A reduction of a system of power series to an equivalent system of polynomials.

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Introduction.

Let us suppose there is given a set of equations

$$f_i(x_1, \dots, x_n; \mu_1, \dots, \mu_p) = 0 \quad (i = 1, \dots, n),$$

in which the f_i are analytic in $x_1, \dots, x_n; \mu_1, \dots, \mu_p$ and vanish with these variables. If the Jacobian $\left| \frac{\partial f_i}{\partial x_j} \right|$ is distinct from zero for

$$x_1 = \dots = x_n = \mu_1 = \dots = \mu_p = 0,$$

then, as is well known, there exists one and but one solution for $x_1, \dots, x_n$ as power series in $\mu_1, \dots, \mu_p$ vanishing for $\mu_1, \dots, \mu_p = 0$. If however the Jacobian vanishes then, in general, there are more solutions than one and $\mu_1 = \dots = \mu_p = 0$ is a branch-point. There is but little information to be found in the literature on the nature of the situation when the Jacobian vanishes, beyond the fact that it is a branch-point.

There is one case however that is amenable to a general treatment when the Jacobian vanishes, and that is when not all of its first minors vanish. Let us suppose the leading first minor is not zero. Then the equations $f_i = 0$ ($i = 2, \dots, n$) can be solved for the variables $x_2, \dots, x_n$ as power series in $x_1, \mu_1, \dots, \mu_p$. The values thus obtained substituted in the first equation, $f_1 = 0$, give a single equation $g(x_1, \mu_1, \dots, \mu_p) = 0$ which is satisfied for $x_1 = \dots = \mu_p = 0$. But since the original Jacobian vanishes it follows that $\frac{\partial g}{\partial x_1} = 0$ for $x_1 = \mu_1 = \dots = \mu_p = 0$, and it may happen that derivatives of higher order also vanish.

The number of solutions of the equation $g = 0$ can be ascertained

by the aid of the factor theorem of Weierstrass which can be stated as follows*):

Let $f(x; \mu_1, \dots, \mu_p)$ be a convergent power series in $x; \mu_1, \dots, \mu_p$ such that $f(x; 0, \dots, 0)$ begins with a term of degree m . Then a convergent power series $\varphi(x; \mu_1, \dots, \mu_p)$ having a constant term different from zero can be found such that the product

$$f(x; \mu_1, \dots, \mu_p) \cdot \varphi(x; \mu_1, \dots, \mu_p) = x^m + a_1 x^{m-1} + \dots + a_m = p^{(m)}$$

is a polynomial in x of degree m with coefficients, $a_1, \dots, a_m$, which are convergent power series in $\mu_1, \dots, \mu_p$ vanishing for $\mu_1 = \dots = \mu_p = 0$, and the coefficient of x^m is unity.

Clearly, since φ is distinct from zero for all $|x|, |\mu_1|, \dots, |\mu_p|$ sufficiently small, the equations $f = 0$ and $x^m + a_1 x^{m-1} + \dots + a_m = p^{(m)} = 0$ admit the same solutions for x as a function of $\mu_1, \dots, \mu_p$ vanishing for $\mu_1 = \dots = \mu_p = 0$, and in this respect the polynomial $p^{(m)}$ is equivalent to the power series f . By the application of this theorem the power series $g(x; \mu_1, \dots, \mu_p)$ can be reduced to an equivalent polynomial and the number of solutions of this polynomial set equal to zero is given at once by the theory of algebraic equations.

It is seen therefore that the general theory as at present developed is sufficient not only for the case where the Jacobian is distinct from zero, but also for the case where the Jacobian is zero if at least one of its first minors is not zero. If the first minors are all zero the present theory is not sufficient to give either the number of branches passing through the origin or the forms of their expansions, or even to tell us, whether or not the solutions are expansible as power series in the parameters in the neighborhood of the origin.

The transformation of a transcendental equation in one dependent variable to an equivalent algebraic equation suggests the possibility of the transformation of a system of n such equations in n dependent variables to an equivalent system of algebraic equations. If this could be done the question as to the number of branches passing through the origin would be answered by the general algebraic theory, and the question as to the forms of the solutions whether answered or not would be an algebraic one.

*) For this statement of the theorem, which is not quite the same as that of Weierstrass, see paper by Mac Millan in the Bulletin of the American Mathematical Society, December 1910. Other recent proofs of this theorem have been given by Goursat, Bulletin de la Société Mathématique de France 36 (1908); Bliss, Bulletin of the American Mathematical Society, April (1910); Hartogs, Sitzungsberichte der Akademie München 1909; Brill, Mathematische Annalen 69 (1910), p. 538.

In the paper which follows I propose to show that this reduction from a system of transcendental equations to an equivalent system of algebraic equations can actually be effected, so that the whole theory of the solutions in the neighborhood of the origin becomes an algebraic theory. I shall show that the theorem which has just been stated can be generalized from one power series in one dependent variable to any number of power series in the same number of dependent variables, and how the equivalent polynomials can be obtained.

The proof of the possibility of the reduction is effected in three distinct steps; first, it is shown that if $f_i^{(d_i)}(x_1, \dots, x_n)$, $i=1, \dots, n$, is a system of polynomials, homogeneous of degree d_i satisfying a certain condition which is related to the non-vanishing of their eliminant, then a system of multipliers $\varphi_i^{(d-d_i)}(x_1, \dots, x_n)$, which are homogeneous polynomials of degree $d-d_i$, can be found such that

$$\sum_{i=1}^n \varphi_i^{(d-d_i)} \cdot f_i^{(d_i)} = p^{(d)},$$

where $p^{(d)}$ is an assigned polynomial homogeneous of degree d , and d is equal to or greater than a certain number depending upon the d_i ; second, if $F_i^{[d_i]}$, $i=1, \dots, n$, is a system of power series in $x_1, \dots, x_n$ without parameters, the terms of lowest degree being homogeneous polynomials

$$f_i^{(d_i)}(x_1, \dots, x_n)$$

which we call the characteristic polynomials, and if the characteristic polynomials satisfy the same conditions as in the first step, then it is proved that multipliers $\Phi_i^{[d-d_i]}$ which are power series of order $d-d_i$ exist such that

$$\sum_{i=1}^n \Phi_i^{[d-d_i]} \cdot F_i^{[d_i]} = F^{[d]},$$

where $F^{[d]}$ is an assigned power series in $x_1, \dots, x_n$, the terms of lowest degree being of degree d ; third, if $F_i^{[d_i]}$ are power series in $x_1, \dots, x_n$, and also in the parameters $\mu_1, \dots, \mu_p$, it is possible to find multipliers $\Phi_i^{[d-d_i]}$ and a polynomial $P^{(d-1)}$ such that

$$\sum_{i=1}^n \Phi_i^{[d-d_i]} \cdot F_i^{[d_i]} + F = P^{(d-1)},$$

where F is an assigned power series in $x_1, \dots, x_n; \mu_1, \dots, \mu_p$. The second step is effected by successive applications of step one, and the third step is effected by successive applications of step two. These results can be accomplished in infinitely many ways. Finally it is shown, by means

of dominant functions, that at least one of these infinitely many ways leads to series which are convergent.

In the foregoing manner one polynomial with convergent coefficients is obtained. The others are obtained similarly. The process is so conducted that each one of the polynomials $P_i^{(d-1)}$ is associated with a certain one of the power series $F_i^{[di]}$ in such a way that the terms in $P_i^{(d-1)}$ which are independent of the parameters are identical with the corresponding terms in the series $F_i^{[di]}$, and that all the terms in $P_i^{(d-1)}$ which are of degree less than q in $x_1, \dots, x_n$, whatever may be their degree in the parameters, are identical with the corresponding terms in the series $F_i^{[di]}$, q being a certain number which is determined in the process. These properties of the polynomials enable us to show that the number of solutions is exactly equal to the product of the degrees of the characteristic polynomials.

Part I.

The formal reduction.

§ 1.

Systems of homogeneous polynomials.

Let us suppose that we have a system of polynomials*)

$$f_i^{(di)}(x_1, \dots, x_n), \quad (i = 1, \dots, n),$$

and let us take

$$d = \sum_{i=1}^n d_i - (n-1).$$

Cayley**) has shown that if we multiply each $f_i^{(di)} = 0$ by each of the power products $x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$, where the letters j satisfy the equation

$$j_1 + j_2 + \dots + j_n = d - d_i,$$

for the purpose of eliminating the x_k we obtain more equations than are necessary for the elimination. The equations thus obtained however are not all distinct, for certain identical relations exist among them. But if we suppress the superfluous equations then the determinant formed from the coefficients of the remaining equations, if it is not zero, is a multiple***) of the eliminant of the equations $f_i^{(di)} = 0$. There is a certain

*) Round brackets on the indices will be used throughout the paper to indicate the degree of the polynomial. The meaning of square brackets will be defined further on.

**) Cambridge and Dublin Mathematical Journal 2, p. 59, 1847.

***) See also Netto, Vorlesungen über Algebra, Vol. II, p. 147.

amount of choice in determining the equations to be suppressed; accordingly the determinant thus obtained is not unique. Let us call any one of these determinants an *eliminating determinant*. They are all zero if the eliminant is zero. But since, if $n > 2$, each of these determinants contains an extraneous factor any one of them may vanish even though the eliminant does not vanish. Cayley*) proposed a method for removing this extraneous factor to which however Netto**) takes exception. Whether or not all of the eliminating determinants can vanish if the eliminant is distinct from zero seems to have been left an open question.

Leaving this question unsettled we proceed to the proof of the following theorem:

Theorem I. *If $f_i^{(d_i)}(x_1, \dots, x_n)$, $i = 1, \dots, n$, is a system of homogeneous polynomials of degree d_i in $x_1, \dots, x_n$, and if $p^{(d+\lambda)}(x_1, \dots, x_n)$ is an assigned polynomial homogeneous of degree $d + \lambda$ where*

$$d = \sum_{i=1}^n d_i - (n-1)$$

and λ equals zero or any positive integer, then there exist polynomial multipliers $\varphi_i^{(d-d_i+\lambda)}(x_1, \dots, x_n)$ homogeneous in $x_1, \dots, x_n$ of degree $d - d_i + \lambda$ such that

$$(1) \quad \sum_{i=1}^n \varphi_i^{(d-d_i+\lambda)} \cdot f_i^{(d_i)} = p^{(d+\lambda)},$$

provided the system of polynomials $f_i^{(d_i)}$ has an eliminating determinant which is distinct from zero.

In order to show this for the case $\lambda = 0$ it is only necessary to give the polynomials $\varphi_i^{(d-d_i)}$ undetermined coefficients α_{ik} , form the equation

$$\sum_{i=1}^n \varphi_i^{(d-d_i)} \cdot f_i^{(d_i)} = p^{(d)},$$

and equate the coefficients of the same powers in the x 's on the two sides of the equation. This gives us a system of equations which are linear in the α_{ik} ; but if $n > 2$ the number of equations is less than the number of undetermined coefficients α_{ik} which are at our disposal. The matrix of the coefficients of the α_{ik} is precisely the same as the matrix obtained in Cayley's method of elimination except that the lines and co-

*) Cambridge and Dublin Mathematical Journal 3, p. 116, 1848. See also Salmon, Modern Higher Algebra, p. 87.

**) Netto, Vorlesungen über Algebra, Vol. II, p. 152.

lums are interchanged. As this matrix contains more lines than columns so here it contains more columns than lines. By suppressing the proper columns (that is, taking the corresponding α_{ik} equal to zero) the matrix can be reduced to the eliminating determinant which is distinct from zero. These equations then determine the remaining α_{ik} uniquely, so that equations (1) are satisfied. The solution thus obtained is not, however, the only solution, for the α_{ik} which we chose equal to zero could have been left arbitrary and the remaining α_{ik} determined in terms of them. Obviously, if $\varphi_i^{(d-d_i)}$ is one system of multipliers which satisfies (1) and $\varphi_i^{(d-d_i)} + \psi_i^{(d-d_i)}$ is another system, then the polynomials $\psi_i^{(d-d_i)}$ must satisfy the identity

$$\sum_{i=1}^n \psi_i^{(d-d_i)} \cdot f_i^{(d_i)} \equiv 0.$$

The proposition which we have just proved is equally true if the assigned polynomial which is the right member of (1) is of degree greater than d , and if the degrees of the multipliers are correspondingly greater than $d - d_i$, but it is not convenient to prove it in the same manner since the order of the determinant increases very rapidly as the degree of the polynomial $p^{(d+\lambda)}$ increases. It is preferable to break up the polynomial $p^{(d+\lambda)}$ when its degree is greater than d into a sum of polynomials of degree d multiplied by power products of the x_j . Let us suppose that the degree of $p^{(d+\lambda)}$ is $d + \lambda$. Then we can write

$$p^{(d+\lambda)} = \sum_j x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} p_{j_1 \cdots j_n}^{(d)} \quad (j_1 + j_2 + \cdots + j_n = \lambda).$$

From what has just been proved we know that for each polynomial $p_{j_1 \cdots j_n}^{(d)}$ there exist multipliers $\psi_{i, j_1 \cdots j_n}^{(d-d_i)}$ such that

$$(2) \quad \sum_{i=1}^n \psi_{i, j_1 \cdots j_n}^{(d-d_i)} \cdot f_i^{(d_i)} = p_{j_1 \cdots j_n}^{(d)}.$$

On multiplying through by $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ and taking the sum of all the parts, we find

$$\sum_{i=1}^n \left(\sum_j x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \psi_{i, j_1 \cdots j_n}^{(d-d_i)} \right) f_i^{(d_i)} = \sum_j x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} p_{j_1 \cdots j_n}^{(d)}$$

Consequently

$$\varphi_i^{(d-d_i+\lambda)} = \sum_{j \cdots} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \psi_{i, j_1 \cdots j_n}^{(d-d_i)}$$

are multipliers which satisfy

$$(3) \quad \sum_{i=1}^n \varphi_i^{(d-d_i+\lambda)} \cdot f_i^{(d_i)} = p^{(d+\lambda)},$$

and the theorem is proved.

§ 2.

Systems of power series without parameters.

In the preceding section certain results have been established with respect to a system of homogeneous polynomials. With proper modifications these results can be extended to systems of power series with or without parameters. As the simpler case we consider first a system of power series without parameters.

We will say that a power series*) $F^{[k]}(x_1, \dots, x_n)$ is of order k if its terms of lowest degree in $x_1, \dots, x_n$ are of degree k . Let $f^{(k)}(x_1, \dots, x_n)$, a homogeneous polynomial of degree k , be these terms. We will call $f^{(k)}$ the *characteristic polynomial* of the power series $F^{[k]}$. We proceed then to the proof of the following theorem:

Theorem II. *If $F_i^{[d_i]}(x_1, \dots, x_n)$, $i = 1, \dots, n$, is a system of power series in $x_1, \dots, x_n$ of order d_i , and if their characteristic polynomials $f_i^{(d_i)}$ have an eliminating determinant which is distinct from zero, then multipliers $\Phi_i^{[d-d_i]}(x_1, \dots, x_n)$, which are power series in $x_1, \dots, x_n$ of order $d - d_i$, where*

$$d = \sum_{i=1}^n d_i - (n-1),$$

can be found such that

$$(4) \quad \sum_{i=1}^n \Phi_i^{[d-d_i]} \cdot F_i^{[d_i]} = F^{[d]},$$

where $F^{[d]}$ is an assigned power series in $x_1, \dots, x_n$ of order d .

In order to show this it will be convenient to write

$$\begin{aligned} F_i^{[d_i]} &= f_i^{(d_i)} + f_i^{(d_i+1)} + f_i^{(d_i+2)} + \dots, \\ F^{[d]} &= f^{(d)} + f^{(d+1)} + f^{(d+2)} + \dots, \\ \Phi_i^{[d]} &= \varphi_i^{(d_i)} + \varphi_i^{(d_i+1)} + \varphi_i^{(d_i+2)} + \dots, \end{aligned}$$

where the symbols in the right members denote homogeneous polynomials of degrees indicated by the superscripts. For simplicity of notation we take

$$\delta_i = d - d_i.$$

*) Square brackets on the upper indices will be used exclusively to denote order.

Expanding equation (4), there results

$$(5) \quad 0 = \left[\sum_{i=1}^n \varphi_i^{(\delta_i)} f_i^{(\delta_i)} - f^{(d)} \right] + \left[\sum_{i=1}^n \varphi_i^{(\delta_i+1)} f_i^{(\delta_i)} + \sum_{i=1}^n \varphi_i^{(\delta_i)} f_i^{(\delta_i+1)} - f^{(d+1)} \right] + \\ \dots \dots \dots + \left[\sum_{i=1}^n \varphi_i^{(\delta_i+j)} f_i^{(\delta_i)} + \sum_{l=1}^j \sum_{i=1}^n \varphi_i^{(\delta_i+j-l)} f_i^{(\delta_i+l)} - f^{(d+j)} \right] + \\ \dots \dots \dots$$

where the first bracket is homogeneous of degree d , the second is homogeneous of degree $d+1$, and so on. Each of these brackets separately can be made equal to zero. For by Theorem I multipliers $\varphi_i^{(\delta_i)}$ can be found which will satisfy the first bracket set equal to zero. Substitute the values thus found in the second bracket and one can then find multipliers $\varphi_i^{(\delta_i+1)}$ which will satisfy the bracket set equal to zero, and so on. The multipliers $\varphi_i^{(\delta_i+j)}$ can be determined in succession so as to satisfy each bracket separately set equal to zero. The theorem holds obviously if the order of the right member is greater than d and the order of the multipliers is correspondingly increased.

§ 3.

Systems of power series with parameters.

In section 2 the characteristic polynomials of a system of power series without parameters were defined to be the homogeneous polynomials of lowest degree in the power series. If the power series have not only the arguments $x_1, \dots, x_n$, but also the arguments $\mu_1, \dots, \mu_p$, which are regarded as parameters, then the characteristic polynomials will be defined to be the characteristic polynomials of the power series when the parameters μ_j all have the value zero. That is, the characteristic polynomials of the system of power series

$$F_i^{[d]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p)$$

are the same as the characteristic polynomials of the series

$$F_i^{[d]}(x_1, \dots, x_n; 0, \dots, 0).$$

We will also define the order of

$$F_i^{[d]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p)$$

to be the same as the order of*)

$$F_i^{[d]}(x_1, \dots, x_n; 0, \dots, 0).$$

*) The definitions here given of characteristic polynomials and of order for a system of power series apply equally well to a system of non-homogeneous polynomials the coefficients of which are functions of a parameter, and will be so used whenever they are needed.

Let us consider now a system of power series

$$F_i^{[d_i]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p)$$

of order d_i , $i=1, \dots, n$. If the terms of this power series be arranged according to their degree in the dependent variables $x_1, \dots, x_n$ we shall have

$$F_i^{[d_i]} = \sum_{j=0}^{\infty} F_i^{(j)},$$

where the $F_i^{(j)}$ are homogeneous polynomials in $x_1, \dots, x_n$ of degree j with coefficients which are power series in $\mu_1, \dots, \mu_p$. For $j=0, 1, \dots, d-1$, the $F_i^{(j)}$ vanish with $\mu_1, \dots, \mu_p$. The polynomials $F_i^{(d)}$ do not vanish for $\mu_1 = \dots = \mu_p = 0$ but reduce to the characteristic polynomials $f_i^{(d)}$. As before, we will take

$$d = \sum_{i=1}^n d_i - (n-1)$$

and $\delta_i = d - d_i$ so that $d_i + \delta_i = d$.

We proceed now to the proof of the following theorem:

Theorem III. *If $F_i^{[d_i]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p)$, $i=1, \dots, n$, is a system of power series in $x_1, \dots, x_n; \mu_1, \dots, \mu_p$ of order d_i , and if the system of characteristic polynomials, $f_i^{(d)}(x_1, \dots, x_n)$, has an eliminating determinant which is distinct from zero, then multipliers*

$$\Phi_i^{[\delta_i]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p) \quad (i=1, \dots, n),$$

which are power series in $x_1, \dots, x_n; \mu_1, \dots, \mu_p$ of order δ_i and a polynomial

$$P_k^{(d-1)}(x_1, \dots, x_n; \mu_1, \dots, \mu_p)$$

of degree $d-1$ and of order d_k , can be found such that

$$(6) \quad (1 + \Phi_k^{[\delta_k]}) F_k^{[d_k]} + \sum_{\substack{i=1 \\ i \neq k}}^n \Phi_i^{[\delta_i]} F_i^{[d_i]} = P_k^{(d-1)}(x_1, \dots, x_n; \mu_1, \dots, \mu_p).$$

The subscript k can be chosen to have any one of the values $1, \dots, n$. It will be seen later that the multipliers of the $F_i^{[d_i]}$ must not all vanish for $x_1 = \dots = x_n = \mu_1 = \dots = \mu_p = 0$. For this reason we give the multiplier of $F_k^{[d_k]}$ a constant term which we choose equal to unity in order that certain terms in the polynomial $P_k^{(d-1)}$ shall be identical with the corresponding terms in the power series $F_k^{[d_k]}$. In other words, so that we can associate the polynomial P with a particular one of the system of power series.

Equation (6) can be written

$$(7) \quad \sum_{i=1}^n \Phi_i^{[\delta_i]} F_i^{[d_i]} + F_k^{[d_k]} = P_k^{(d-k)}$$

and the series

$$F_k^{[d_k]} = \sum_{j=0}^{\infty} F_k^{(j)} = \sum_{j=0}^{d-1} F_k^{(j)} + \sum_{j=d}^{\infty} F_k^{(j)},$$

the first term of which, $\sum_{j=0}^{d-1}$, is a polynomial of degree $d-1$ in $x_1, \dots, x_n$

with coefficients which are power series in $\mu_1, \dots, \mu_p$. Let us take therefore

$$P_k^{(d-k)} = \sum_{j=0}^{d-1} F_k^{(j)} + Q_k^{(d-1)},$$

$$\sum_{j=d}^{\infty} F_k^{(j)} = G_k^{[d]},$$

where $Q_k^{(d-1)}$ is a polynomial in $x_1, \dots, x_n$, of degree $d-1$ which, we will see in a moment, vanishes for $\mu_1, \dots, \mu_p = 0$. With these substitutions equation (7) becomes

$$(8) \quad \sum_{i=1}^n \Phi_i^{[\delta_i]} F_i^{[d_i]} + G_k^{[d]} = Q_k^{(d-1)}.$$

For convenience of notation let us take $\mu_j = m_j \mu$. We can then arrange each of the power series of equation (8) according to powers of μ . Thus

$$\begin{aligned} F_i^{[d_i]} &= F_{i0}^{[d_i]} + F_{i1}^{[d_i]} \mu + F_{i2}^{[d_i]} \mu^2 + \dots, \\ \Phi_i^{[\delta_i]} &= \Phi_{i0}^{[\delta_i]} + \Phi_{i1}^{[\delta_i]} \mu + \Phi_{i2}^{[\delta_i]} \mu^2 + \dots, \\ G_k^{[d]} &= G_{k0}^{[d]} + G_{k1}^{[d]} \mu + G_{k2}^{[d]} \mu^2 + \dots, \\ Q_k^{(d-1)} &= Q_{k0}^{(d-1)} + Q_{k1}^{(d-1)} \mu + Q_{k2}^{(d-1)} \mu^2 + \dots. \end{aligned}$$

The order of $F_{i0}^{[d_i]}$ is obviously d_i , but no assumptions are made with respect to the order of the coefficients of the higher powers of μ in $F_i^{[d_i]}$. In general they will contain terms which are independent of $x_1, \dots, x_n$, that is, in general they will be of order zero. The multipliers $\Phi_i^{[\delta_i]}$ are assumed to have no terms of degree less than δ_i in $x_1, \dots, x_n$, for any power of μ , that is, every $\Phi_{ij}^{[\delta_i]}$ is assumed to be of order δ_i . From the

satisfy the remaining terms of the equation. The series $\Phi_{i0}^{[\delta_i]}$ do not contain any terms of degree less than the smallest δ_i in $x_1, \dots, x_n$, and the series F_{i1} do not contain any terms of degree less than zero. It follows therefore that $Q_{k1}^{(d-1)}$ does not contain any terms in $x_1, \dots, x_n$ of degree less than the smallest δ_i .

Let us suppose that we have computed $\Phi_{i0}^{[\delta_i]}, \dots, \Phi_{i,j-1}^{[\delta_i]}$, and let us bear in mind that they are all of order δ_i . Then from the coefficient of μ^j we find

$$\sum_{i=1}^n \Phi_{ij}^{[\delta_i]} F_{i0}^{[d_i]} = -G_{kj}^{[d]} - \sum_{i=1}^j \sum_{i=1}^n \Phi_{ii} F_{i,j-i} + Q_{kj}^{(d-1)},$$

in which everything is known except the polynomial $Q_{kj}^{(d-1)}$ and the multipliers $\Phi_{ij}^{[\delta_i]}$. After taking $Q_{kj}^{(d-1)}$ equal to the terms in

$$\sum_{i=1}^j \sum_{i=1}^n \Phi_{ii}^{[\delta_i]} F_{i,j-i}$$

which are of degree less than d , the remaining equation can be solved for the multipliers $\Phi_{ij}^{[\delta_i]}$. Since the $F_{i,j-i}$ do not contain terms of degree less than zero and the $\Phi_{ii}^{[\delta_i]}$ do not contain terms of degree less than the smallest δ_i it is seen that the polynomial $Q_{kj}^{(d-1)}$ does not contain any terms of degree less than the smallest δ_i . Since this is true for every $j = 0, \dots, \infty$ it is seen that $Q_k^{(d-1)}$ is a polynomial of degree $d-1$ and that it contains no terms of degree less than the smallest δ_i . From this it follows that the polynomial

$$P_k^{(d-1)[d_k]} = \sum_{j=0}^{d-1} F_k^{(j)} + Q_k^{(d-1)}$$

is identical with the series $F_k^{[d_k]}$ in all terms of degree less than the smallest δ_i in $x_1, \dots, x_n$ for all degrees in $\mu_1, \dots, \mu_p$ from $1, \dots, \infty$.

Furthermore since $Q_{k0}^{(d-1)} \equiv 0$ it follows that the terms in $P_k^{(d-1)[d_k]}$ which are independent of $\mu_1, \dots, \mu_p$ are the same as the corresponding terms in the series $F_k^{[d_k]}$, and in particular $P_k^{(d-1)[d_k]}$ and $F_k^{[d_k]}$ have the same characteristic polynomial.

$f_i^{(d)} = 0$ and by hypothesis is not zero, it is seen that the multiplicity is exactly $\prod_{i=1}^n d_i$.*) The same is therefore true for the system of equations $F_i^{[d]} = 0$.

Subject to the convergence of the series employed, we have then the following theorem:

Theorem IV. *If $F^{[d]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p)$ ($i = 1, \dots, n$) is a system of power series in $x_1, \dots, x_n; \mu_1, \dots, \mu_p$, of order d_i and if the system of characteristic polynomials $f_i^{(d)}(x_1, \dots, x_n)$ has an eliminating determinant which is distinct from zero, then the system of equations*

$$F_i^{[d]}(x_1, \dots, x_n; \mu_1, \dots, \mu_p) = 0 \quad (i = 1, \dots, n)$$

admits exactly $\prod_{i=1}^n d_i$ solutions for $x_1, \dots, x_n$ as functions of $\mu_1, \dots, \mu_p$ which reduce to $x_1 = \dots = x_n = 0$ for $\mu_1 = \dots = \mu_p = 0$.

Part II.

Convergence.

§ 5.

A dominant system of polynomials of degree d .

There are infinitely many ways of reducing a system of power series to a system of polynomials by the method which has just been described. In order to complete the proof of Theorem IV it is necessary for us to show that at least one of these ways results in series which are convergent when the original series are convergent. The proof of convergence which is here given is by means of dominant functions.

As a first step in this proof we construct a system of polynomials homogeneous of degree d_i corresponding to the $f_i^{(d)}$ of § 1, which has an eliminating determinant distinct from zero, and a polynomial $M\sigma^{(d)}$ corresponding to $p^{(d)}$ of § 1, such that the equation corresponding to (1) admits solutions which dominate the solutions of (1). Let the new system of polynomials corresponding to the $f_i^{(d)}$ be $\varepsilon x_i^{d_i}$ ($i = 1, \dots, n$), and let the multipliers corresponding to the $\varphi_i^{(d-d_i)}$ be $X_i^{(\delta)}$ ($d - d_i = \delta_i$). Then the equations corresponding to (1) are

$$(14) \quad \sum_{i=1}^n \varepsilon x_i^{d_i} X_i^{(\delta)} = M\sigma^{(d)},$$

*) I have not been able to find a theorem covering this last statement. A proof for it will be given in a subsequent paper.

where ε and M are constants, and we will take

$$\sigma^{(d)} = \sum_{i=1}^n x_i^{d_i} (x_1 + x_2 + \dots + x_n)^{\delta_i}.$$

If we put

$$x_1 + x_2 + \dots + x_n = s,$$

then

$$\sigma^{(d)} = \sum_{i=1}^n x_i^{d_i} s^{\delta_i},$$

and an obvious solution of (14) is

$$(15) \quad X_i^{(\delta_i)} = \frac{M}{\varepsilon} s^{\delta_i}.$$

We shall show that ε (depending upon the $f_i^{(d_i)}$ alone) can be chosen so small and M (depending upon the $p^{(d)}$ alone) can be chosen so large that equations (1) admit solutions which are dominated by (15). We cannot prove that all of the solutions of (1) are dominated by (15) for the $\varphi_i^{(d-d_i)}$ contain arbitrary constants so that no polynomial dominates all of the solutions of (1). It will be sufficient to prove that some of the solutions of (1) are dominated by (15) and to show how such solutions can be obtained.

Let us consider further the solutions of (1)

$$(1) \quad \sum_{i=1}^n \varphi_i^{(\delta_i)} f_i^{(d_i)} = p^{(d)} \quad (d_i + \delta_i = d),$$

where the $\varphi_i^{(\delta_i)}$ are homogeneous polynomials of degree δ_i with undetermined coefficients α_{ik} . We assume that the system $f_i^{(d_i)}$ has at least one eliminating determinant Δ which is distinct from zero. This determinant Δ will contain as many rows and columns as there are terms in the complete homogeneous polynomial of degree d in n variables, viz. *)

$$\frac{n(n+1) \cdots (n+d-1)}{d!} = R.$$

Except in the simple cases where $n=2$ or where $d_i=1$ ($i=1, \dots, n$), the total number of coefficients α_{ik} in the system of complete polynomials $\varphi_i^{(\delta_i)}$ is greater than R . In the exceptional cases it is exactly equal to R . The determinant $\Delta \neq 0$ having been chosen the superfluous α_{ik} can be taken equal to zero. Then the remaining α_{ik} will be determined uniquely. It will be convenient to let the subscript k on the undetermined coefficients α_{ik} run from 1 to R . Let the coefficients of the polynomial $p^{(d)}$ be denoted by $C_1, \dots, C_R$. Then the linear equations in the α_{ik} can be so arranged that the solutions have the notation

*) Netto, Algebra, Vol. II, p. 2.

$$(16) \quad \alpha_{ik} = \sum_{l=1}^R \frac{\Delta_{lk}}{\Delta} C_l \quad (k=1, \dots, R),$$

where Δ_{lk} is the minor obtained by suppressing the l^{th} row and the k^{th} column and attaching the proper sign.

The polynomial

$$\sigma^{(d)} = \sum_{i=1}^n x_i^{d_i} (x_1 + x_2 + \dots + x_n)^{s_i} = \sum_{i=1}^n x_i^{d_i} s^{s_i}$$

is a complete polynomial of degree d , for s^{s_i} is a complete polynomial of degree s_i . Hence if there is a term in the complete homogeneous polynomial of degree d which is not in $\sigma^{(d)}$, its degree in x_i is less than or at most equal to $d_i - 1$ for every $i = 1, \dots, n$. Its degree in $x_1, \dots, x_n$ is therefore at most equal to

$$\sum_{i=1}^n (d_i - 1) = d - 1,$$

which is of degree less than d . Hence there is no such term and $\sigma^{(d)}$ is a complete homogeneous polynomial of degree d , the coefficients of which are positive and not less than unity. It can be written

$$(17) \quad \sigma^{(d)} = \sum_j A_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \quad (j_1 + j_2 + \dots + j_n = d),$$

where $A_{j_1 \dots j_n}$ are positive integers.

The polynomial $p^{(d)}$ can also be written

$$(18) \quad p^{(d)} = C \sum_j C_{j_1 \dots j_n} A_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \quad (j_1 + j_2 + \dots + j_n = d),$$

where C is a positive constant, $|C_{j_1 \dots j_n}| \leq 1$ and at least one $|C_{j_1 \dots j_n}| = 1$. The $A_{j_1 \dots j_n}$ are the coefficients of the corresponding terms in $\sigma^{(d)}$. Hence $\sigma^{(d)} \geq p^{(d)}/C$. Dividing through equation (16) by C we have

$$(19) \quad \frac{\alpha_{ik}}{C} = \sum_{l=1}^R \frac{\Delta_{lk}}{\Delta} \frac{C_l}{C}.$$

The fraction $\frac{C_l}{C}$ is one of the quantities $C_{j_1 \dots j_n} A_{j_1 \dots j_n}$, and the fraction Δ_{lk}/Δ is a certain number which depends only on the polynomials $f_i^{(d)}$. Let D be the maximum $|\Delta_{lk}/\Delta|$ and A the maximum $A_{j_1 \dots j_n}$. Then

$$\frac{\alpha_{ik}}{C} < \sum D A = R D A = H \quad (k=1, \dots, R),$$

and consequently

$$(20) \quad \varphi_i^{(\delta_i)} \leq CH \sum_j x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \quad (j_1 + j_2 + \cdots + j_n = \delta_i).$$

If then we take $M = C$ the right member of (14) dominates the right member of (1) as is shown by (17) and (18). If we take also

$$\frac{1}{\varepsilon} = H,$$

the solutions (15) become

$$(21) \quad X_i^{(\delta_i)} = CH s^{\delta_i}.$$

Comparing (20) and (21) it is clear that

$$\varphi_i^{(\delta_i)} \leq X_i^{(\delta_i)}.$$

§ 6.

A dominant system of polynomials of degree greater than d .

If the degree of the polynomial in the right member is greater than d we have from equations (3)

$$(3) \quad \sum_{i=1}^n \varphi_i^{(\delta_i + \lambda)} f_i^{(d_i)} = p^{(d + \lambda)}.$$

Let us compare the solutions of this with the solutions of

$$(22) \quad \sum_{i=1}^n \varepsilon x_i^{d_i} X_i^{(\delta_i + \lambda)} = M \sigma^{(d)} s^\lambda \gg p^{(d + \lambda)}$$

where ε is the same constant as before. One solution of (22) is

$$(23) \quad X_i^{(\delta_i + \lambda)} = \frac{M}{\varepsilon} s^{\delta_i + \lambda}.$$

We wish to show that (3) admits solutions which are dominated by (23).

The polynomial $p^{(d + \lambda)}$ can be broken up into polynomials of degree d multiplied by power products of the x 's of degree λ , thus

$$(24) \quad p^{(d + \lambda)} = \sum_j {}^{(d)} p_{j_1 \dots j_n} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} \quad (j_1 + j_2 + \cdots + j_n = \lambda),$$

where $p_{j_1 \dots j_n}^{(d)}$ is a homogeneous polynomial of degree d and $B_{j_1 \dots j_n}$ is the polynomial coefficient of $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ in s^λ , or stated otherwise

$$\sum_j B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} = s^\lambda \quad (j_1 + j_2 + \cdots + j_n = \lambda).$$

This is by no means a unique process. It can be done in infinitely many ways. Likewise the polynomial $M\sigma^{(d)}s^2$ can be broken up in the same manner,

$$M\sigma^{(d)}s^2 = M\sigma^{(d)} \sum B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} = \sum \pi_{j_1 \dots j_n}^{(d)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n},$$

where $\pi_{j_1 \dots j_n}^{(d)}$ are polynomials of degree d . The method of breaking up the polynomials, obviously, can be so chosen that $\pi_{j_1 \dots j_n}^{(d)} = M\sigma^{(d)}$, that is to say, they are all alike. If then for all j 's such that $j_1 + j_2 + \dots + j_n = \lambda$ we have $p_{j_1 \dots j_n}^{(d)} \ll \pi_{j_1 \dots j_n}^{(d)}$ it follows that

$$\varphi_i^{(\delta_i + \lambda)} \ll X_i^{(\delta_i + \lambda)}.$$

In order to see this consider the solutions of the equations*)

$$\sum_{i=1}^n f_i^{d_i} \psi_{i, j_1 \dots j_n}^{(\delta_i)} = p_{j_1 \dots j_n}^{(d)}$$

and

$$\sum_{i=1}^n \varepsilon x_i^{d_i} Y_{i, j_1 \dots j_n}^{(\delta_i)} = \pi_{j_1 \dots j_n}^{(d)} = M\sigma^{(d)} \gg p_{j_1 \dots j_n}^{(d)}.$$

Since the right members are polynomials of degree d it follows from § 5 that (26) admits solutions such that

$$(27) \quad \psi_{i, j_1 \dots j_n}^{(\delta_i)} \ll Y_{i, j_1 \dots j_n}^{(\delta_i)} = \frac{M}{\varepsilon} s^{\delta_i}.$$

Multiplying (25) by $B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$ and adding with respect to the combinations of the letters j we get

$$\begin{aligned} \sum \psi_{i, j_1 \dots j_n}^{(\delta_i)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} &\ll \sum Y_{i, j_1 \dots j_n}^{(\delta_i)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} \\ &= \frac{M}{\varepsilon} s^{\delta_i} \sum B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}, \end{aligned}$$

or finally

$$\varphi_i^{(\delta_i + \lambda)} \ll \frac{M}{\varepsilon} s^{\delta_i + \lambda} = X_i^{(\delta_i + \lambda)}$$

since

$$\sum \psi_{i, j_1 \dots j_n}^{(\delta_i)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} = \varphi_i^{(\delta_i + \lambda)}.$$

It remains to show that we can determine the polynomials

$$p_{j_1 \dots j_n}^{(d)} \ll \pi_{j_1 \dots j_n}^{(d)} = M\sigma^{(d)}.$$

Consider any one of the polynomials

$$M\sigma^{(d)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$$

*) See proof of Theorem I.

of (25). The coefficient of any term in it is positive and less than the coefficient of the same term in $M\sigma^{(d)}s^{\lambda}$. Each coefficient is therefore a certain fractional part of the coefficient of the same term in $M\sigma^{(d)}s^{\lambda}$. For the corresponding coefficient in the corresponding polynomial

$$p_{j_1 \dots j_n}^{(d)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$$

of (24) take the same fractional part of the coefficient of the same term in $p^{(d+\lambda)}$. Since the coefficient of the term in $M\sigma^{(d)}s^{\lambda}$ is positive and greater than the absolute value of the corresponding coefficient in $p^{(d+\lambda)}$ it follows that each coefficient in

$$M\sigma^{(d)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} = \pi_{j_1 \dots j_n}^{(d)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$$

is greater than the coefficient of the corresponding term in

$$p_{j_1 \dots j_n}^{(d)} B_{j_1 \dots j_n} x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}.$$

That is

$$M\sigma^{(d)} = \pi_{j_1 \dots j_n}^{(d)} \geq p_{j_1 \dots j_n}^{(d)}$$

for every combination of the letters j . Therefore if the solutions are obtained in the manner indicated it follows that

$$(28) \quad \varphi_i^{(\delta_i+\lambda)} \leq \frac{M}{\varepsilon} s^{\delta_i+\lambda} = X_i^{(\delta_i+\lambda)}.$$

§ 7.

Construction of dominant power series.

Let us suppose that the power series $F_i^{[d_i]}$ of Theorem III converge for $x_i = \mu_j = 1$. Let us set

$$V = \frac{1}{(1-\mu_1)(1-\mu_2)\dots(1-\mu_n)} - 1.$$

Then there exists a positive constant M such that the series

$$(29) \quad L_i^{[d_i]} = \frac{-\varepsilon x_i^{d_i} + M s_i^{d_i+1} + M V}{1-s}$$

dominates the series $F_i^{[d_i]}$ aside from the characteristic polynomials; and the series

$$(30) \quad L^{[d]} = n M \sigma^{(d)} + \frac{C \varepsilon \sigma^{(d)} - n M s^d}{\omega - C s}$$

dominates the series

$$G_k^{[d]} = \sum_{j=d}^{\infty} F_k^{(j)}$$

of equations (8) provided

$$C > \frac{nB+1}{\varepsilon} M.$$

The constant ε is defined as in § 5, B is the greatest coefficient in $s^d = (x_1 + x_2 + \dots + x_n)^d$ and

$$\omega = 1 - \nu,$$

where $\nu = KV$ and $K \geq 1$. It will be seen later, equation (36), that K is a power series in V with positive coefficients satisfying this condition.

By the process which has been previously described multipliers $\Lambda_i^{[d]}$ of order δ_i can be determined such that

$$(31) \quad \sum_{i=1}^n \Lambda_i^{[\delta_i]} L_i^{[d_i]} + L^{[d]} = \Pi^{(d-1)},$$

where $\Pi^{(d-1)}$ is a polynomial of degree $d-1$.

§ 8.

Proof of convergence.

It is sufficient to prove that the solution of equation (8) converges, for if this converges so also does the solution of equation (6) converge. Equation (31) corresponds to equation (8). We wish to show that (31) admits a convergent solution S , and that equation (8) admits a solution which is dominated by S and is therefore convergent.

In order to find a convergent solution of (31) let us take

$$(32) \quad \Lambda_i^{[\delta_i]} = \frac{C(1-s)s^{\delta_i}}{\omega - Cs}.$$

With this value for $\Lambda_i^{[\delta_i]}$ the expression

$$\sum_{i=1}^n \Lambda_i^{[\delta_i]} L_i^{[d_i]} + L^{[d]}$$

becomes

$$\sum_{i=1}^n \left[\frac{-C\varepsilon x_i^{d_i} s^{\delta_i} + C M s^{d+1} + C M V s^{\delta_i}}{\omega - Cs} \right] + n M s^d + \frac{C\varepsilon \sigma^{(d)} - n M s^d}{\omega - Cs}.$$

If we add and subtract $M(1-\omega)s^d$ to the numerator of each fraction included under the symbol Σ we obtain

$$\begin{aligned} \sum_{i=1}^n \left[\frac{-C\varepsilon x_i^{d_i} s^{\delta_i} + M s^d - M s^d (\omega - Cs) + \{-(1-\omega) M s^d + C M V s^{\delta_i}\}}{\omega - Cs} \right] \\ + n M s^d + \frac{C\varepsilon \sigma^{(d)} - n M s^d}{\omega - Cs}, \end{aligned}$$

which is equal to

$$\left[-nMs^d - \frac{C\varepsilon \sum_{i=1}^n x_i^{d_i} s^{\delta_i} - nMs^d}{\omega - Cs} + \frac{(1-\omega)M}{C} \cdot \frac{ns^d - \frac{CV}{1-\omega} \sum_{i=1}^n s^{\delta_i}}{s - \frac{\omega}{C}} \right] \\ + nMs^d + \frac{C\varepsilon \sigma^{(d)} - nMs^d}{\omega - Cs}.$$

Since

$$\sigma^{(d)} = \sum_{i=1}^n x_i^{d_i} s^{\delta_i}$$

this expression at once reduces to

$$(33) \quad \frac{(1-\omega)M}{C} \cdot \frac{ns^d - \frac{CV}{1-\omega} \sum_{i=1}^n s^{\delta_i}}{s - \frac{\omega}{C}},$$

which by the remainder theorem of algebra will be a polynomial in s of degree $d-1$ provided ω satisfies the relation

$$(34) \quad n\left(\frac{\omega}{C}\right)^d - \frac{CV}{1-\omega} \sum_{i=1}^n \left(\frac{\omega}{C}\right)^{\delta_i} = 0.$$

We have already seen that for $V=0$, ω must reduce to unity. Let us take therefore $\omega = 1 - \nu$. On dividing through by $n\left(\frac{\omega}{C}\right)^d$ and multiplying by $(1-\omega)$ there results

$$(35) \quad \nu = \frac{C}{n} V \sum_{i=1}^n \left(\frac{C}{1-\nu}\right)^{d_i} \quad (\text{since } d_i + \delta_i = d),$$

which has a unique solution for ν as a power series in V vanishing for $V=0$. Since the right member of (35) is expansible as a power series in ν with positive coefficients one sees that the solution for ν is also a power-series in V with positive coefficients, thus:

$$(36) \quad \nu = \left(\frac{C}{n} \sum_{i=1}^n C^{d_i}\right) V + \text{higher degree terms.}$$

The polynomial which is the right member of (31) is therefore

$$(37) \quad \Pi^{(d-1)} = \frac{M}{C} \left\{ n\nu \frac{s^d - \left(\frac{1-\nu}{C}\right)^d}{s - \left(\frac{1-\nu}{C}\right)} - CV \sum_{i=1}^n \frac{s^{\delta_i} - \left(\frac{1-\nu}{C}\right)^{\delta_i}}{s - \left(\frac{1-\nu}{C}\right)} \right\}.$$

The coefficients of this polynomial are themselves polynomials in ν and therefore convergent power series in V . The coefficient of the general term, s^{d-r-1} , is the sum of the terms in

$$(38) \quad \frac{M}{C} \left\{ n \nu \left(\frac{1-\nu}{C} \right)^r - C V \sum_{i=1}^n \left(\frac{1-\nu}{C} \right)^{r-d_i} \right\}$$

the exponents of which are positive or zero. But since from (35) the sum of all terms in (38) is zero it follows that the coefficients of s^{r-d-1} are equal to the terms in

$$(39) \quad \frac{M}{C} \left\{ C V \sum_{i=1}^n \left(\frac{C}{1-\nu} \right)^{d_i-r} \right\}$$

which have positive exponents, and these terms are obviously expansible as power series in V with positive coefficients. For example, suppose d_1 is smaller than any other d_i . Then the coefficient of s^{d-d_1-1} is

$$\frac{M}{C} \left\{ n \nu \left(\frac{1-\nu}{C} \right)^{d_1} - C V \right\},$$

and by (35) this is equal to

$$\frac{M}{C} \left\{ C V \sum_{i=1}^n \left(\frac{C}{1-\nu} \right)^{d_i-d_1} \right\}.$$

If in (38) $r \geq d_g$, the greatest d_i , then there are no terms in (38) with negative exponents. These coefficients therefore vanish by (35), and the polynomial $\Pi^{(d-1)}$ has no terms of degree less than $d-d_g$ in s agreeing in this respect with the polynomial $Q_k^{(d-1)}$ of § 3, since d_g is the smallest of the d_i .

We find therefore that multipliers $\Lambda^{[d_i]}$ which are expansible as convergent power series in $x_1, \dots, x_n; \mu_1, \dots, \mu_p$ and a polynomial $\Pi^{(d-1)}$ in $x_1, \dots, x_n$ with coefficients which are convergent power series in $\mu_1, \dots, \mu_p$ exist such that

$$(40) \quad \sum_{i=1}^n \Lambda_i^{[d_i]} L_i^{[d_i]} + L^{[d]} = \Pi^{(d-1)}.$$

It is clear that equation (40) can be solved in the same manner as was equation (8). Since the series L dominate the series F and since the series Λ (32) are expansible as power series in s with positive coefficients and since ε has the value indicated in § 5 it follows that solutions

for (8) can be found which, step by step, are dominated by the solutions which have been obtained for equations (40). The polynomial $Q_k^{(d-1)}$ is dominated by the polynomial $\Pi^{(d-1)}$ the coefficients of which are convergent. Hence the multipliers $\Phi_i^{[d]}$ and the coefficients of the polynomial $P_k^{(d-1)}$ obtained in the manner pointed out in §§ 5 and 6 are convergent power series. The proof of the theorems stated in Part I is therefore complete.

The University of Chicago, March 27th, 1911.

Note: The following theorem has been given by Hugo Kistler: *Jedes analytische Gebilde $2(n-k)$ ter Dimension wird ausgeschnitten durch Funktionen, die in $n-k$ Variablen Potenzreihen, in k Variablen ganz rational sind* (Über Funktionen von mehreren komplexen Veränderlichen, S. 19. Inaugural-Dissertation, Göttingen 1905).