$$

Hence $R_{\eta\alpha} = \frac{\alpha\beta}{\frac{d\alpha}{d\eta}}$; similarly $R_{\eta\gamma} = \frac{\beta\gamma}{\frac{d\gamma}{d\eta}}$.

These values of the radii of curvature are true for any system of orthogonal surfaces, but since in this case $\alpha = \gamma$, we find

$$R_{\eta\alpha} = R_{\eta\gamma};$$

or the principal radii of curvature of the surface η are equal to each other at every point of the surface. Hence the surfaces η must be spherical.

Since $\alpha = \beta = \gamma$, the other families of surfaces ξ and ζ must also be spherical.

Now take one of the points where three spherical surfaces meet at right angles for the origin, and invert the system. These spheres when inverted become three planes intersecting at right angles, and the other spheres become spheres, each intersecting at right angles two of these planes. Hence their centres lie in the lines of intersection of the planes. Let a, b, c be the distances, from the point of intersection of the three planes, of the centres of spheres belonging each to one of the three systems, and let their radii be p, q, r respectively. Then the conditions that these spheres intersect at right angles are $q^2 + r^2 = b^2 + c^2$, $r^2 + p^2 = c^2 + a^2$, $p^2 + q^2 = a^2 + b^2$,

whence

$$p^2 = a^2, \quad q^2 = b^2, \quad r^2 = c^2;$$

or all the spheres pass through the origin, and each system has a common tangent plane. Hence if we take these three planes as coordinate planes, and write

$$x^2 + y^2 + z^2 = r^2, \\ \xi^2 + \eta^2 + \zeta^2 = \rho^2;$$

we have $r\rho = R^2$, a constant quantity,

$$\frac{x}{\xi} = \frac{y}{\eta} = \frac{z}{\zeta};$$

whence

$$x = \xi \frac{R^2}{\rho^2}, \quad y = \eta \frac{R^2}{\rho^2}, \quad z = \zeta \frac{R^2}{\rho^2};$$

$$\xi = x \frac{R^2}{r^2}, \quad \eta = y \frac{R^2}{r^2}, \quad \zeta = z \frac{R^2}{r^2};$$

$$\alpha = \beta = \gamma = \frac{R^2}{\rho^2} = \frac{r^2}{R^2} = \frac{r}{\rho}.$$

It appears from this that the only method of transforming a figure of three dimensions, so that every angle in the system shall remain of the same magnitude, is the method of inversion or of reciprocal radii vectores.

[Reference may be made to "Méthodes de Transformation en Géométrie et en Physique mathématique," par J. N. Haton de Goupillière, especially § vii.; "*Journal de l'École Polytechnique*," tom. xxv., p. 188.]

June 13th, 1872.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Sir Charles Wheatstone, F.R.S., was amongst the visitors present.

Mr. W. Paice, M.A., was admitted into the Society; and the Rev. C. Taylor, M.A., and Mr. R. W. Genese, B.A., were ballotted for in the usual manner, and elected members.

Prof. Cayley, V.P., gave an account of his paper "On the Surfaces divisible into Squares by their Curves of Curvature."

Sir W. Thomson, Mr. Merrifield, and Prof. Clifford put a few questions to the Author upon points suggested by his paper.

Mr. S. Roberts, V.P., gave some illustrative details of his communication "On Prof. Cremona's Transformation between Two Planes, and Tables relating thereto."

Dr. Hirst briefly described a few results he had obtained bearing upon "a Manifold Correspondence of Two Plaues." Prof. Clifford made a few remarks on Dr. Hirst's communication.

Sir W. Thomson then explained the object of his short paper on "the Simultaneous Reduction of Two Polynomial Quadratics to Sums of Squares." The Author cited particularly Cayley, "Cambridge and Dublin Mathematical Journal," Feb. 1849, and "Quarterly Mathematical Journal," 1858; and the celebrated problem of Cauchy's, treated in vol. iv. of his "Exercices," under the title "Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des planètes." The Hon. J. W. Strutt referred the Author to a memoir bearing on the subject of his paper.

On the Surfaces divisible into Squares by their Curves of Curvature.

By A. CAYLEY, F.R.S.

[Read 13th June, 1872.]

Prof. Cayley gave an account of an investigation recently communicated by him to the Academy of Sciences at Paris. The fundamental theorem is that, if the coordinates x, y, z of a point on a surface are expressed as functions of two parameters p, q (such expressions, of course, replacing the equation of the surface); and if these parameters are such that $p = \text{const.}$, $q = \text{const.}$ are the equations of the two sets of curves of curvature respectively; then (writing for shortness $\frac{dx}{dp} = x_1$, $\frac{dx}{dq} = x_2$, $\frac{d^2x}{dp^2} = x_3$, $\frac{d^2x}{dpdq} = x_4$, $\frac{d^2x}{dq^2} = x_5$, and the like for y, z), the coordinates x, y, z , considered always as functions of p, q , satisfy the equations

$$x_1x_2 + y_1y_2 + z_1z_2 = 0,$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0.$$

The last equation is equivalent to

$$x_4 + Ax_1 + Bx_2 = 0,$$

$$y_4 + Ay_1 + By_2 = 0,$$

$$z_4 + Az_1 + Bz_2 = 0;$$

and if in the notation of Gauss we write

$$x_1^2 + y_1^2 + z_1^2 = E,$$

$$x_2^2 + y_2^2 + z_2^2 = G,$$

then adding the equations multiplied by x_1, y_1, z_1 respectively, and also adding the equations multiplied by x_2, y_2, z_2 respectively, we find

$$A = -\frac{1}{2} \frac{1}{E} \frac{dE}{dq}, \quad B = -\frac{1}{2} \frac{1}{G} \frac{dG}{dq}$$