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LXIII. On the kinetic criterion of potential energy

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The following table shows the readings :—

TABLE IX.

Spark Lengths.	Voltmeter Reading.	
	No Air Blast.	With Air Blast.
3·0 mm.	2·4	2·7
2·5 „	2·4	2·7
2 „	1·85	2·2
1·5 „	1·7	1·7
1·0 „	1·65	2·0
·5 „	·75	1·0
1·0 „	2.0	2·25
1·5 „	2·1	2·4
2·0 „	2·05	2·35
3·0 „	2·3	2·45

The Table shows in every case an increase in the current circulating in the condenser circuit when the spark-gap is subjected to the air-blast, but does not indicate the much greater steadiness of the voltmeter-needle which is seen with the air-blast. Without the air-blast the voltage was by no means constant, and the readings given in the first column are therefore a mean reading from which the extreme readings may differ by 10 per cent.

The final result of the experiments is to show that when using spark-gaps, 1, 2, or 3 mm. in width, in a condenser circuit, for the purpose of exciting oscillations, much greater uniformity in the discharge current can be obtained if the spark-gap is subjected to an air-blast as described.

LXIII. *On the Kinetic Criterion of Potential Energy.* *By* C. V. BURTON, *D.Sc.**

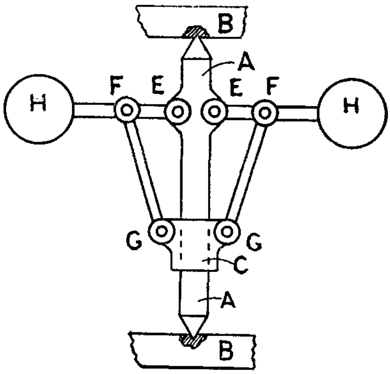
1. IN a large number of cases, if not in all, energy which we find it justifiable and convenient to treat as potential is found on a closer scrutiny to be essentially kinetic; and the object of this note is to supply an answer to the question: in what circumstances may kinetic energy be treated as potential?

* Communicated by the Author.

2. The potential energy of a conservative dynamical system is sometimes defined as that portion of the total energy which is a function of the coordinates of the system only, and is independent of the rates of change of those coordinates. Such a definition, however, is too inclusive, for in the energy-expressions of many dynamical systems there are terms which, although independent of the time-fluxes of the (working) coordinates *, do not fall into the category of potential energy.

3. Before treating the question more generally, it will be convenient to consider an example. Fig. 1 represents what

Fig. 1.



may be shortly called a governor, though we are not here concerned with its capacity for "governing." The spindle A A is frictionlessly journalled in a fixed frame B B, and the only other freedom of the system corresponds to a motion of the block C lengthwise of the spindle, with accompanying motion at the pivots E, F, G. For convenience of description, suppose the axis A A to be vertical, and as coordinates of the system take χ defining the azimuth of the governor, and z a length measured vertically downwards from a fixed horizontal plane to some point of the block C. Using dots over the coordinates to signify their time-fluxes, the kinetic energy may be expressed in the form

$$E=\tfrac{1}{2}M\dot{z}^2+\tfrac{1}{2}I\dot{\chi}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where I, the moment of inertia about A A, is a function of z, and M (a coefficient of the same kind as a mass) is also a function of z.

4. Let the only external force acting be a vertical force Z

* *Cf.* § 8 below.

(measured downward) on the block C; then the equations of motion are

$$\frac{d}{dt}(\mathrm{I}\dot{\chi})=0, \qquad \ldots \ldots \ldots \qquad (2)$$

$$\mathrm{Z}=\frac{d}{dt}\cdot\frac{\partial \mathrm{E}}{\partial \dot{z}}-\frac{\partial \mathrm{E}}{\partial z}$$

$$=\mathrm{M}\ddot{z}+\tfrac{1}{2}\dot{z}^2\frac{d\mathrm{M}}{dz}-\tfrac{1}{2}\dot{\chi}^2\frac{d\mathrm{I}}{dz}. \qquad \ldots \ldots \qquad (3)$$

Now

$$\frac{d}{dz}(\tfrac{1}{2}\mathrm{I}\dot{\chi}^2)_{\mathrm{I}\dot{\chi}\text{ constant}}=\frac{d}{dz}\left(\frac{1}{2}\frac{(\mathrm{I}\dot{\chi})^2}{\mathrm{I}}\right)_{\mathrm{I}\dot{\chi}\text{ constant}}$$

$$=-\tfrac{1}{2}\dot{\chi}^2\frac{d\mathrm{I}}{dz}. \qquad \ldots \ldots \qquad (4)$$

Hence, on the understanding that the differentiation with respect to z is to be performed with $\mathrm{I}\dot{\chi}$ constant (that is, with no turning moment applied to the governor, as expressed by (2)), we may replace (3) by

$$\mathrm{Z}=\mathrm{M}\ddot{z}+\tfrac{1}{2}\dot{z}^2\frac{d\mathrm{M}}{dz}+\frac{d}{dz}(\tfrac{1}{2}\mathrm{I}\dot{\chi}^2). \qquad \ldots \qquad (5)$$

5. This is precisely the form which the equation of motion corresponding to the z-coordinate would take if the velocity $\dot{\chi}$ were permanently zero, while the system, in place of the kinetic energy $\tfrac{1}{2}\mathrm{I}\dot{\chi}^2$, possessed potential energy of like amount. In other words, provided the angular momentum $\mathrm{I}\dot{\chi}$ remains constant, and so long as we confine our attention to the coordinate z, the energy of rotation may be treated as potential. The axial motion of the block C under given axial force (including of course the special case of free oscillations) will be the same as if the rotation were abolished, and suitable springs of negligible mass introduced; the equilibrium position of the governor under the action of such springs being approximately that shown in fig. 1.

6. The energy of rotation of the governor considered in the last three paragraphs is more palpably kinetic than anything which we are accustomed to classify as potential energy; but when we come to consider the dynamical criterion of potential energy, we shall find that, under the limitations above defined, the rotational energy in question is as much entitled to be called potential, as are the forms more commonly included under that designation.

7. It need hardly be pointed out that the potential energy (energy of rotation) with which we are concerned in §§ 3–5 may lose its potential character if the conditions of the motion are modified: for example, if the governor is acted on by forces having a moment about its axis of rotation, so that the angular momentum about that axis no longer remains constant; or if the axis A A is allowed to change its direction in space. In such cases the kinetic character of the rotational energy must be explicitly recognized in the dynamical equations.

8. In any conservative dynamical system let $\psi, \phi, \theta, \ldots$ be the coordinates which we employ to define the configuration at each instant, and let $\dot\psi, \dot\phi, \dot\theta, \ldots$ stand for the time-fluxes of these *working coordinates* (as they may be called). Then the kinetic energy of the system as ordinarily understood may be expressed as a h.q.f.* of $\dot\psi, \dot\phi, \dot\theta, \ldots$, with coefficients in general functions of $\psi, \phi, \theta, \ldots$, while the potential energy is a function of $\psi, \phi, \theta, \ldots$ only. As soon as we admit the kinetic nature of the energy which we treat as potential, we realize that in addition to the working coordinates $\psi, \phi, \theta, \ldots$ there must be others (say) $\chi, \chi', \chi'', \ldots$ whose time-fluxes $\dot\chi, \dot\chi', \dot\chi'', \ldots$ are involved in this so-called potential energy, and which may be distinguished as "ignored coordinates"†.

9. The Lagrangian function for any conservative dynamical system is the *difference* of the kinetic and potential energies, the energy ($\mathfrak{T}$) being expressed in terms of the generalized velocities $\dot\psi, \dot\phi, \dot\theta, \ldots$; on the other hand, the Hamiltonian reciprocal function is the *sum* of the kinetic and potential energies, the kinetic energy being expressed in terms of the momenta $\partial\mathfrak{T}/\partial\dot\psi, \partial\mathfrak{T}/\partial\dot\phi, \ldots$ In the first place, then, it is apparent that potential energy is energy expressed in the Hamiltonian form, in terms of momenta, not in the Lagrangian form, in terms of velocities.

10. Suppose at the outset that the entire energy T is recognized as kinetic, and is expressed as a h.q.f. of $\dot\psi, \dot\phi, \dot\theta, \ldots$ and the $\dot\chi$'s. T is then the Lagrangian function for the system, and the equations of motion corresponding to $\psi, \phi, \theta, \ldots$ are of the type

$$\frac{d}{dt}\cdot\frac{\partial T}{\partial\dot\psi}-\frac{\partial T}{\partial\psi}=\Psi, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$\Psi, \Phi, \Theta, \ldots$ being the impressed forces of types corresponding to $\psi, \phi, \theta, \ldots$ respectively.

* Here and below h.q.f. stands for "homogeneous quadratic function."

† Thomson (Kelvin) and Tait's 'Natural Philosophy,' Part 1. § 319.

11. Following Routh *, let us now *modify* the function T with respect to the coordinates $\chi, \chi', \ldots$, the modified function will be

$$T' = T - \frac{\partial T}{\partial \dot{\chi}}\dot{\chi} - \frac{\partial T}{\partial \dot{\chi}'}\dot{\chi}' - \ldots, \quad . \quad . \quad . \quad . \quad (7)$$

the $\dot{\chi}$'s being supposed eliminated from the right-hand of (7) by means of the equations

$$\partial T/\partial \dot{\chi} = C, \qquad \partial T/\partial \dot{\chi}' = C', \ldots \quad . \quad . \quad . \quad (8)$$

so that the C's are the momenta corresponding to the χ's respectively.

12. In a system such as we contemplate, the energy is made up of two parts: one a h.q.f. ($\mathfrak{T}$) of the velocities $\dot{\psi}, \dot{\phi}, \dot{\theta}, \ldots$, and the other (K) independent of those velocities; that is

$$T = \mathfrak{T} + K. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In this case $\mathfrak{T}$ is the energy which we recognize as kinetic, and K is that which we call potential.

13. It is of course understood, when $\psi, \phi, \theta, \ldots$ are the only "working coordinates," that the system is acted upon by no external forces except $\Psi, \Phi, \Theta, \ldots$; in these circumstances it is shown in Kelvin and Tait that (9) will be fulfilled provided the χ's do not appear in the coefficients of the energy expression T; and it is easy to show that (9) will not be fulfilled otherwise. Hence in place of (9) we may write the conditions

$$\partial T/\partial \chi = 0, \qquad \partial T/\partial \chi' = 0, \ldots \quad . \quad . \quad . \quad (9a)$$

14. Kelvin and Tait's analysis † relating to the ignoration of coordinates is therefore applicable, and we have

$$C, C', \ldots \text{ all constant}; \quad . \quad . \quad . \quad . \quad (10)$$

while K is a h.q.f. of the C's.

15. From the modified function T′ the equations of motion of the system may be obtained in the form

$$\frac{d}{dt}\frac{\partial T'}{\partial \dot{\psi}} - \frac{\partial T'}{\partial \psi} = \Psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

with similar equations for $\Phi, \Theta, \ldots$; and these by a process

* 'Rigid Dynamics,' vol. i. chap. viii.
† *Loc. cit.*

similar to that followed in Kelvin and Tait, may be reduced to the form in which they are obtained in that treatise, namely,

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{\psi}}\right)-\frac{\partial \mathfrak{T}}{\partial \psi}+\mathrm{C}\left\{\left(\frac{\partial \mathrm{M}}{\partial \phi}-\frac{\partial \mathrm{N}}{\partial \psi}\right)\dot{\phi}+\left(\frac{\partial \mathrm{M}}{\partial \theta}-\frac{\partial \mathrm{O}}{\partial \psi}\right)\dot{\theta}+\ldots\right\} \\ \quad +\mathrm{C}'\left\{\left(\frac{\partial \mathrm{M}'}{\partial \phi}-\frac{\partial \mathrm{N}'}{\partial \psi}\right)\dot{\phi}+\left(\frac{\partial \mathrm{M}'}{\partial \theta}-\frac{\partial \mathrm{O}'}{\partial \psi}\right)\dot{\theta}+\ldots\right\}+\ldots+\frac{\partial \mathrm{K}}{\partial \psi} \\ \qquad =\Psi, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad (12)$$

Here M, N, O,... M′, N′, O′,... are best defined for our purpose by the relations

$$\left.\begin{array}{l} \dot{\chi}=\frac{\partial \mathrm{K}}{\partial \mathrm{C}}-(\mathrm{M}\dot{\psi}+\mathrm{N}\dot{\phi}+\mathrm{O}\dot{\theta}+\ldots)=\frac{\partial \mathrm{K}}{\partial \mathrm{C}}+\dot{\chi}_1 \text{ (say)}, \\ \dot{\chi}'=\frac{\partial \mathrm{K}}{\partial \mathrm{C}'}-(\mathrm{M}'\dot{\psi}+\mathrm{N}'\dot{\phi}+\mathrm{O}'\dot{\theta}+\ldots)=\frac{\partial \mathrm{K}}{\partial \mathrm{C}'}+\dot{\chi}'_1. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad (13)$$

16. It is clear that (12) are precisely the equations of of motion of a system in which $\mathfrak{T}$ is the kinetic and K the potential energy, *provided that for all values of* $\dot{\psi}$, $\dot{\phi}$, $\dot{\theta}$,...

$$\left.\begin{array}{l} \Sigma\mathrm{C}\left\{\left(\frac{\partial \mathrm{M}}{\partial \phi}-\frac{\partial \mathrm{N}}{\partial \psi}\right)\dot{\phi}+\left(\frac{\partial \mathrm{M}}{\partial \theta}-\frac{\partial \mathrm{O}}{\partial \psi}\right)\dot{\theta}+\ldots\right\}=0, \\ \Sigma\mathrm{C}\left\{\left(\frac{\partial \mathrm{N}}{\partial \psi}-\frac{\partial \mathrm{M}}{\partial \phi}\right)\dot{\psi}+\left(\frac{\partial \mathrm{N}}{\partial \theta}-\frac{\partial \mathrm{O}}{\partial \phi}\right)\dot{\theta}+\ldots\right\}=0, \\ \Sigma\mathrm{C}\left\{\left(\frac{\partial \mathrm{O}}{\partial \psi}-\frac{\partial \mathrm{M}}{\partial \theta}\right)\dot{\psi}+\left(\frac{\partial \mathrm{O}}{\partial \phi}-\frac{\partial \mathrm{N}}{\partial \theta}\right)\dot{\phi}+\ldots\right\}=0. \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\} \quad . \quad (14)$$

Let m be the number of the working coordinates ψ, ϕ, θ, ...; then since in each of the m equations (14) the coefficients of $\dot{\psi}$, $\dot{\phi}$, $\dot{\theta}$, ... will have to vanish separately, and since $\dot{\psi}$ is absent from the first equation, $\dot{\phi}$ from the second, and so on, we obtain $m(m-1)$ equations which must be satisfied if K is to be treated as the potential energy of the system. These conditional equations, however, are not all independent, but consist of $\frac{1}{2}m(m-1)$ independent equations, each occurring twice ; thus the conditions sought for, in addition to (9*a*), are

$$\left.\begin{aligned}\Sigma C\left(\frac{\partial M}{\partial \phi}-\frac{\partial N}{\partial \psi}\right)=0, \quad \Sigma C\left(\frac{\partial M}{\partial \theta}-\frac{\partial O}{\partial \psi}\right)=0,\ldots \\ \Sigma C\left(\frac{\partial N}{\partial \theta}-\frac{\partial O}{\partial \phi}\right)=0,\ldots\end{aligned}\right\} . \quad (15)$$

. . . .

17. *Special Class* i. An interesting case is that in which there is but a single working coordinate, say ψ; so that the system, regarded from our standpoint, has only one degree of freedom. In this case the sole equation of motion is

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{T}}{\partial \dot{\psi}}\right)-\frac{\partial \mathfrak{T}}{\partial \psi}+\frac{\partial K}{\partial \psi}=\Psi; \quad . \quad . \quad . \quad . \quad (16)$$

which indicates that the energy K, due to the "ignored" momenta, necessarily behaves as potential energy, without any condition having to be satisfied beyond those expressed by (9*a*).

18. Returning now to (7) and making use of (8) and (13), we see that

$$\begin{aligned}T'&=T-C\frac{\partial K}{\partial C}-C'\frac{\partial K}{\partial C'}-\ldots \\ &\quad +C(M\dot{\psi}+N\dot{\phi}+\ldots)+C'(M'\dot{\psi}+N'\dot{\phi}+\ldots)+\ldots \\ &=T-C\frac{\partial K}{\partial C}-C'\frac{\partial K}{\partial C'}-\ldots-C\dot{\chi}_1-C\dot{\chi}_1'-\ldots \quad . \quad (17)\end{aligned}$$

Remembering that K is a h.q.f. of C, C′,..., we have accordingly

$$\begin{aligned}T'&=T-2K-C\dot{\chi}_1-C'\dot{\chi}_1'-\ldots \\ &=\mathfrak{T}-K-C\dot{\chi}_1-C'\dot{\chi}_1'-\ldots, \quad . \quad . \quad . \quad (18)\end{aligned}$$

by (9). Thus the modified function T′, from which the equations of motion (11) are derived, becomes identical with $\mathfrak{T}-K$ provided

$$C\dot{\chi}_1+C'\dot{\chi}_1'+\ldots=0 ; \quad . \quad . \quad . \quad . \quad . \quad (19)$$

that is, provided

$$\Sigma C(M\dot{\psi}+N\dot{\phi}+\ldots)=0, \quad . \quad . \quad . \quad . \quad (19a)$$

a condition which is the same as

$$\dot{\psi}\Sigma CM+\dot{\phi}\Sigma CN+\ldots=0. \quad . \quad . \quad . \quad (19b)$$

So long as this condition is satisfied, we may treat K as the potential energy of the system.

19. The condition (19), (19*a*) or (19*b*) necessarily implies (14), which must in any case be fulfilled if the energy K is to be treated as potential; in general, however, it includes more than is strictly demanded. But in most of the examples which readily present themselves, and in which K behaves as potential energy, the condition in question is in fact satisfied. If (19*b*) is to hold good *without any restriction being imposed on the velocities* $\dot{\psi}$, $\dot{\phi}$, ..., we shall have

$$\Sigma CM = 0, \qquad \Sigma CN = 0. \quad . \quad . \quad . \quad . \quad (20)$$

20. The C's being constants, it is evident that (14) will be satisfied provided

$$\Sigma CM, \quad \Sigma CN, \ldots \text{ are all independent of } \psi, \phi; \quad . \quad (21)$$

which relations, though in general expressing more than the requisite conditions (14), impose less restriction than do (20).

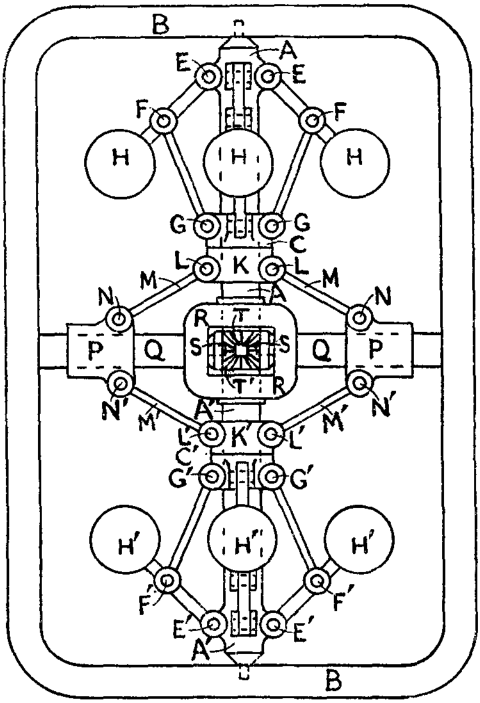
21. *Special Class* ii. A simple case, illustrating the condition (20), is when the M's, N's, ... all vanish, or in other words when, for given values of the working coordinates ψ, ϕ, θ, ... the velocities $\dot{\chi}$, $\dot{\chi}'$, ... of the ignored coordinates are determined solely by the momenta C, C′..., independently of the velocities $\dot{\psi}$, $\dot{\phi}$, $\dot{\theta}$, ...

22. *Special Class* iii. Another simple case is when the state of the system, as defined by the working coordinates ψ, ϕ, θ, ..., is one of *continued rest*. For when $\dot{\psi}$, $\dot{\phi}$, $\dot{\theta}$, ... are all constantly zero, (14) are satisfied. In any system, therefore, for which (9 *a*) hold good, the energy K—due to the momenta of the ignored coordinates—may be treated as potential energy in computing what forces, corresponding to the working coordinates, must be applied to the system to maintain it "at rest." If, however, the system is moved from one configuration to another, even infinitely slowly, although at each instant the generalized forces required to maintain (infinitely nearly) equilibrium will only differ infinitesimally from $\partial K/\partial\psi$, $\partial K/\partial\phi$, ... yet the time-integrals of the forces in question will in general be finitely different from those of $\partial K/\partial\psi$, $\partial K/\partial\phi$, ... unless (20), or failing that (21), or in any case unless (14) are satisfied.

23. Though the rotational energy of the governor represented in fig. 1 can no longer be treated as potential when the frame B B is permitted to turn without restriction of direction, it is easy to devise a pair of governors carried by a single frame, and so connected or so set in motion that their

energy of rotation may properly be treated as potential energy. In fig. 2 let B be a rigid frame, and let there be

Fig. 2.



two equal and similar governors journalled in this frame, so as to be capable of rotation about a common (geometrical) axis. The construction of the governors A C E F G H, A′C′E′F′G′H′ is essentially similar to that of the governor shown in fig. 1, like letters indicating like parts. Each governor, however, has four arms (instead of only two), each carrying a ball, and all symmetrically arranged about the axis A A. The rotatable system which constitutes the governor proper has of course moments of inertia which vary according to the inclination of the arms E H; but the axis A A is always a principal axis of inertia, and the moment of inertia is the same about all axes which intersect A A perpendicularly at a given point. This condition has to be fulfilled in order that the angular coordinates χ, χ' (defining the integral angular rotations of the governors relatively to the frame) may not appear in the coefficients of the expression for the kinetic energy of the system.

24. The collar K can partake of the axial motion of the block C, as the latter slides lengthwise of the axis A A; K is incapable of rotating with respect to the frame B, but is so connected to C as not to interfere with C's rotation. The parts in the second governor are indicated by accented letters, and are in all respects equal and similar to those in the first governor. The guides Q Q are integral with the frame B on the one hand, and with the inner frame R on the other hand, and upon these guides the blocks P P can slide without rotation. The pieces P K P′K′ are connected up as shown by the four links M M M′ M′, which are equivalent to a jointed parallelogram. For the present we leave out of consideration the mitre-gears S S T T′, to which reference will be made later.

25. It is evident that, in addition to any freedom of motion which the frame B may have, and exclusive of the rotational motions of the governors, the system made up of the two governors and the parallelogram linkwork has one degree of freedom, corresponding to a motion of each of the blocks P P (say) towards the centre of the framework, with simultaneous motion of K K′ away from the centre, and increase of the angle θ which the arm E H makes with the axis A A′. This angle θ may conveniently be taken as the coordinate corresponding to the freedom just defined. For the remaining coordinates of the system we may take x, y, z, the Cartesian coordinates of the centroid, and three angular coordinates defining the orientation of the frame B, together with the angles χ, χ' (measured from a standard configuration) through which the respective governors E H G, E′H′G′ have turned with respect to the frame B.

26. Let the system, *with the exception of the rotatable governors*, have the moment of inertia I_1 about the axis A A′, and at any instant (t) let the frame B be turning about the axis A A′ with angular velocity ω_1. Let I_2 be the moment of inertia of the whole system *including the governors* about the axis Q Q; and ω_2 the angular velocity of the frame B about the instantaneous axis Q Q; I_3, ω_3 being the corresponding quantities referred to an axis through the centroid of the system perpendicular to the plane of fig. 2. If M is the mass of the entire system, and $\mathfrak{I}$ the moment of inertia of either governor about its mechanical axis, the kinetic energy may be written

$$T=\tfrac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\tfrac{1}{2}I_1\omega_1{}^2+\tfrac{1}{2}I_2\omega_2{}^2+\tfrac{1}{2}I_3\omega_3{}^2+\tfrac{1}{2}I_0\dot{\theta}^2$$
$$+\tfrac{1}{2}\mathfrak{I}(\dot{\chi}+\omega_1)^2+\tfrac{1}{2}\mathfrak{I}(\dot{\chi}'+\omega_1)^2. \quad (22)$$

27. Here I_0 is a quantity of the same nature as a moment

of inertia, and it is to be understood that ω_1, ω_2, ω_3 are to be replaced by their values in terms of the time-fluxes of the angular coordinates of the system. Of the coefficients, M and I_1 alone are constants, I_2, I_3, I_0, and $\mathfrak{I}$ being functions of θ. In the absence of any external forces acting on the system, T is the Lagrangian function, which we proceed to *modify* with respect to the coordinates χ, χ'. The angular momenta of the governors about their axis are

$$\left.\begin{aligned} \partial T/\partial\dot\chi &= \mathfrak{I}(\dot\chi+\omega_1)=C \text{ (say)}, \\ \text{and}\quad \partial T/\partial\dot\chi' &= \mathfrak{I}(\dot\chi'+\omega_1)=C' \quad ,, \end{aligned}\right\}; \quad \ldots \quad (23)$$

Thus

$$\dot\chi=\frac{C}{\mathfrak{I}}-\omega_1, \quad \dot\chi'=\frac{C'}{\mathfrak{I}}-\omega_1. \quad \ldots \quad (24)$$

The modified Lagrangian function is thus

$$\begin{aligned} T' &= T - C\dot\chi - C'\dot\chi' \\ &= \tfrac{1}{2}M(\dot x^2+\dot y^2+\dot z^2)+\tfrac{1}{2}I_1\omega_1^2+\tfrac{1}{2}I_2\omega_2^2+\tfrac{1}{2}I_3\omega_3^2+\tfrac{1}{2}I_0\dot\theta^2 \\ &\quad +\frac{1}{2}\frac{C^2}{\mathfrak{I}}+\frac{1}{2}\frac{C'^2}{\mathfrak{I}}-C\left(\frac{C}{\mathfrak{I}}-\omega_1\right)-C'\left(\frac{C'}{\mathfrak{I}}-\omega_1\right); \end{aligned}$$

i. e.

$$\begin{aligned} T' &= \tfrac{1}{2}M(\dot x^2+\dot y^2+\dot z^2)+\tfrac{1}{2}I_1\omega_1^2+\tfrac{1}{2}I_2\omega_2^2+\tfrac{1}{2}I_3\omega_3^2+\tfrac{1}{2}I_0\dot\theta^2 \\ &\quad -\frac{1}{2}\frac{C^2}{\mathfrak{I}}-\frac{1}{2}\frac{C'^2}{\mathfrak{I}}+(C+C')\omega_1. \quad \ldots \quad (25) \end{aligned}$$

28. Since the rotatable governors are acted upon by no forces having a moment about the axes A A or A′A′, and since the coordinates χ, χ' do not appear in the coefficients of the expression T, the whole kinetic energy may be divided into two parts: one (to be called $\mathfrak{T}$) being a h.q.f. of the velocities $\dot x$, $\dot y$, $\dot z$, ω_1, ω_2, ω_3, $\dot\theta$, and the other (to be called K) being a function of the coordinate θ, and involving besides only the constant momenta C, C′. (25) may in fact be written

$$T'=\mathfrak{T}-K+(C+C')\omega_1; \quad \ldots \quad (26)$$

and we shall accordingly be able to treat the rotational energy K of the governors as potential, *provided the constant angular momenta* C, C′ *are equal and opposite.*

29. Alternatively, instead of the two governors being rotatable independently of one another, we may suppose them to be positively connected in such a way that their rotations with respect to the frame B are always equal and opposite. The

mitre-wheels T, T′, fixed upon the spindles A A, A′A′ respectively and gearing with a mitre-wheel S, would effect this result; a fourth mitre-wheel S being introduced for the sake of symmetry and balance. Though the omission is not essential to the case now considered, it will be assumed for simplicity that the rotational energy of the mitre-wheels S, S is relatively small enough to be left out of account.

30. The expression for the whole kinetic energy is obtained from (22) by substituting $-\chi$ for χ'; while the momentum corresponding to the χ-coordinate is now

$$\partial T/\partial\dot{\chi} = 2\mathfrak{K}\dot{\chi} = u \text{ (say)}; \quad \ldots \quad (27)$$

and the modified Lagrangian function is accordingly

$$\begin{aligned} T' &= T - u\dot{\chi} \\ &= \tfrac{1}{2}M(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \tfrac{1}{2}(I_1+2\mathfrak{K})\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2 \\ &\quad + \tfrac{1}{2}I_0\dot{\theta}^2 - \frac{1}{2}\frac{u^2}{2\mathfrak{K}}. \quad \ldots \quad (28) \end{aligned}$$

If the momentum u remains constant, the last term on the right-hand of (28) is the energy of rotation of the governors about their axes, with sign reversed. When u is given, this term is a function of θ only, and T′, given by (28), may accordingly be taken as made up of the difference of the kinetic and potential energies of the system. As regards the equations of motion corresponding to the working coordinates, the present example differs from that of § 28 in that the moment of inertia I_1 of the frame B about the axis A A′ is now effectively increased by $2\mathfrak{K}$, the sum of the moments of inertia of the two governors about their axes A A, A′A′.

31. An interesting example of potential energy is furnished by a system of perforated solids, immersed in a frictionless incompressible fluid which is circulating irrotationally through their various apertures. Let each of the solids be in the form of a thin rigid wire or wires, forming a closed loop or a framework. Then, provided no two solids approach one another very closely, the component of fluid motion contributed by any one of the solids is due almost exclusively to the cyclic constants of circulation associated with that solid, and is appreciably the same as if the remaining solids were non-existent. In the present case the working coordinates are any such as serve to define at each instant the position and orientation of every one of the solids, and each of the coordinates $(\chi, \chi', \ldots)$ to be subsequently ignored is the volume of liquid which, starting from a definite configuration, has flowed across one of the ideal geometrical surfaces required to close the various

apertures in the solids. We shall be able to treat the energy of the circulation-momenta as potential if certain conditions are satisfied which are equivalent to (9a) and (19). The condition (9a) (that the coordinates χ shall not appear in the coefficients of the expression for the total kinetic energy of the system) is obviously realized. But (19) will only be consistently fulfilled when the sum of the $C\dot{\chi}_1$'s for each solid is always zero ; the momentum C corresponding to the coordinate χ being $\kappa\rho$ where κ is the cyclic constant of circulation for the aperture in question, and ρ is the density of the liquid.

32. *For each solid* therefore there is a condition to be satisfied of the form

$$\Sigma\kappa\rho\dot{\chi}_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

the $\dot{\chi}_1$'s being homogeneous linear functions of $\dot{x}, \dot{y}, \dot{z}, \omega_1, \omega_2, \omega_3$, where ω_1, ω_2, ω_3 are the angular velocities of the solid about axes instantaneously coincident with a set of rectangular axes moving with the solid, and x, y, z are the Cartesian coordinates of the origin of those moving axes. For the particular solid under consideration, let

$$\left.\begin{array}{l}\dot{\chi}_1 = [\dot{x}]\dot{x} + [\dot{y}]\dot{y} + [\dot{z}]\dot{z} + [\omega_1]\omega_1 + [\omega_2]\omega_2 + [\omega_3]\omega_3 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\end{array}\right\} . \quad (30)$$

Then (19) or (19b) for that solid is equivalent to the six conditions

$$\left.\begin{array}{l}0 = \Sigma\kappa\rho[\dot{x}] = \Sigma\kappa\rho[\dot{y}] = \Sigma\kappa\rho[\dot{z}], \\ 0 = \Sigma\kappa\rho[\omega_1] = \Sigma\kappa\rho[\omega_2] = \Sigma\kappa\rho[\omega_3].\end{array}\right\} \quad . \quad . \quad (31)$$

33. Let S, S$'$, ... be geometrical surfaces invariably related to the solid with which we are dealing, and sufficing to close all its apertures. Then since $[\dot{x}]\dot{x}$, $[\dot{x}]'\dot{x}$, ... are the volumes of liquid flowing per unit time past the surfaces S, S$'$, .., owing to the velocity-component $\dot{x}$ of the body, we easily see that

$$[\dot{x}] = -\int\cos\nu\, dS, \quad [\dot{x}]' = -\int\cos\nu'\, dS', \quad . \quad . \quad (32)$$

where ν is the angle which the positively drawn normal at any point of the surface S makes with the axis of x, and so on. Thus the first of the conditions (31) may be written

$$\Sigma\,.\,\kappa\rho\int\cos\nu\, dS = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

34. Remembering that the $\kappa\rho$'s measure the impulsive pressures which must be applied over the surfaces S, S$'$, ... to

produce the circulations κ, we see that (33) amounts to this: that when all the impulsive pressures $\kappa\rho$ are so applied, there must, on the whole, be no component of impulse parallel to the axis of x (or of course to the axis of y or of z).

35. Similarly $\Sigma\kappa\rho[\omega_1]=0$, which is one of the conditions (31), may be put in the form

$$\Sigma\,.\,\kappa\rho\int r\cos\vartheta dS=0, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where r is the perpendicular distance of any point on the surface S from the axis about which ω_1 is measured, and ϑ is the angle which the normal to S at the point in question makes with a line perpendicular both to r and to the axis of ω_1, the positive direction of this latter line corresponding (let us suppose for definiteness) to the positive sense of the angular velocity ω_1. Now (34) expresses the condition that, when all the impulsive pressures $\kappa\rho$ act over the surfaces S, there shall be no resultant impulsive moment about the axis of ω_1 (or of course about the axis of ω_2 or of ω_3).

36. The results obtained in §§ 31–35 may be summarized as follows: If in a frictionless liquid free from vortex motion a number of solids are immersed, each consisting of a rigid framework of thin wires, then the energy of the circulation-momenta may be treated as potential energy, provided that for each single solid the impulses required to initiate all the circulations of that solid are such as, being applied to a rigid body, would be in equilibrium.

37. A dynamical system of the kind just considered may also be made to furnish an example of the fulfilment of the conditions (21). With our previous stipulation as to the thinness of the wires of which the solids are built up, it is evident that (21) will be satisfied, provided only that each body of the system is limited to translational freedom, without the possibility of rotation. In this case, no matter what may be the values of the circulation-momenta, the energy due to those momenta may be treated as potential energy, although in general the translational movement of any solid will involve reactions against the constraints arising from what may be called "want of balance" of the circulation-momenta.

38. The case of a mass of gas whose pressure is varied adiabatically may serve as a final example; isothermal conditions are excluded from consideration, as in such case the system is not properly speaking conservative, although simulating a conservative system in its general dynamical behaviour. For simplicity let the gas be monatomic and be contained in a fixed cylinder, in which works a gas-tight

frictionless piston, both cylinder and piston being impermeable to heat, and the axis of the cylinder being vertical with the piston uppermost. If, as a first approximation, we neglect the inertia of the gas in comparison with that of the piston, the single working coordinate is the height (z) of the piston above the bottom of the cylinder, the kinetic energy being made up of $\frac{1}{2}M\dot{z}^2$ (where M is the mass of the piston) and the $\dot{\chi}$'s, the χ's being the coordinates necessary for the complete specification of the distribution of gas-molecules when z is given. On differentiating with respect to the various $\dot{\chi}$'s it is evident that the momenta thus obtained are homogeneous linear functions of the $\dot{\chi}$'s, so that the $\dot{\chi}$'s are likewise homogeneous linear functions of the momenta, and the whole kinetic energy is equal to $\frac{1}{2}M\dot{z}^2$ together with a h.q.f. of the χ-momenta. The further condition which is sufficient to ensure that the energy of the χ-momenta shall have the potential character is that, when these momenta are given, none of the χ's shall involve $\dot{z}$ (though they may and do involve z). This is consonant with the assumptions which we make when we propose to treat as potential energy the translational energy of the gas-molecules.

39. Similar considerations are readily applied to a differential volume-element of a gas through which sound-waves are travelling. That energy of the element which we commonly treat as kinetic is its energy of translational motion, corresponding to velocity-components $\dot{x}$, $\dot{y}$, $\dot{z}$ of its mass-centre: while the energy of the momenta corresponding to the remaining (ignored) coordinates of the gas-molecules which make up the element is independent of $\dot{x}$, $\dot{y}$, $\dot{z}$, as are also the velocities of the ignored coordinates when the corresponding momenta have assigned values.

LXIV. *Direct Application of the Electron Theory to Induction Currents.* By R. J. A. Barnard, *M.A., Melbourne* *.

A MOVING charge of electricity in a magnetic field is acted on by the electromagnetic force $e(\mathbf{v}\times\mathbf{H})$ in Gibbs's Vector notation, where $\mathbf{v}$ is the velocity. Consequently, if electrons are moving about in a conductor, even when no current is flowing an electromagnetic force is acting on each electron when there is an external magnetic field. But since the electrons are moving in such a case impartially in all directions, there can be no resultant effect produced by these forces. Even if a current is flowing, the resultant effect of the electromagnetic force will be perpendicular to

* Communicated by the Author.