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V. *On the Foci and Directrices of the Line of the Second Order.* By T. S. DAVIES, F.R.S. Lond. and Ed., F.S.A., &c., Royal Military Academy.

IN vol. viii. (pp. 317–21.) of Gergonne's *Annales des Mathématiques*, M. Bret has proposed a method of discovering the foci of a conic section, defined by the usual general equation of the second degree between x and y . He adopts as his definition of the focus, that of Euler: the focus is a point in the plane of the curve, such that its distance from any point in the curve is a linear function of the corresponding coordinates of the curve. The investigation, however, is not carried beyond rectangular coordinates, and he does not solve his resulting equations except for two very simple cases. Those equations are certainly, to the eye, very simple; but on attempting their solution, it will be found that they involve an amount of reduction such as we should be little prepared to expect. Moreover, the conclusion he draws from his equation (that there are *four* foci, two real and two imaginary) does not seem to be justified by the investigation about to be here offered.

Sir John Lubbock has also investigated, under a view which is virtually the same as Euler's, but by a very elegant process peculiar to himself, the existence of a focus in the line of the second order*. As the subject is introduced incidentally and briefly in Sir John's paper, there is nothing to lead us to any conclusion as to how far he had pushed these researches.

These two are all the discussions of the problem of the foci with which I am acquainted, though possibly others might be scattered in some of the continental periodicals which I have no means of consulting. As the method which I have employed is perfectly general, and I have completely resolved the resulting equations, giving at the same time both the foci and directrices, instead of the foci alone, I am disposed to think it will not be without interest to those geometers who have been in the habit of studying the conic sections by means of their coordinate equations. One or two applications of the formulæ here obtained may be given in a future Magazine.

THEOREM.

A point and a line can be found in the plane of a line of the second order, referred to any system of rectilinear coordinates, such that any point in the line of the second order shall have its distances from the point and line to be found in a constant ratio, which can also be found.

* Phil. Mag. S. 3. vol. x. Aug. 1831, p. 86.

First. THE ELLIPSE AND HYPERBOLA.

Denote the equations of the given curve and of the line to be found, respectively, by

$$a y^2 + b x y + c x^2 + d y + e x + f = 0 \quad . \quad . \quad (1.)$$

$$p y + q x + r = 0 \quad . \quad . \quad . \quad . \quad (2.)$$

Also, let $(h k)$ be the point to be found, m the constant ratio, and α the given angle of ordination.

Retaining the same angular directions of the coordinate-axes, transfer the origin to the centre of the curve; then the preceding equations become

$$a y^2 + b x y + c x^2 + f' = 0 \quad . \quad . \quad . \quad (3.)$$

$$p y + q x + r' = 0 \quad . \quad . \quad . \quad . \quad (4.)$$

$$\text{where } r' = \frac{2 c d - b e}{b^2 - 4 a c} p + \frac{2 a e - b d}{b^2 - 4 a c} q + r \quad . \quad . \quad . \quad (5.)$$

$$f' = \frac{a e^2 + c d^2 - b d e}{b^2 - 4 a c} + f \quad . \quad . \quad . \quad (6.)$$

If now D and P represent the respective distances of a point in the curve from the point and line to be found, the proposition affirms that $D = m P$, or $D^2 = m^2 P^2$. Also, if $h' k'$ be the point in reference to the new axes of coordinates, this condition will be expressed by

$$\left. \begin{aligned} (x - h')^2 + 2 (x - h') (y - k') \cos \alpha + (y - k')^2 \\ = \frac{m^2 (p y + q x + r)^2 \sin^2 \alpha}{p^2 - 2 p q \cos \alpha + q^2} \end{aligned} \right\} \quad . \quad . \quad (7.)$$

It will be convenient, before proceeding further, to lay down the following notation. The functions, indeed, from their perpetual occurrence in researches concerning the conic sections, ought to be designated in some uniform manner. Those here employed, from their not interfering with any of the notations generally used in this class of inquiries, may perhaps be found worthy of adoption.

$$\left. \begin{aligned} u^2 &= p^2 - 2 p q \cos \alpha + q^2, \quad . \quad . \quad . \quad . \\ R^2 &= (a - b \cos \alpha + c)^2 + (b^2 - 4 a c) \sin^2 \alpha, \quad . \\ Q &= a - b \cos \alpha + c, \quad . \quad . \quad . \quad . \\ H &= Q - 2 a \sin^2 \alpha = a \cos 2 \alpha - b \cos \alpha + c, \\ K &= Q - 2 c \sin^2 \alpha = a - b \cos \alpha + e \cos 2 \alpha, \end{aligned} \right\} \quad . \quad (8.)$$

Expand (7.) and identify the result with (3.) by equating the homologous coefficients; then we have

$$a = u^2 - m^2 p^2 \sin^2 \alpha \quad . \quad . \quad . \quad . \quad (9.)$$

$$b = 2 u^2 \cos \alpha - 2 m^2 p q \sin^2 \alpha \quad . \quad . \quad . \quad . \quad (10.)$$

$$c = u^2 - m^2 q^2 \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

$$0 = u^2 (k' + h' \cos \alpha) + m^2 p r' \sin^2 \alpha \quad . \quad . \quad . \quad . \quad (12.)$$

$$0 = u^2 (h' + k' \cos \alpha) + m^2 q r' \sin^2 \alpha \quad . \quad . \quad . \quad (13.)$$

$$f = u^2 (h'^2 + 2 h' k' \cos \alpha + k'^2) - m^2 r'^2 \sin^2 \alpha \quad . \quad (14.)$$

The quantities h', k', p, q, r', m being determined from these equations, the proposition will be established.

Multiply (10.) by $\cos \alpha$ and subtract the result from the sum of (9.) and (11.); then

$$a - b \cos \alpha + c = 2u^2 \sin^2 \alpha - m^2 \sin^2 \alpha (pr - 2pq \cos \alpha + q^2), \text{ or}$$

$$Q = (2 - m^2)u^2 \sin^2 \alpha \quad . \quad . \quad . \quad (15.)$$

$$Q = (2 - m^2) u^2 \sin^2 \alpha \quad , \quad , \quad , \quad (15.)$$

Again, from the same three equations, we have

$$m^2 p^2 \sin^2 \alpha = u^2 - a \quad . \quad . \quad . \quad . \quad (16.)$$

$$2m^2pq\sin^2\alpha = 2u^2\cos\alpha - b \quad . \quad . \quad . \quad (17.)$$

$$m^2 q^2 \sin^2 \alpha = u^2 - c \quad . \quad . \quad . \quad . \quad (18.)$$

From these, since

$(2 m^2 p q \sin^2 \alpha)^2 = 4 (m^2 p^2 \sin^2 \alpha) (m^2 q^2 \sin^2 \alpha)$, we get

$$(2u^2 \cos \alpha - b)^2 = 4(u^2 - a)(u^2 - c), \text{ or}$$

$$4 u^4 \sin^2 \alpha - 4 (a - b \cos \alpha + c) u^2 = b^2 - 4 a c; \text{ or again,}$$

$$2u^2 \sin^2 \alpha = Q + R, \text{ and } u^2 = \frac{Q + R}{2 \sin^2 \alpha} \quad . \quad . \quad . \quad (19.)$$

Resolve (15.) for m^2 , and insert (19.) in the result; then

$$m^2 = \frac{2R}{Q+R} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (20.)$$

the constant ratio, or "determining ratio" is hence found.

For the determination of p and q , substitute the values of m^2 and u^2 from (19, 20.) in (16, 18.); then after very slight reduction we obtain

$$p^2 = \frac{u^2 - a}{m^2 \sin^2 \alpha} = \frac{(Q - 2a \sin^2 \alpha + R)(Q + R)}{4R \sin^4 \alpha} = \frac{(H + R)(Q + R)}{4R \sin^4 \alpha} \quad (21.)$$

$$q^2 = \frac{u^2 - c}{m^2 \sin^2 \alpha} = \frac{(Q - 2c \sin^2 \alpha + R)(Q + R)}{4R \sin^4 \alpha} = \frac{(K + R)(Q + R)}{4R \sin^4 \alpha} \quad (22.)$$

The signs of p and q deduced from these are double, and they are in each case to be so taken (like or unlike) as to fulfil (17.), viz. to be alike when $2u^2 \cos \alpha - b$, or $(a + c + R) \cos \alpha - b \cos 2\alpha$ is positive, and unlike when this quantity is negative. In either case the two values express sameness of inclination to either axis of coordinates, and hence parallelism of position. Wherefore in the ellipse and hyperbola there are two such lines as that of which the proposition affirms the existence.

It would also be easy to show that these lines are perpen-

dicular to one of the conjugate rectangular axes, did space allow us to dilate.

We shall now proceed to the other three equations (12, 13, 14.) which, together with m , p , q already found, involve h' , k' , r' .

From (12, 13.) resolved for h' and k' , we get

$$h' = - \frac{m^2 r' (q - p \cos \alpha)}{u^2} \quad . \quad . \quad . \quad (23.)$$

$$k' = - \frac{m^2 r' (p - q \cos \alpha)}{u^2} \quad . \quad . \quad . \quad (24.)$$

Also, multiply (12.) by k' and (13.) by h' , and add; then

$$u^2 (h'^2 + 2 h' k' \cos \alpha + k'^2) = -m^2 r' (p k' + q h') \sin^2 \alpha. \quad (25.)$$

Insert (23, 24.) in (25.); whence there results on reduction,

$$u^2 (h'^2 + 2 h' k' \cos \alpha + k'^2) = m^4 r'^2 \sin^2 \alpha \quad . \quad (26.)$$

Write for the left side of this equation its value on the right in (14.); then we immediately obtain by means of (20.),

$$r'^2 = \frac{f'}{m^2 (m^2 - 1) \sin^2 \alpha} = \frac{(R + Q)^2 f'}{2 R (R - Q) \sin^2 \alpha} \quad . \quad (27.)$$

This gives two values of r' , and since they are equal, the two lines to be found are symmetrically situated with respect to the centre.

Again, substitute the values of r' , p , q , u^2 in (23, 24.), and we obtain the values of h' and k' ; viz.

$$h' = \pm \sqrt{\frac{\frac{1}{2} f'}{b^2 - 4 a c}} \cdot \frac{\sqrt{H + R} \cos \alpha - \sqrt{K + R}}{\sin^2 \alpha} \quad . \quad (28.)$$

$$k' = \pm \sqrt{\frac{\frac{1}{2} f'}{b^2 - 4 a c}} \cdot \frac{\sqrt{K + R} \cos \alpha - \sqrt{H + R}}{\sin^2 \alpha} \quad . \quad (29.)$$

These show that the points to be found are also symmetrically situated with respect to the centre, and with respect to the axes of coordinates. It also appears from these equations that h' and k' are always of *different signs*. Also, since

$$h = h' + \frac{2 a c - b d}{b^2 - 4 a c} \quad \text{and} \quad k = k' + \frac{2 c d - b e}{b^2 - 4 a c},$$

we get by merely substituting the value of $\frac{1}{2} f'$ from (6.), the value of r from (5, 27.), and those of p , q from (21, 22.). The complete solution of the system of equations, and therefore the demonstration of the theorem, is effected.

With respect to the conditions of reality of these expressions, little need be said here, as the subject is discussed in several treatises on lines of the second order with sufficient amplification, the functions here involved occurring in almost all inquiries relating to these curves.

Throughout I have written R with the positive sign; but this is done to avoid the interference of the double sign derived from this source with others that occur in the investigation: but the change is easily made for the other case from the forms here given.

When Q is absolutely greater than R , or $(a - b \cos \alpha + c)^2$ greater than $(a - b \cos \alpha + c)^2 + (b^2 - 4ac) \sin^2 \alpha$; or again, $b^2 - 4ac$ negative, the curve (3.) is the ellipse referred to its centre, and f' is known in this case to be *essentially negative*. Wherefore R must have the sign that will render $H + R$ and $K + R$ both positive; that is, R is to be taken positive.

When, on the contrary, $b^2 - 4ac$ is positive, or the curve is the hyperbola, f' may be either negative or positive, according as the hyperbola is the primary or the conjugate one. These two cases will require that R shall be respectively negative or positive.

Second. THE PARABOLA.

The preliminary transformation of coordinates cannot in this case be made in the same manner as in the preceding one. I have therefore deduced the truth of the theorem without the aid of any transformation. However, as the expressions, though greatly simplified in the parabola by the relation $b^2 - 4ac = 0$, are not so simple as when the origin is transposed to a point $x_1 y_1$ in the curve, I shall here give the investigation according to this latter method. The general investigation might, indeed, be conducted in an analogous manner; but for the ellipse and hyperbola this would have led to expressions much more complex than we have had occasion to use in the method already developed.

The equations to be resolved become, under this transformation,

$$a = u^2 - m^2 p^2 \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

$$b = 2 u^2 \cos \alpha - 2 m^2 p q \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

$$c = u^2 - m^2 q^2 \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

$$d' = -2 u^2 (h' + k' \cos \alpha) - 2 m^2 p r' \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

$$e' = -2 u^2 (h' + k' \cos \alpha) - 2 m^2 q r' \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

$$0 = u^2 (h'^2 + 2 h' k' \cos \alpha + k'^2) - m^2 r'^2 \sin^2 \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

$$\text{where } \left. \begin{array}{l} d' = 2 a y_1 + b x_1 + d \\ e' = 2 c x_1 + b y_1 + e \end{array} \right\} \quad \left. \begin{array}{l} h' = h - x_1 \\ k' = k - y_1 \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

From (1, 2.) and (2, 3.) we get, as in the former case,

$$u^2 = \frac{Q}{\sin^2 \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

$$m^2 = \frac{2 R}{Q + R} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

since here $Q = R$.

Again, from (1, 3, 8, 9.) we have, after slight reduction by means of the relation $b = 2\sqrt{a}c$, the equations

$$p = \frac{\sqrt{a} \cdot \cos \alpha - \sqrt{c}}{\sin^2 \alpha} \quad . \quad . \quad . \quad (10.)$$

$$q = -\frac{\sqrt{c} \cdot \cos \alpha - \sqrt{a}}{\sin^2 \alpha} \quad . \quad . \quad . \quad (11.)$$

the sameness or difference of the signs of which are to be determined by the relation (2.).

From (4, 5.) and (9.), we have

$$2u^2(h' + h' \cos \alpha) = -(d' + 2pr' \sin^2 \alpha) \quad . \quad . \quad . \quad (12.)$$

$$2u^2(h' + h' \cos \alpha) = -(e' + 2qr' \sin^2 \alpha) \quad . \quad . \quad . \quad (13.)$$

Resolving which for h' , h' , and putting Q for $u^2 \sin^2 \alpha$,

$$h' = \frac{d' - e' + 2(p \cos \alpha - q)r' \sin^2 \alpha}{2Q} \quad . \quad . \quad . \quad (14.)$$

$$h' = \frac{e' - d' + 2(q \cos \alpha - p)r' \sin^2 \alpha}{2Q} \quad . \quad . \quad . \quad (15.)$$

Now we have from (10, 11.),

$$p \cos \alpha - q = \frac{\sqrt{a} \cdot \cos^2 \alpha - \sqrt{c} \cdot \cos \alpha + \sqrt{c} \cdot \cos \alpha - \sqrt{a}}{\sin^2 \alpha} = -\sqrt{a} \quad (16.)$$

$$q \cos \alpha - p = -\frac{\sqrt{c} \cdot \cos^2 \alpha - \sqrt{a} \cdot \cos \alpha + \sqrt{a} \cdot \cos \alpha - \sqrt{c}}{\sin^2 \alpha} = +\sqrt{c} \quad (17.)$$

Insert (16.) in (14.), and (17.) in (15.); then we get

$$h' = \frac{d' - e' - 2\sqrt{a} \cdot r' \sin^2 \alpha}{2Q} \quad . \quad . \quad . \quad (18.)$$

$$h' = -\frac{d' - e' - 2\sqrt{c} \cdot r' \sin^2 \alpha}{2Q} \quad . \quad . \quad . \quad (19.)$$

Express by means of (18, 19.) the value of $h'^2 + 2h'h' \cos \alpha + k'$, and insert in (6.); then after very slight reduction,

$$r' = \frac{d' - e'}{2(\sqrt{a} + \sqrt{c}) \sin^2 \alpha} \quad . \quad . \quad . \quad (20.)$$

Insert the value of $2r' \sin^2 \alpha$ from (20.) in (18, 19.), and there results

$$h' = \frac{(d' - e')\sqrt{c}}{2Q(\sqrt{a} + \sqrt{c})} \quad . \quad . \quad . \quad (21.)$$

$$h' = \frac{(d' - e')\sqrt{a}}{2Q(\sqrt{a} + \sqrt{c})} \quad . \quad . \quad . \quad (22.)$$

We have hence only to correct these results for the coordi-

nates of transformation, to obtain the several values required by the proposition.

It may be remarked that the coordinates x, y , are only connected by a single equation, that of the parabola; and hence another condition might be introduced, some one perhaps which would usefully simplify the results. Several such may be suggested; as for instance, x, y , to be such a point in the curve as should render P a minimum: but this is not the proper place for such details.

Postscript.

I take this opportunity (the earliest and the most proper one that has occurred) of correcting a mistake into which Professor De Morgan has fallen with respect to the authorship of a paper on the works of Alexander Anderson, which was published in the 'Ladies' Diary' for 1840 (Companion to the Almanac, 1843, p. 9.). That paper was written by my late friend Dr. Gregory, and not by me, as stated in the place referred to. It was, I am pretty sure, the very last dissertation he ever published. I am the more anxious to disclaim it early, as the authority of Mr. De Morgan would otherwise render it a matter of history, and I am by no means desirous to have the works of other geometers, however valuable, ascribed to me.

Whilst I am on this subject, I have to correct a mis-statement of my own in the Philosophical Magazine for August last. I had attributed the theorem of which I there gave a demonstration to Dr. Wallace, mainly on his own authority. I have since found that the same theorem had been given about thirty years earlier by Lambert (in his *Insigniores orbitæ cometarum proprietates*, section i.), together with the other theorems published in the 'Mathematical Repository'. (Poncelet, *Traité de Propriétés Projectives*, p. 268.).

It may also be worth noticing, that amongst the foreign proofs of Pascal's Hexagram, I have found since my own was printed (Phil. Mag., July 1842.), two very elegant ones, which deserve to be known in this country; one by Magnus in Crelle's Journal, which is copied into the 'Ladies' Diary' for 1843; and the other by Levy in *tom. iv.* Quetelet's Correspondence, p. 24. Five others also are given in the 'Diary' by English writers.

Royal Military Academy, Dec. 3, 1842.