On the Flexure of an Elastic Plate. By HORACE LAMB, M.A., F.R.S.

[*Read Dec.* 12*th*, 1889.]

The following paper, which was written out in substance more than a year ago, is an attempt to put in as simple and straightforward a form as possible the theory of the flexure of a plane elastic plate of uniform isotropic substance, with comments and illustrations designed to elucidate points of difficulty. I should hardly venture to submit it to the Mathematical Society, except for a certain novelty in the treatment of the boundary conditions, where attention is drawn to a point of some importance not hitherto, I believe, noticed. It is hoped that the method here given may help to remove some of the obscurity which, if we may judge from some recent writings, is still held to attach to this part of the subject.

In a future paper I hope to discuss the case of a curved plate or shell, with special reference to the points in controversy between Mr. A. E. H. Love and Lord Rayleigh.*

1. A few remarks on the nature of the problem may be in place at the outset. It is one of a large class in mathematical physics, where the difficulties of an exact solution are evaded by means of special assumptions based either on observation or on a sort of induction from the known results in such particular cases as admit of rigorous treatment. As instances, we may cite the flexure of a stretched membrane, the theory of "long" waves in canals, the conduction of electricity in a wire of varying section, and so on. In many of these questions, it is true, the auxiliary assumptions required are of so obvious a character that no dispute about them is possible, and it is therefore not always thought necessary to state them expressly. The present problem is, however, by far the most difficult of its class, and, although it has been the subject of many elaborate investigations, there cannot be said to be at present any general agreement as to what is the proper foundation for the theory. Without entering into a detailed criticism of the investigations referred to, we may remark that many of them,† at least as ordinarily presented, appear to set undue

---

* See *Phil. Trans.*, 1888 (A), pp. 491–546, and *Proc. Roy. Soc.*, Dec. 13, 1888, p. 105.

† More especially those which start from the general equations of elasticity.

limits to the scope of the theory by restrictions as to the continuity of the distribution of impressed force, and the magnitude of its rate of variation from point to point of the substance. For example, it can make no sensible difference to the general form assumed by the plate whether a given force be applied as a pressure on one face or as a tension on the other, or whether it be a bodily force acting on a small volume of the substance—whether, again, it be diffused and continuous, or concentrated and discontinuous, although the distribution of strain in the immediate neighbourhood may be very different in the several cases.

It appears most natural to regard as the object of the theory the determination of the general form of the middle surface, ignoring such minute features as may arise from the particular mode of application of the external forces, in the manner indicated. With this view we shall understand by the "deflection" at any point of the middle surface the mean deflection over an area including this point, whose dimensions are of the same order as the thickness of the plate. By an "element" of the plate, we shall understand the portion cut off by a cylindrical surface drawn through the contour of such an area normal to the middle surface. In like manner, by an element of a normal section we shall understand the portion corresponding to a linear element on the middle surface, of length comparable with the thickness.

This being premised, it will appear that a complete theory can be based on the following principle:

If the external forces on each element of the plate are by themselves in equilibrium, the deflection will be everywhere zero, and the stresses across any element of a normal section will be in equilibrium.

By the method of superposition of stresses and strains, this is equivalent to the following:

The external forces on an element of the plate are sufficiently represented by their force- and couple-resultants; *i.e.*, we need not attend to the particular way in which the forces are distributed within the element. The same applies to the stress across an element of a normal section, and to the applied force on an element of the boundary.

2. It is necessary, as various writers have insisted, to distinguish between the state of things in the main body of the plate and that which obtains near the edges. As regards the former, it is easy to show, on the above principle, that the essential features of the deformation can be represented by the superposition of two particular

states of strain, to be described. The plan of the following investigation is, first, to ascertain the distribution of applied force which would accurately maintain the resulting strains; and then to show that, if the functions at our disposal are properly chosen, the deflection will not sensibly alter when for the distribution in question we substitute that which actually holds.

The first of the two states of strain referred to is that assumed by Kirchhoff as the basis of his first memoir on the subject, and afterwards employed by Thomson and Tait. It is defined by the properties (i.) that the middle surface is unextended, (ii.) that the particles originally in a line normal to this surface still form a straight line normal to it in the strained condition, and (iii.) that there is no traction across planes parallel to the middle surface.

Taking, as usual, the axes of x, y in the middle surface, in the unstrained condition, let w denote the deflection at any point (x, y) of this surface. In the state of strain above defined, the coordinates of any point of the substance are changed from (x, y, z) to (x', y', z'), where

$$x' = x - \frac{dw}{dx} z, \quad y' = y - \frac{dw}{dy} z,$$

so that the projections of a linear element $(dx, dy, 0)$ become

$$dx' = \left(1 - \frac{d^2w}{dx^2} z\right) dx - \frac{d^2w}{dx\,dy} z \,.\, dy,$$

$$dy' = -\frac{d^2w}{dx\,dy} z \,.\, dx + \left(1 - \frac{d^2w}{dy^2} z\right) dy.$$

We have here neglected the square of the angle which the normal to the middle surface makes with its original direction. Hence, for the elongations parallel to x, y, we have

$$e_{xx} = -\frac{d^2w}{dx^2} z, \quad e_{yy} = -\frac{d^2w}{dy^2} z,$$

while the shear in the plane xy is

$$e_{xy} = -2 \frac{d^2w}{dx\,dy} z.$$

The elongation e_{zz} parallel to z is to be found from the condition that there is no traction across any plane parallel to the middle surface, that is

$$0 = p_{zz} = (\lambda + 2\mu)\, e_{zz} + \lambda\, (e_{xx} + e_{yy}),$$

the notation for the elastic constants being that introduced by Lamé, and now generally accepted by continental writers.* Substituting the value of e_{zz}, hence obtained, in the expressions for the remaining component tractions, we find

$$p_{xx} = \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}(e_{xx}+\sigma e_{yy}),$$

$$p_{yy} = \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}(e_{yy}+\sigma e_{xx}),$$

$$p_{xy} = \mu e_{xy},$$

$$p_{zx} = p_{yz} = 0,$$

where $\sigma, = \lambda/2(\lambda+\mu)$, is Poisson's ratio.

The applied forces necessary to maintain this state of strain are, per unit volume,

$$(X) = -\left(\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}\right)$$

$$= \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}\,.\,z\frac{d}{dx}\nabla^2 w,$$

$$(Y) = \frac{4(\lambda+\mu)\mu}{\lambda+2\mu}\,.\,z\frac{d}{dy}\nabla^2 w,$$

$$(Z) = 0,$$

where $$\nabla^2 = d^2/dx^2+d^2/dy^2.$$

Replacing these by their force- and couple-resultants $\overline{X}$, $\overline{Y}$, $\overline{Z}$ and

---

* A few words on this difficult question of notation. As regards the elastic constants, a special symbol for the rigidity is in any case required, whilst it is comparatively unimportant in what manner the second elastic constant is defined. So many combinations of the two constants are needed—*e.g.*, the cubical compressibility, Young's modulus, Poisson's ratio, &c.—that it is impossible to devise a notation which shall give simple expressions to them all. Lamé's notation is convenient, and has found wider acceptance than any other. That of Thomson and Tait is in itself very neat, but is open to the objection that the letters m, n are so often wanted in other senses.

With respect to strains and stresses, some form of double suffix notation appears to the writer to be for many purposes almost indispensable. Such forms have at all events the advantage that they define themselves, and so give no trouble to readers who are accustomed to other notations. F. Neumann's symbols for the component stresses (X_x, X_y, X_z, &c.) are very expressive, but the corresponding notation devised by Kirchhoff for the strains (x_x, x_y, x_z, &c.) seems far less happy.

$\overline{L}$, $\overline{M}$, $\overline{N}$, reckoned per unit area of the plate, we find

$$\overline{X} = \int_{-h}^{h} (X)\, dz = 0, \quad \overline{Y} = 0, \quad \overline{Z} = 0,$$

$$\overline{L} = -\int_{-h}^{h} z\,(Y)\, dz = -A \frac{d}{dy} \nabla^2 w,$$

$$\overline{M} = \int_{-h}^{h} z\,(X)\, dz = A \frac{d}{dx} \nabla^2 w,$$

$$\overline{N} = 0,$$

where $2h$ is the thickness, and

$$A = \frac{8}{3} \frac{(\lambda+\mu)\,\mu}{\lambda+2\mu} h^3.$$

Next, for the stresses across normal sections of the plate. The stresses across a section perpendicular to x reduce to a couple whose components are, per unit length,

$$G_{xx} = -\int_{-h}^{h} p_{xy} z\, dz = (1-\sigma) A \frac{d^2 w}{dx\, dy}, \quad \text{about } Ox,$$

and
$$G_{xy} = \int_{-h}^{h} p_{xx} z\, dz = -A \left(\frac{d^2 w}{dx^2} + \sigma \frac{d^2 w}{dy^2}\right), \quad \text{about } Oy.$$

In like manner the stresses across a section perpendicular to y reduce to a couple whose components are

$$G_{yx} = -\int_{-h}^{h} p_{yy} z\, dz = A \left(\frac{d^2 w}{dy^2} + \sigma \frac{d^2 w}{dx^2}\right), \quad \text{about } Ox,$$

and
$$G_{yy} = \int_{-h}^{h} p_{xy} z\, dz = -(1-\sigma) A \frac{d^2 w}{dx\, dy}, \quad \text{about } Oy.*$$

We notice that
$$\frac{dG_{xx}}{dx} + \frac{dG_{yx}}{dy} = -\overline{L},$$

$$\frac{dG_{xy}}{dx} + \frac{dG_{yy}}{dy} = -\overline{M},$$

as evidently ought to be the case, since the stresses on the boundary of any portion of an elastic solid must balance the applied forces on

* In the notation of Thomson and Tait,

$$G_{xx} = -G_{yy} = \Pi,$$
$$G_{xy} = -\mathrm{K}, \quad G_{yx} = \Lambda.$$

the interior, the portion considered in the present instance being the element of the plate corresponding to the rectangular element $dx\,dy$ of the middle surface.

We shall require, presently, expressions for the stress-couples across *any* normal section. Let the normal to the plane of the section make an angle ϕ with the axis of x, and let the component couples about this normal and about the line in which the plane of the section meets the middle surface be $\mathfrak{N}$ and $\mathfrak{T}$, respectively. Considering the equilibrium of a triangular element of the plate bounded by this section, and by sections perpendicular to x and y, respectively, we find, taking moments about Ox, Oy,

$$G_{xx}\cos\phi + G_{yx}\sin\phi = \mathfrak{N}\cos\phi - \mathfrak{T}\sin\phi,$$

$$G_{xy}\cos\phi + G_{yy}\sin\phi = \mathfrak{N}\sin\phi + \mathfrak{T}\cos\phi,$$

whence

$$\mathfrak{N} = (1-\sigma)\,A\,(\cos^2\phi - \sin^2\phi)\,\frac{d^2w}{dx\,dy} + (1-\sigma)\,A\cos\phi\sin\phi\left(\frac{d^2w}{dy^2} - \frac{d^2w}{dx^2}\right),$$

$$\mathfrak{T} = -\sigma A\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right)$$

$$-(1-\sigma)\,A\left(\frac{d^2w}{dx^2}\cos^2\phi + 2\,\frac{d^2w}{dx\,dy}\cos\phi\sin\phi + \frac{d^2w}{dy^2}\sin^2\phi\right).$$

3. The second state of strain which we shall consider is as follows. Keeping the middle surface fixed, let the particles on either side undergo displacements relative to it whose components are of the forms

$$\alpha = P\left(z - \frac{z^3}{3h^2}\right),\quad \beta = Q\left(z - \frac{z^3}{3h^2}\right),\quad \gamma = 0,$$

where P, Q are functions of x and y as yet undetermined. We will further suppose that the variations of P, Q, when x and y receive increments of the order h, may be neglected in comparison with the values of these functions themselves. This is to be regarded merely as a provisional assumption which may be justified *à posteriori* in any particular problem. On this supposition we may write, for the components of stress,

$$p_{xx} = 0,\quad p_{yy} = 0,\quad p_{zz} = 0,$$

$$p_{yz} = \mu Q\left(1 - \frac{z^2}{h^2}\right),\quad p_{xz} = \mu P\left(1 - \frac{z^2}{h^2}\right),\quad p_{xy} = 0;$$

and we find that the stresses across an element of a normal section perpendicular to x reduce practically to a force parallel to z, whose value per unit length is

$$P_{xz}, = \int_{-h}^{h} \mu P \left(1 - \frac{z^2}{h^2}\right) dz, = \tfrac{4}{3}\mu Ph.$$

Similarly, the action across an element of a normal section perpendicular to y reduces to a shearing force

$$P_{yz} = \tfrac{4}{3}\mu Qh,$$

per unit length. Hence the above formulæ for α, β, γ may be written

$$\alpha = \frac{3}{4} \cdot \frac{P_{xz}}{\mu h}\left(z - \frac{z^3}{3h^2}\right), \quad \beta = \frac{3}{4} \cdot \frac{P_{yz}}{\mu h}\left(z - \frac{z^3}{3h^2}\right), \quad \gamma = 0.$$

By considering a triangular element of the plate as before, and resolving parallel to z, we find that the shearing force $\mathfrak{Z}$ across a section the normal to which makes an angle ϕ with Ox is, per unit length,

$$\mathfrak{Z} = P_{xz} \cos\phi + P_{yz} \sin\phi.$$

The shearing forces on the whole contour of an element $dx\,dy$ reduce to a force

$$\left(\frac{dP_{xz}}{dx} + \frac{dP_{yz}}{dy}\right) dx\,dy,$$

normal to the plate, and to a couple whose components are

$$P_{yz}\, dx \,.\, dy, \quad -P_{xz}\, dy \,.\, dx, \quad 0.$$

Since the values of α, β have been chosen so as to give zero stress on the two faces, this force and couple represent the stresses on the whole boundary of an element of the plate. Hence, if we reverse the signs, we get the force- and couple-resultants of the system of applied forces which must act on the interior of the element in order to maintain the strain in question.

4. Combining our results, we learn that the strain resulting from the superposition of the two states described in §§ 2 and 3, respectively, requires for its maintenance a certain distribution of applied force which is equivalent, for the element $dx\,dy$, to a force whose components are

$$0, \quad 0, \quad -\left(\frac{dP_{xz}}{dx} + \frac{dP_{yz}}{dy}\right) dx\,dy,$$

and a couple $\quad (\overline{L} - P_{yz})\, dx\,dy, \quad (\overline{M} + P_{xz})\, dx\,dy, \quad 0.$

In virtue of our fundamental assumption, no sensible change in the general form of the middle surface, or in the resultant stress across an element of a normal section, will ensue when for this distribution we substitute that which actually obtains, provided

$$\frac{dP_{xz}}{dx}+\frac{dP_{yz}}{dy}=-Z,$$

$$\bar{L}-P_{yz}=L,$$

$$\bar{M}+P_{xz}=M,$$

where Z represents the applied force normal to the plate, and L, M the components of applied couples about Ox, Oy, per unit area of the middle surface, in the sense explained in § 1.*

The last two equations give

$$P_{xz}=-A\frac{d}{dx}\nabla^2 w+M,$$

$$P_{yz}=-A\frac{d}{dy}\nabla^2 w-L,$$

and, substituting in the former, we obtain†

$$A\nabla^4 w=Z+\frac{dM}{dx}-\frac{dL}{dy},$$

the usual differential equation.

We have also, for the shearing force across any normal section,

$$\begin{aligned}\mathfrak{Z}&=P_{xz}\cos\phi+P_{yz}\sin\phi\\&=-A\left\{\frac{d}{dx}\nabla^2 w\,.\cos\phi+\frac{d}{dy}\nabla^2 w\,.\sin\phi\right\}+M\cos\phi-L\sin\phi.\end{aligned}$$

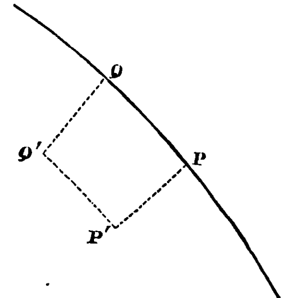
5. We come now to the boundary conditions. The difficulty here is, that in the immediate neighbourhood of the edge the mean state of strain is no longer represented, even approximately, by the assump-

* The most general specification of applied force would consist of forces X, Y, Z and couples L, M, 0; but the group Z, L, M acts quite independently of the group X, Y, the former producing curvature of the plate without extension of the middle surface, the latter extension without curvature. As we are here concerned only with the theory of flexure, we suppose X, Y to be zero. A like restriction is made as to the applied forces on the boundary.

† It is here assumed that L, M are continuous functions of x and y; Z, on the other hand, may be discontinuous. It is to be remembered that these functions merely express, as it were, the *average* distribution of applied force in the neighbourhood of the point (x, y).

tions of §§ 2, 3. In particular, a line of particles originally normal to the middle surface will not, in general, be normal to it after the deformation; it will not, in general, even remain straight. On the other hand, this exceptional state of things is practically confined to a narrow zone of the plate whose breadth is only a very moderate multiple of the thickness; beyond this region the statements in question apply, in the sense already explained.*

Now let PQ be an element ds of the boundary, and draw parallel to it, at a sufficient distance to fall outside the zone referred to, a line $P'Q'$, and let PP', QQ' be at right angles to the edge. The boundary conditions are obtained by considering the equilibrium of the element $PQQ'P'$ of the plate. Across $P'Q'$ we have a shearing force $\mathfrak{Z}\,ds$, and couples $\mathfrak{N}\,ds$, $\mathfrak{T}\,ds$, whose values are given by the formulæ of § 2, in which ϕ now denotes the angle which the normal to the edge makes with the axis of x. At the edge PQ we have the external forces, which may be specified by a force $\mathfrak{Z}_0\,ds$ parallel to z, and couples $\mathfrak{N}_0\,ds$, $\mathfrak{T}_0\,ds$ whose axes are the normal and tangent to the edge respectively. Across PP' (and this appears to be the key to the whole matter) we have a shearing force ζ which, unlike the shearing forces in the body of the plate, is specified not by its amount per unit length, but by its integral over a line extending inwards from the edge. This integral is finite in spite of the infinitely short length PP' over which it is taken.† The



* See Thomson and Tait, §§ 727–9.

† This point, which does not appear to have been hitherto expressly noted, is illustrated by the problem discussed in Thomson and Tait, *loc. cit.* It is there shown that, if a certain distribution of force, constituting a uniform couple $\mathfrak{N}_0$ per unit length, be applied to the plane edge ($y=0$) of an otherwise infinite plate bounded by the planes $z=\pm h$, the component displacements α, β, γ at any point (x, y, z) are given by

$$\alpha = 3\,\frac{\mathfrak{N}_0}{\mu h}\left(\frac{2}{\pi}\right)^3 \Sigma \frac{(-1)^n}{(2n+1)^3}\, e^{-(2n+1)\pi y/2h} \sin\frac{(2n+1)\pi z}{2h},$$

$$\beta = 0, \quad \gamma = 0;$$

the plate being supposed to lie on the positive side of the plane zx. (The notation has been modified to correspond to that of this paper.) Hence

$$\zeta = \int_0^\eta \int_{-h}^{h} \mu \frac{d\alpha}{dz}\,dx\,dz = \frac{96}{\pi^4}\mathfrak{N}_0 \,.\, \Sigma \frac{1-e^{-(2n+1)\pi\eta/2h}}{(2n+1)^4},$$

where η stands for the breadth PP' in the above figure. Since $e^{-10} = \cdot 0000454$, we see that, if η exceeds three or four times the thickness, the series practically reduces to $\Sigma\,(2n+1)^{-4}$, or $\pi^4/96$, so that

$$\zeta = \mathfrak{N}_0,$$

as the argument in the text, applied to this particular case, would indicate.

corresponding shearing force across QQ' is $\zeta + d\zeta/ds \, . \, ds$. Hence, resolving parallel to z,

$$\mathfrak{Z}_0 - \mathfrak{Z} + \frac{d\zeta}{ds} = 0,$$

and taking moments about the tangent and normal, respectively,

$$\mathfrak{T}_0 - \mathfrak{T} = 0,$$

$$\mathfrak{N}_0 - \mathfrak{N} + \zeta = 0.$$

If we substitute the values of $\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{Z}$, given in §§ 2, 4, in the equations which remain after elimination of ζ, viz., in

$$\mathfrak{T} = \mathfrak{T}_0,$$

$$\mathfrak{Z} - \frac{d\mathfrak{N}}{ds} = \mathfrak{Z}_0 - \frac{d\mathfrak{N}_0}{ds},$$

we obtain Kirchhoff's boundary conditions.

It was pointed out by Thomson and Tait that, if $\mathfrak{Z}_0$ and $d\mathfrak{N}_0/ds$ be increased by equal amounts, the state of things beyond the immediate neighbourhood of the edge is unaltered; and this remark forms the basis of their treatment of the boundary conditions.* We now see that the effect is on the distribution of the shearing force which we have denoted by ζ; viz., if $\phi'(s)$ be the common increment of the above functions, then ζ is increased by $-\phi(s)$.

The precise character of the boundary conditions laid down by Poisson and Kirchhoff respectively has often been discussed, and there is, perhaps, even yet room for comment on the subject. It is certainly remarkable, as a point of mathematical history, that the exceptional state of strain near the edge, on which, as we have seen, so much depends, should not have been recognized by either writer.

In the work of Poisson the fundamental assumption is, that when the various functions expressing the strain are expanded in powers of z, the distance from the middle surface, all terms after the first two may be neglected; and it is shown that, in the absence of stress on the two surfaces of the plate, the state of strain will then everywhere conform to the description given above in § 2. If this state is to hold right up to the edge, the three boundary conditions given by Poisson, viz. (in our notation)

$$\mathfrak{Z} = \mathfrak{Z}_0, \quad \mathfrak{T} = \mathfrak{T}_0, \quad \mathfrak{N} = \mathfrak{N}_0,$$

* [*April*, 1890. The argument is presented in a very clear form by Boussinesq in his book entitled *Application des Potentiels à l'Étude de l'Équilibre .... des Solides Élastiques*, §§ 71, 72.]

are certainly necessary. It has been proved, however, by Kirchhoff that it is, in general, impossible to satisfy three independent boundary conditions. We infer that, with arbitrary values of $\mathfrak{Z}_0$, $\mathfrak{T}_0$, $\mathfrak{N}_0$, there will, in general, be the peculiar state of strain near the edge, and the accompanying distribution of the shearing force ζ, above described. That Poisson's assumption may be here violated is well shown in the case worked out by Thomson and Tait, to which reference has already been made, where the variation of strain through the thickness is given by circular functions of $\pi z/2h$.

Kirchhoff, in his original investigation, begins by postulating the state of strain of § 2, and proceeds to calculate the potential energy in terms of the curvature of the middle surface. He then applies the energy condition of equilibrium, equating the increment of potential energy in any slight change of deformation to the work done by the applied forces. In this process the internal forces, including the shearing force ζ, do not, of course, explicitly appear; but to make the argument accord with our present knowledge it would be necessary to recognize the existence of a narrow zone along the edge to which the fundamental postulates do not apply, and to assume that the potential energy of this zone may be neglected in comparison with that of the rest of the plate.

6. A simple but very instructive illustration of the point to which special attention has been directed above is furnished by a problem discussed by Thomson and Tait (§ 656) from a somewhat different point of view. Taking the case of a rectangular plate whose edges are parallel to x and y, let us ascertain what is the simplest* system of applied forces which will produce the uniform anticlastic curvature

$$w = Cxy.$$

It appears at once that we may suppose P_{xz}, P_{yz}, $\mathfrak{Z}$, Z, L, M to be zero. Again, at an edge perpendicular to x, we have

$$\mathfrak{N} = (1-\sigma)\,AC, \quad \mathfrak{T} = 0,$$

whilst at an edge perpendicular to y

$$\mathfrak{N} = -(1-\sigma)\,AC, \quad \mathfrak{T} = 0;$$

* The problem, to find a system of forces which will produce a given deformation, is of course indeterminate, the only conditions being that

$$Z + \frac{dM}{dx} - \frac{dL}{dy}, \quad \mathfrak{T}_0, \quad \mathfrak{Z}_0 - \frac{d\mathfrak{N}_0}{ds}$$

are to have prescribed values.

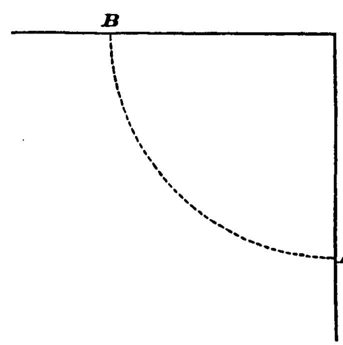
so that the boundary conditions are satisfied by $\mathfrak{T}_0 = 0$, $\mathfrak{N}_0 = 0$, $\mathfrak{Z}_0 = 0$, except at the actual corners. These points must be excepted on account of the term $d\mathfrak{N}/ds$ in the second condition, which is there infinite. If we suppose the corners to be rounded off, and integrate the condition in question over an infinitely short length of the edge including a corner, we find

$$-[\mathfrak{N}] = \int \mathfrak{Z}_0 ds,$$

the square brackets being used to indicate that the difference of the values of $\mathfrak{N}$ at the two limits is to be taken. For one corner these limits will correspond to $\phi = 0$, $\phi = \pi/2$; for a second to $\phi = \pi/2$, $\phi = 3\pi/2$; and so on. Hence

$$\int Z_0 ds = \pm 2(1-\sigma) AC;$$

so that we have the case of a plate bent by two contrary pairs of equal forces at the extremities of the two diagonals. If we consider the equilibrium of a portion of the plate cut off by a line AB drawn as in the figure, the applied force at the corner is balanced by the shearing forces which we have denoted by ζ [$=(1-\sigma) AC$] acting at the points A, B, and by the flexural couples across the dotted line.



7. The number of problems which have been solved in the present subject is so small that I may be allowed to append a discussion of one or two very simple questions.

In the first place, let us take the case of waves propagated along an infinitely long band bounded by straight parallel edges $y = \pm b$, there being no external forces. The boundary conditions then reduce to

$$\frac{d}{dy}\left\{\frac{d^2w}{dy^2} + (2-\sigma)\frac{d^2w}{dx^2}\right\} = 0,$$

$$\frac{d^2w}{dy^2} + \sigma\frac{d^2w}{dx^2} = 0.$$

The general differential equation is, by d'Alembert's principle,

$$\frac{d^2w}{dt^2} + c^4\nabla^4 w = 0,$$

where $c^4 = A/\rho$, ρ being the mass per unit area. If we assume

$$w = Qe^{ik(x-vt)},$$

where Q is a function of y only, this gives

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 Q = \frac{k^2v^2}{c^4} Q.$$

The solution of this, appropriate to our present purpose, is

$$Q = C_1 \cosh m_1 y + C_2 \cosh m_2 y,$$

where m_1, m_2 are determined by

$$m_1^2 = k^2 + \frac{kv}{c^2}, \quad m_2^2 = k^2 - \frac{kv}{c^2}.$$

The boundary conditions then give

$$C_1 m_1 \{m_1^2 - (2-\sigma) k^2\} \sinh m_1 b + C_2 m_2 \{m_2^2 - (2-\sigma) k^2\} \sinh m_2 b = 0,$$

$$C_1 (m_1^2 - \sigma k^2) \cosh m_1 b + C_2 (m_2^2 - \sigma k^2) \cosh m_2 b = 0,$$

whence, remembering that $m_1^2 + m_2^2 = 2k^2$, we find

$$m_1 (m_2^2 - \sigma k^2)^2 \tanh m_1 b = m_2 (m_1^2 - \sigma k^2)^2 \tanh m_2 b,$$

the equation to determine v, the velocity of propagation.

If $m_1 b$, $m_2 b$ are small, this reduces to

$$m_1^2 (m_2^2 - \sigma k^2)^2 = m_2^2 (m_1^2 - \sigma k^2)^2,$$

whence we easily find $\quad v = (1-\sigma^2)^{\frac{1}{2}} . kc^2.$

This coincides with the velocity of propagation of waves of length $2\pi/k$ along a uniform straight bar, as might have been anticipated from the fact that we have virtually assumed the wave-length to be great in comparison with the width of the band. Continuing the approximation a stage further, we find

$$v^2/k^2c^4 = 1 - \sigma^2 + \tfrac{2}{3} (1-\sigma) \sigma^2 k^2 b^2.$$

If, on the other hand, $m_1 b$, $m_2 b$ are moderately large, we may put

$$\tanh m_1 b = 1, \quad \tanh m_2 b = 1,$$

so that $\quad m_1 (m_2^2 - \sigma k^2)^2 = m_2 (m_1^2 - \sigma k^2)^2.$

Rearranging, dividing by $m_1 - m_2$, and putting

$$m_1^2 + m_2^2 = 2k^2,$$

we get $\quad m_1^2 m_2^2 + 2(1-\sigma) k^2 m_1 m_2 - \sigma^2 k^4 = 0,$

a quadratic to determine $m_1 m_2$. Since we have virtually taken m_1, m_2 to be positive, the positive root of this is to be chosen, and the wave-velocity is then given by

$$v^2/k^2c^4 = 1 - m_1^2 m_2^2/k^4$$
$$= (1-\sigma)\left\{3\sigma - 1 + 2(1-2\sigma+2\sigma^2)^{\frac{1}{2}}\right\}.$$

The following table gives, in the second column, the values of v/kc^2 —*i.e.*, the ratio of the velocity of propagation to that of waves of the same length in an infinite plate—for various values of σ. The fourth column enables us to estimate the ratio which the breadth of the

σ	v/kc^2	m_1/k	m_2/k
0	1·000000	1·4142	·0000
·1	·999984	1·4142	·0040
·2	·999697	1·4141	·0174
·3	·998102	1·4135	·0436
·4	·992639	1·4116	·0858
·5	·978318	1·4065	·1472

band must bear to the wave-length in order that the above approximation may be valid. Thus, since $\tanh \pi$ falls short of unity by about 1/150000, we see that for $\sigma = \frac{1}{2}$ the approximation is ample so soon as the breadth exceeds three or four times the wave-length. For smaller values of σ much higher values of the ratio may be required. It is to be noticed that, however short the waves may be in comparison with the breadth, the velocity of propagation (except in the extreme case $\sigma = 0$) falls below its value for an unlimited plate by a finite, though not a very great, amount. The circumstances are, in fact, different; owing to the occurrence of the hyperbolic functions of $m_1 y$, $m_2 y$ in the value of w, the amplitude increases as we pass outwards from the centre of the band towards either edge, instead of remaining uniform. Thus the ratio of the amplitudes at the edge and at the centre is

$$\frac{C_1 \cosh m_1 b + C_2 \cosh m_2 b}{C_1 + C_2},$$

which
$$= \frac{(m_1^2 - m_2^2) \cosh m_1 b \,.\, \cosh m_2 b}{(m_1^2 - \sigma k^2) \cosh m_1 b - (m_2^2 - \sigma k^2) \cosh m_2 b}.$$

This is sensibly equal to unity in the case first considered, where $m_1 b$, $m_2 b$ were small; but when $m_1 b$, $m_2 b$ are considerable, and m_1

large in comparison with m_2, it is of the order $e^{m_2 b}$, and therefore large.* It appears that the waves propagated in the form of cylindrical corrugations, which are contemplated in the investigation for an infinite plate, form an ideal case which cannot be realized in a plate of finite dimensions, except in so far as the extensions (neglected in the above theory) which are called into play by the increase of amplitude towards the edges tend to keep this increase in check.

8. The vibrations of a rectangular plate have not yet been investigated in a rigorous manner. Part of the difficulty is in satisfying the conditions at the corners. It may be worth while to write down these conditions, as one conclusion of some interest can at once be drawn. Taking the origin at a corner, and the axes of x, y along the edges which meet there, we have, as before, for all points of the axis of x,

$$\frac{d}{dy}\left\{\frac{d^2w}{dy^2}+(2-\sigma)\frac{d^2w}{dx^2}\right\}=0,$$

$$\frac{d^2w}{dy^2}+\sigma\frac{d^2w}{dx^2}=0;$$

whilst, along the axis of y,

$$\frac{d}{dx}\left\{\frac{d^2w}{dx^2}+(2-\sigma)\frac{d^2w}{dy^2}\right\}=0,$$

$$\frac{d^2w}{dx^2}+\sigma\frac{d^2w}{dy^2}=0.$$

The further condition to be satisfied at the corner has already been obtained in § 6, viz., it is

$$[\mathfrak{N}]=-\int \mathfrak{Z}_0\,ds=0,$$

whence

$$\frac{d^2w}{dx\,dy}=0$$

at the origin. Since the first pair of boundary conditions may be differentiated with respect to x, and the second with respect to y, it easily follows that the differential coefficients of w of the second and

* The case, at which we have virtually arrived, of waves travelling along the straight edge of an otherwise unlimited plate may be treated more directly by taking the origin in this edge ($y = 0$), and assuming

$$w=(C_1e^{-m_1 y}+C_2e^{-m_2 y})\,e^{ik(x-vt)}.$$

The results coincide with those found above.

third orders all vanish at the origin. Hence near the corner the form of the plate will differ from a plane by small quantities of the *fourth* order. This may be compared with the state of things at the free end of a bar.

Appendix.

In the paper as originally drafted, there followed a discussion of one or two problems which can be worked out rigorously from the general equations of elasticity, and so serve in some measure as a test of the assumptions made in the theory of thin plates. The most important of these, viz., the propagation of waves in an infinite plate, has been fully treated by Lord Rayleigh in a paper* which has in the meantime appeared; I therefore confine myself to the following problem, which may serve as an illustration of the action of applied force:

An infinite plate, bounded by the planes $z = \pm h$, is subject to a force $C \cos kx$ per unit area of the middle surface, parallel to z; to find the deformation.

We assume for the component displacements

$$\alpha = \frac{d\phi}{dx} + \frac{d\psi}{dz}, \quad \beta = 0, \quad \gamma = \frac{d\phi}{dz} - \frac{d\psi}{dx} \quad \ldots\ldots\ldots\ldots\ldots (1),$$

where ϕ, ψ do not contain y. Let us first suppose that the external force is applied in the way of a bodily force of amount $C/2h \,.\, \cos kx$ per unit volume. The equations to be satisfied in the interior of the plate then are

$$\left.\begin{aligned} (\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2\alpha = 0, \\ (\lambda+\mu)\frac{d\delta}{dz} + \mu\nabla^2\gamma + \frac{C}{2h}\cos kx = 0, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2),$$

where
$$\delta = \frac{d\alpha}{dx} + \frac{d\gamma}{dz},$$

and ∇^2 now stands for $d^2/dx^2 + d^2/dz^2$. By differentiation we obtain

$$\left.\begin{aligned} \nabla^2\delta = 0, \\ \mu\nabla^2\left(\frac{d\alpha}{dz} - \frac{d\gamma}{dx}\right) + \frac{kC}{2h}\sin kx = 0, \end{aligned}\right\}$$

* *Lond. Math. Soc. Proc.*, April, 1889.

or, if we assume that $\alpha \propto \sin kx$, $\gamma \propto \cos kx$,

$$\left.\begin{aligned}\left(\frac{d^2}{dz^2}-k^2\right)^2\phi = 0,\\ \mu\left(\frac{d^2}{dz^2}-k^2\right)^2\psi+\frac{kC}{2h}\sin kx = 0,\end{aligned}\right\} \qquad (3).$$

A particular solution is

$$\left.\begin{aligned}\phi &= (F\sinh kz+Gkz\cosh kz)\cos kx,\\ \psi &= -\frac{C}{2\mu k^3 h}\sin kx+(H\cos kz+Kkz\sinh kz)\sin kx,\end{aligned}\right\} \qquad (4);$$

and on substitution we find that the original equations (2) are also satisfied, provided

$$(\lambda+2\mu)\,G = \mu K \qquad (5).$$

The boundary conditions are

$$p_{zz} = 0, \quad p_{xz} = 0 \qquad (6),$$

or

$$\lambda\delta+2\mu\frac{d\gamma}{dz} = 0, \quad \frac{d\alpha}{dz}+\frac{d\gamma}{dx} = 0,$$

for $z = \pm h$. The first of these gives, in conjunction with (5),

$$(F-H)\sinh kh+(G-K)\,kh\cosh kh = 0 \qquad (7),$$

whilst the second leads to

$$(F-H)\cosh kh+(G-K)(\cosh kh+kh\sinh kh)+\frac{C}{4\mu k^3 h} = 0 \qquad (8).$$

Solving,

$$\left.\begin{aligned}F-H &= \frac{C}{4\mu k^3}\cdot\frac{\cosh kh}{\sinh kh\cosh kh-kh},\\ G-K &= -\frac{C}{4\mu k^3}\cdot\frac{\sinh kh/kh}{\sinh kh\cosh kh-kh},\end{aligned}\right\} \qquad (9).$$

After a little reduction the consequent expressions for the displacements are found to be

$$\alpha = -\frac{C}{4\mu k}\left\{\cosh kh\,.\sinh kz-\frac{\sinh kh}{kh}\left(kz\cosh kz+\frac{\lambda+2\mu}{\lambda+\mu}\sinh kz\right)\right\}\sin kx$$
$$\div\{\sinh kh\cosh kh-kh\} \qquad (10),$$

$$\gamma = \frac{C}{2\mu k^2 h}\cos kx$$
$$+\frac{C}{4\mu k}\left\{\cosh kh\,.\cosh kz-\frac{\sinh kh}{kh}\left(kz\sinh kz-\frac{\mu}{\lambda+\mu}\cosh kz\right)\right\}\cos kx$$
$$\div\{\sinh kh\cosh kh-kh\} \qquad (11).$$

If kh be small—*i.e.*, if the variation of the applied force within a space compared with the thickness may be neglected—we find, to a first approximation,

$$\left.\begin{aligned} a &= \frac{3}{8}\frac{\lambda+2\mu}{(\lambda+\mu)\mu}\cdot\frac{C}{k^3h^3}\,z\sin kx, \\ \gamma &= \frac{3}{8}\frac{\lambda+2\mu}{(\lambda+\mu)\mu}\cdot\frac{C}{k^4h^3}\cos kx, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (12).$$

The latter formula agrees with the result given by the ordinary theory, viz.,

$$w = \frac{C}{k^4 A}\cos kx,$$

where A has the value given in § 2. We further notice that

$$a = -z\frac{d\gamma}{dx},$$

so that a line of particles originally straight and normal to the middle surface retains these properties after the flexure. This statement needs correction when we proceed to a second approximation; but, since a is an odd function of z, the curve into which such a line of particles is deformed has a point of inflexion where it crosses the middle surface; moreover, it is orthogonal at its extremities to the faces of the plate, since the shear in the plane zx there vanishes. It follows that the correction in question is very slight for moderate values of kh.

The formulæ for the component stresses give, on substitution from (10), (11),

$$p_{xx} = \lambda\delta + 2\mu\frac{da}{dx}$$

$$= \frac{C}{2}\left\{\frac{2\sinh kh}{kh}\sinh kz + \frac{\sinh kh}{kh}\,.\,kz\cosh kz - \cosh kh\,.\,\sinh kz\right\}\cos kx$$

$$\div\left\{\sinh kh\cosh kh - kh\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

$$p_{xz} = \mu\left(\frac{da}{dz} + \frac{d\gamma}{dx}\right)$$

$$= \frac{C}{2}\left\{\frac{\sinh kh/kh\,.(\cosh kz + kz\sinh kz) - \cosh kh\cosh kz}{\sinh kh\cosh kh - kh} - \frac{1}{kh}\right\}\sin kx$$

$$\ldots\ldots\ldots\ldots (14),$$

$$p_{zz} = \lambda\delta + 2\mu\frac{d\gamma}{dz}$$

$$= \frac{C}{2}\left\{\cosh kh \,.\, \sinh kz - \frac{\sinh kh}{kh}\,.\, kz\cosh kz\right\}\cos kx$$

$$\div \left\{\sinh kh \cosh kh - kh\right\} \quad \text{......................} \quad (15).$$

For small values of kh these reduce to

$$\left.\begin{aligned} p_{xx} &= \frac{3}{2}\frac{C}{k^2h^2}\cdot\frac{z}{h}\cos kx = -4\frac{(\lambda+\mu)\mu}{\lambda+2\mu}z\frac{d^2w}{dx^2},\\ p_{xz} &= -\frac{3}{4}\frac{C}{kh}\left(1-\frac{z^2}{h^2}\right)\sin kx = -2\frac{(\lambda+\mu)\mu}{\lambda+2\mu}\,.\,(h^2-z^2)\frac{d^3w}{dx^3},\\ p_{zz} &= \frac{1}{4}C\frac{z}{h}\left(1-\frac{z^2}{h^2}\right)\cos kx = \frac{2}{3}\frac{(\lambda+\mu)\mu}{\lambda+2\mu}\,.\,z(h^2-z^2)\frac{d^4w}{dx^4},\end{aligned}\right\} \dots (16).$$

The value of p_{xx} agrees with that obtained on the assumption of § 2; that of p_{xz} is small in comparison, whilst p_{zz} is of a still higher order of smallness. The second forms (in terms of w) are given on account of their greater generality; they hold for any distribution of bodily force which depends on x only,* since any such distribution can be expressed by Fourier's theorem in a series of terms of the forms $\cos kx$, $\sin kx$.

Let us next suppose that the external forces act on the two *faces* of the plate, and are equally divided between them, so that the surface conditions (6) are replaced by

$$\left.\begin{aligned} p_{zz} &= \tfrac{1}{2}C\cos kx, \quad \text{for } z = +h,\\ &= -\tfrac{1}{2}C\cos kx, \text{ for } z = -h,\\ p_{xz} &= 0.\end{aligned}\right\} \quad \text{..............} \quad (17).$$

The equations to be satisfied in the interior now are

$$\left.\begin{aligned} (\lambda+\mu)\frac{d\delta}{dx} + \mu\nabla^2\alpha = 0,\\ (\lambda+\mu)\frac{d\delta}{dz} + \mu\nabla^2\gamma = 0,\end{aligned}\right\} \quad \text{....................} (18);$$

whence

$$\left(\frac{d^2}{dz^2} - k^2\right)^2\phi = 0, \quad \left(\frac{d^2}{dz^2} - k^2\right)^2\psi = 0.$$

* Provided, of course, that its rate of variation within a space comparable with h may be neglected.

Assuming

$$\left.\begin{aligned}\phi &= (F\sinh kz + Gkz\cosh kz)\cos kx,\\ \psi &= (H\cosh kz + Kkz\sinh kz)\sin kx,\end{aligned}\right\}$$

we find, as before, that the equations (18) are satisfied, provided

$$(\lambda+2\mu)\,G = \mu K.$$

Taking account of this, the boundary conditions (17) give

$$\left.\begin{aligned}(F-H)\sinh kh + (G-K)\,kh\cosh kh &= \frac{C}{4\mu k^2},\\ (F-H)\cosh kh + (G-K)\,(\cosh kh + kh\sinh kh) &= 0,\end{aligned}\right\} \quad \dots (19);$$

whence

$$\left.\begin{aligned}F-H &= \frac{C}{4\mu k^2}\cdot\frac{\cosh kh + kh\sinh kh}{\sinh kh\cosh kh - kh},\\ G-K &= -\frac{C}{4\mu k^2}\cdot\frac{\cosh kh}{\sinh kh\cosh kh - kh},\end{aligned}\right\} \quad \dots\dots\dots\dots (20).$$

The values of the displacements are found to be

$$\alpha = \frac{C}{4\mu k}\left\{\cosh kh\left(\frac{\mu}{\lambda+\mu}\sinh kz + kz\cosh kz\right) - kh\sinh kh.\sinh kz\right\}\sin kx$$

$$\div\left\{\sinh kh\cosh kh - kh\right\} \quad \dots\dots\dots\dots\dots\dots\dots (21),$$

$$\gamma = \frac{C}{4\mu k}\left\{\cosh kh\left(\frac{\lambda+2\mu}{\lambda+\mu}\cosh kz - kz\sinh kz\right) + kh\sinh kh\cosh kz\right\}\cos kx$$

$$\div\left\{\sinh kh\cosh kh - kh\right\} \quad \dots\dots\dots\dots\dots\dots\dots (22).$$

These may be compared with (10) and (11). When kh is small, the values of α and γ to a first approximation coincide with those given in (12). The stresses p_{xx}, p_{xz} are also found on examination to retain the same approximate values (16) as before, whilst for p_{zz} I find

$$p_{zz} = \frac{3}{4}\,C\cdot\frac{z}{h}\left(1-\frac{z^2}{3h^2}\right)\cos kx$$

$$= \frac{2}{3}\,\frac{(\lambda+\mu)\,\mu}{\lambda+2\mu}\cdot z\,(3h^2-z^2)\,\frac{d^4w}{dx^4} \quad \dots\dots\dots\dots\dots (23).$$

We might also consider the case where the forces on the two faces are equal and opposite, so that the boundary conditions are

$$p_{zz} = \frac{C}{2} \cos kx, \quad p_{xz} = 0 \quad \text{.................... (24)},$$

for $z = \pm h$. We should find

$$\left.\begin{aligned} \phi &= (F \cosh kz + G \,.\, kz \sinh kz) \cos kx, \\ \psi &= (H \sinh kz + K \,.\, kz \cosh kz) \sin kx, \end{aligned}\right\} \quad \text{........... (25)},$$

where, as before,

$$(\lambda + 2\mu)\, G = \mu K;$$

and the rest of the solution is obtained by simply interchanging the functions cosh kh, sinh kh, in equations (19). Hence

$$\left.\begin{aligned} F - H &= \frac{C}{4\mu k^2} \cdot \frac{\sinh kh + kh \cosh kh}{\sinh kh \cosh kh + kh}, \\ G - K &= -\frac{C}{4\mu k^3} \cdot \frac{\sinh kh}{\sinh kh \cosh kh + kh}. \end{aligned}\right\} \quad \text{........... (26)}.$$

When kh is small, these lead to

$$\alpha = -\frac{C}{8k} \cdot \frac{\lambda}{(\lambda + \mu)\,\mu} \,.\, \sin kx, \quad \gamma = 0 \quad \text{.............. (27)},$$

at the middle surface, as might have been found by a much more elementary process. The effect of the given distribution is to produce an extension of the plate, but the displacement α is of the order k^3h^3 as compared with the value (12) of γ in the former case.

By superposition we can obtain from this case and the preceding the exact solution of the problem when the applied force is divided in any constant ratio between the two faces, but the resulting deformation is seen to be practically the same as in the case of an equal division.

The fundamental assumption of § 1 may be examined by taking kh large. In the case of a bodily force the formula (11) makes

$$\gamma = \frac{C}{2\mu k^3} \cos kx,$$

nearly, at the middle surface; *i.e.*, it varies ultimately as the square of the wave-length. When the force is applied to the two faces, we learn from (22) that γ is of the order e^{-kh} at the middle surface; in fact, the stress is now confined to a thin superficial stratum on each side.

Thursday, January 9th, 1890.

J. J. WALKER, Esq., F.R.S., President, in the Chair.

Mr. J. E. Campbell, B.A., Fellow and Lecturer of Hertford College, and Lecturer at University College, Oxford, was elected a Member.

The following communications were made:—

On the Deformation of an Elastic Shell: Prof. H. Lamb.

On the relation between the Logical Theory of Classes and the Geometrical Theory of Points: Mr. A. B. Kempe.

On the Correlation of Two Spaces, each of Three Dimensions: Dr. T. A. Hirst.

Note on the Simultaneous Reduction of the Ternary Quadric and Cubic to the forms $Ax^2+By^2+Cz^2+Dw^2$, $ax^3+by^3+cz^3+dw^3$: The President.

The following presents were received:—

"Educational Times," for January.

"Nautical Almanac," for 1893, 8vo; London.

"The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S." Vol. II., 4to; Cambridge, 1889. Two copies.

"Problems on the Motion of Atoms," by J. K. Smythies, 4to; London, 1885.

"Bulletin de la Société Mathématique de France," Tome XVII., No. 5.

"Bulletin des Sciences Mathématiques," Tome XIII., December, 1889.

"Beiblätter zu den Annalen der Physik und Chemie," Band XIII., Stück 11.

"Atti della Reale Accademia dei Lincei—Rendiconti," Vol. v., Fasc. 5 and 6.

"Bollettino delle Pubblicazioni Italiane, ricevute per Diritto di Stampa," No. 96.

"Atti del Reale Istituto Veneto," Tomo VII., Ser. VI., Disp. 3 to 10.

"Nieuw Archief voor Wiskunde," Deel XVI., Stuk 2.

"Prace Matematyczno-Fizyczne," Tom. II., Zeszyt I.

"Festschrift, herausgegeben von der Mathematischen Gesellschaft in Hamburg anlässlich ihres 200 jährigen Jubelfestes, 1890," erster Teil.

Ditto, ditto, Sonder-Abzug.

"Vierde Rapport van de Huygens-Commissie," 8vo; Amsterdam, 1889.