*Notes on Quantics of Alternate Numbers, used as a means for determining the Invariants and Covariants of Quantics in general. By the late Prof. CLIFFORD, F.R.S.**

[Read April 10th, 1879.]

The term *alternate numbers* means a set (λ_1, λ_2) or sets of numbers which satisfy the following relations:

$$\lambda_1^2 = 0, \quad \lambda_2^2 = 0, \quad \lambda_1\lambda_2 + \lambda_2\lambda_1 = 0 \dots\dots\dots (1);$$

to which it is usual to add $\lambda_1\lambda_2 = 1$.

The above set is binary, but there may be ternary, &c. sets, or sets consisting of any number of letters $\lambda_1, \lambda_2, \dots$

By a *Quadratic Form* is here meant an expression—lineo-linear in two sets of such numbers regarded as variables, say $\lambda_1, \lambda_2; \mu_1, \mu_2$, such as

$$a_{11}\lambda_1\mu_1 + a_{12}\lambda_1\mu_2 + a_{21}\lambda_2\mu_1 + a_{22}\lambda_2\mu_2 \dots\dots\dots (2).$$

This may be also denoted, for shortness, by the symbol $a \mid 12$, or even by 12, where the 1, 2 refer to the two sets of variables λ, μ .

In this case there is one invariant only which allows of the variables being separately transformed, namely, the discriminant, which is got by squaring the form. We have, in fact, by the properties of alternate numbers, $(a \mid 12)^2 = -2(a_{11}a_{22} - a_{12}a_{21}) = -2D$, suppose $\dots\dots\dots (3)$.

But if the variables are transformed by the same substitution, there is a universal covariant, $\lambda_1\mu_2 - \lambda_2\mu_1$, which may be denoted by (12), or by (21); for $\lambda_1\mu_2 - \lambda_2\mu_1 = \mu_1\lambda_2 - \mu_2\lambda_1$, by the property of alternate numbers.

If we replace the μ 's by x , an ordinary, not an alternate, number, we get a linear function of the λ 's, whose square vanishes in virtue of the relations (1). Thus we have $(a \mid 1x)^2 = 0$, and consequently such a product as $(a \mid 1x)(a \mid 1y)$ must be divisible by $(xy) = x_1y_2 - x_2y_1$

* This is the substance of some fragments found amongst the papers of the late Professor Clifford. The only published explanation of the method with which I am acquainted, is contained in the following short paragraph from a letter to Professor Sylvester (see "American Journal of Pure and Applied Mathematics," Vol. i., p. 127): "I consider forms which are linear in a certain number of sets of k variables each. To fix the ideas, suppose $k = 2$, and that I have altogether 6 sets of 2 variables each, namely,

$$x_1x_2, \quad y_1y_2, \quad z_1z_2, \quad u_1u_2, \quad v_1v_2, \quad w_1w_2.$$

Suppose the forms are $(xyzw), (yzvw), (xv), (uw)$;

viz., $(xyzw)$ means an expression separately linear, and homogeneous in the x , the y , the z , the u , and so for the rest. I observe that in these four forms each set of variables occurs twice. This being so, there is one invariant of the four forms, which is invariant in regard to independent transformations of the six sets of variables . . . To obtain this invariant, regard the variables as alternate numbers, and simply multiply all the forms together. The product of the forms will then be equal to the invariant in question multiplied by the product of all the variables."—[W. SPOTTISWOODE.]

since it vanishes when x and y represent the same point. In fact, if we write the expressions in full, thus,

$$\begin{aligned}(a \mid 1x) &= (a_{11}x_1 + a_{12}x_2) \lambda_1 + (a_{21}x_1 + a_{22}x_2) \lambda_2, \\ (a \mid 1y) &= (a_{11}y_1 + a_{12}y_2) \lambda_1 + (a_{21}y_1 + a_{22}y_2) \lambda_2,\end{aligned}$$

actual multiplication gives

$$\begin{aligned}(a \mid 1x)(a \mid 1y) &= a_{11}x_1 + a_{12}x_2, \quad a_{21}x_1 + a_{22}x_2 = a_{11}, \quad a_{12} \times x_1, \quad x_2 \\ &\quad a_{11}y_1 + a_{12}y_2, \quad a_{21}y_1 + a_{22}y_2 \quad a_{21}, \quad a_{22} \quad y_1, \quad y_2.\end{aligned}$$

But $a_{11}a_{22} - a_{12}a_{21}$ may be regarded as the discriminant of any quadratic form having a 's for its coefficients, say any form $(a \mid 1u)$. And, since it was shown, by equation (3), that the square of such a form is equal to minus twice its discriminant, it follows that the above equation may be written thus,

$$-2(a \mid 1x)(a \mid 1y) = (a \mid 1u)^2(xy) \dots\dots\dots (4).$$

Note.—The transformation

$$y_1, y_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} (x_1, x_2)$$

may also be written

$$a_{11}x_1y_1 + a_{21}x_1y_2 + a_{12}x_2y_1 + a_{22}x_2y_2 = 0 = (a \mid yx),$$

where the y 's are arbitrary functions satisfying identically

$$y_1y_1 + y_2y_2 = 0.$$

In like manner, a second transformation

$$z_1, z_2 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} (y_1, y_2)$$

may be represented by $(b \mid zy) = 0$; and the result of the two by

$$(b \mid zy)(a \mid yx) = 0.$$

There are one or two other formulæ which it will be convenient to notice before proceeding further. If we multiply the form $(a \mid 12)$ by (12), we get

$$(12)(a \mid 12) = a_{21} - a_{12} = s \text{ (suppose) } \dots\dots\dots (5),$$

also, if we suppose the coefficients to remain unaltered, viz., if $(a \mid 21)$ represents $a_{11}\mu_1\lambda_1 + a_{12}\mu_1\lambda_2 + \dots$, then

$$(a \mid 12) + (a \mid 21) = -s(12) \dots\dots\dots (6),$$

or

$$2(a \mid 12) = (a \mid 12) - (a \mid 21) - s(12),$$

the quantity s vanishing when the form is symmetrical, i.e., when $a_{12} = a_{21}$. Again, if the coefficients be supposed to remain unaltered, so that $(a \mid 13) = a_{11}\lambda_1\nu_1 + a_{12}\lambda_1\nu_2 + \dots$, then it will be found that

$$\begin{aligned}(12)(a \mid 23) &= -(a \mid 13) \} \\ (13)(a \mid 12) &= -(a \mid 32) \} \dots\dots\dots (7).\end{aligned}$$

Such multiplication is in fact tantamount to a substitution of variables, from 2 to 1 (*i.e.*, from μ to λ), or from 1 to 3 (*i.e.*, from λ to ν). This theorem, as will readily be seen, is not restricted to quadric forms; but if a , b be any two different forms, and if 1 be a set of variables in a and not in b , and 2 a set of variables in b and not in a , then, making the same supposition as in b , *viz.*, that the constants a , b remain the same on both sides of the equation, we shall find that

$$(12) (a | 1 \dots) (b | 2 \dots) = (a | 2 \dots) (b | 2 \dots) = (a | 1 \dots) (b | 1 \dots).$$

With reference to the universal covariant $(12) = \lambda_1 \mu_2 - \lambda_2 \mu_1$, it may be remarked that we may consider such covariants for more than two sets of variables; and we shall then obtain the following formulæ: *

$$(12) (23) = -(13), \quad (12) (23) (34) = (14), \dots \dots (7),$$

of which the following is a consequence,

$$(12) (23) (31) = 0 \dots \dots \dots (8).$$

It is clear that quadratic forms can only be combined in a *chain*,† which, when open, gives a quadratic covariant; when closed, an invariant. We can now show that a closed chain of $2n$ sides is equal to $\pm 2D^n$, while a chain of an odd number of sides vanishes.

Take a chain of four sides, (12) , (23) , (34) , (41) ; then, since

$$(12) (23) (34) (41) = (12) (23) \times (34) (41);$$

and since, by (3) and (4),

$$2 (12) (23) = (a | 2\psi)^2 (31) = -2D (31),$$

$$2 (34) (41) = (a | 4\psi)^2 (13) = -2D (13),$$

$$(31) (13) = 2,$$

it follows that $(12) (23) (34) (41) = 2D^2 \dots \dots \dots (9).$

In like manner any chain of an even number of sides may be resolved into a power of the discriminant multiplied by a chain of determinants of the alternate variables. The product of these latter is ± 2 , according as the number of terms is even or odd. Hence, generally, a chain of $2n$ sides is $2 (-D)^n$.

* This and other parts of the present paper may be compared with Spottiswoode "On Determinants of Alternate Numbers," "Proceedings of the London Mathematical Society," Vol. vii., p. 100.

† I have, in this paragraph, retained the language of the original MS., although the term *chain* is not here explained. The author appears to have had in his mind a theory which he propounded verbally at the meeting of the British Association at Bristol, and to which, in his latter days, he attached great importance and devoted much time. Some indications of it will be found in the "American Journal of Pure and Applied Mathematics," Vol. i., p. 127. Several notes relating to the subject have been found amongst his papers, but as they are almost exclusively memoranda without explanation, it is still uncertain whether they can be published.

A chain of $2n+1$ sides may, by analogous processes, be reduced to the product of a determinant of alternate variables by a form containing those variables, and must therefore vanish for symmetrical forms. For unsymmetrical it contains s as a factor. In this case the fundamental formula must be modified as follows:

$$(a | x1) (a | 1y) = (a | xy) s - (xy) D;$$

or. with alternate numbers in the place of x, y ,

$$(a | 12) (a | 23) = (a | 13) s - (13) D.$$

Multiplying now into $(a | 31)$, we get

$$(a | 12) (a | 23) (a | 31) = (a | 13) (a | 31) s + sD$$

in virtue of equation (5).

$$\begin{aligned} \text{But } (a | 13) (a | 31) &= (a_{11}\lambda_1\mu_1 + \dots) (a_{11}\mu_1\lambda_1 + \dots) \\ &= (2a_{11}a_{22} - a_{12}^2 - a_{21}^2) = 2D - s^2. \end{aligned}$$

$$\text{Hence } (a | 12) (a | 23) (a | 31) = (3D - s^2) s \dots\dots\dots (10).$$

$$\begin{aligned} \text{Again, } (a | 12) (a | 23) (a | 34) &= (a | 13) (a | 14) s - (13) (a | 34) D \\ &= (a | 14) s^2 - (14) sD + (a | 14) D \\ &= (a | 14) (D + s^2) - (14) sD; \end{aligned}$$

whence also

$$(a | 12) (a | 23) (a | 34) (a | 41) = (2D - s^2) (D + s^2) + s^2 D \dots\dots (11).$$

Suppose, in general, that

$$(a | 12) (a | 23) \dots (a | lm) = A_m (a | 1m) - B_m (1m);$$

then

$$\begin{aligned} (a | 12) (a | 23) \dots (a | mn) &= A_m (a | 1m) (amn) + B_m (a | 1n) \\ &= A_m \{ (a | 1n) s - (1n) D \} + B_m (a | 1n) \\ &= (sA_m + B_m) (a | 1n) - A_m D (1n). \end{aligned}$$

$$\text{Hence } A_{m+1} = sA_m + B_m, \quad B_{m+1} = A_m D;$$

$$\text{and consequently } A_{m+1} = sA_m + DA_{m-1} \dots\dots\dots (12).$$

The form $\lambda_1\mu_1 + 0\lambda_1\mu_2 + s\lambda_2\mu_1 + D\lambda_2\mu_2$ is a form having s and D for invariants; and it may therefore be taken as the canonical form for a bipartite quantic such as we have been here considering. Again, another useful form is the following

$$\sqrt{(s^2 - 4D)} (\lambda_1\mu_2 + \lambda_2\mu_1) + s(\lambda\mu).$$

We may, however, combine not only two quadratic forms having like coefficients, say, two forms a ; but we may also combine two having different coefficients, say, two forms a and b . Two such forms give

rise to an invariant, namely, their product. In fact, we have

$$-(a | 12)(b | 12) = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11} = D_{ab}, \text{ suppose ;}$$

and from (5), $(a | 12)(b | 12) + (a | 21)(b | 12) = -s_a s_b$,

or $(a | 21)(b | 12) = D_{ab} - s_a s_b \dots \dots \dots (13).$

The product of $(a | 12) = a_{11}\lambda_1\mu_1 + \dots$ and $(b | 13) = b_{11}\lambda_1\nu_1 + \dots$ gives a covariant; namely,

$$\begin{aligned} (a | 12)(b | 13) &= \begin{vmatrix} a_{11}\mu_1 + a_{12}\mu_2 & a_{21}\mu_1 + a_{22}\mu_2 \\ b_{11}\nu_1 + b_{12}\nu_2 & b_{21}\nu_1 + b_{22}\nu_2 \end{vmatrix} \\ &= \begin{vmatrix} a_{11}b_{21} - a_{21}b_{11} & a_{11}b_{22} - a_{21}b_{12} \\ a_{12}b_{21} - a_{22}b_{11} & a_{12}b_{22} - a_{22}b_{12} \end{vmatrix} (\mu_1, \mu_2)(\nu_1, \nu_2) \\ &= J_{ab} | 23, \text{ suppose } \dots \dots \dots (14). \end{aligned}$$

If, in the covariant, we make the forms a and b coincident, it becomes

$$(a | 12)(a | 13) = D_a(23), \quad (a | 12)(a | 23) = s_a(a | 13) - D_a(13);$$

and, multiplying this by, or into, b , we get

$$\begin{aligned} (a | 12)(a | 13)(b | 43) &= (b | 43)(a | 12)(a | 13) = -D_a(b | 42), \\ (a | 12)(a | 13)(b | 34) &= (b | 43)(a | 12)(a | 13) = -D_a(b | 24), \\ \text{and} \quad (a | 12)(a | 13)(b | 23) &= D_a s_b, \\ (a | 12)(a | 13)(b | 32) &= D_a s_b \dots \dots \dots (15). \end{aligned}$$

The remaining form to be investigated is $(a | 12)(b | 13)(a | 43)$. The value of this is

$$\begin{aligned} &(a_{11}b_{21} - a_{21}b_{11})\mu_1 + (a_{12}b_{21} - a_{22}b_{11})\mu_2, \quad (a_{11}b_{22} - a_{21}b_{12})\mu_1 + (a_{12}b_{22} - a_{22}b_{12})\mu_2 \\ &\quad a_{11}\rho_1 + \quad a_{21}\rho_2, \quad a_{12}\rho_1 + \quad a_{22}\rho_2 \\ &= (-a_{21}a_{12}b_{11} + a_{11}a_{21}b_{12} + a_{11}a_{12}b_{21} - a_{11}a_{11}b_{22})\mu_1\rho_1 \\ &\quad + (-a_{21}a_{22}b_{11} + a_{21}a_{21}b_{12} + a_{11}a_{22}b_{21} - a_{11}a_{21}b_{22})\mu_1\rho_2 \\ &\quad + (a_{12}a_{12}b_{21} - a_{22}a_{12}b_{21} - a_{11}a_{12}b_{22} + a_{11}a_{22}b_{12})\mu_2\rho_1 \\ &\quad + (a_{22}a_{12}b_{21} - a_{22}a_{22}b_{11} - a_{21}a_{12}b_{22} + a_{21}a_{22}b_{12})\mu_2\rho_2 \\ &= (-a_{11}D_{ab} + b_{11}D_{aa})\mu_1\rho_1 \\ &\quad + (-a_{21}D_{ab} + b_{21}D_{aa})\mu_1\rho_2 \\ &\quad + (-a_{12}D_{ab} + b_{12}D_{aa})\mu_2\rho_1 \\ &\quad + (-a_{22}D_{ab} + b_{22}D_{aa})\mu_2\rho_2 \\ &= D_{ab}(a | 42) - D_{aa}(b | 42) \dots \dots \dots (16). \end{aligned}$$

Multiplying this into $(b | 45)$, we obtain

$$\begin{aligned} (a | 12)(b | 13)(a | 43)(b | 45) \\ = D_{ab}(a | 42)(b | 45) - D_{aa}(b | 42)(b | 45); \end{aligned}$$

that is, referring to equation (14), and dropping the suffix ab ,

$$(\mathfrak{J} \mid 23)(\mathfrak{J} \mid 35) = D_{ab}(\mathfrak{J} \mid 25) - D_{aa} \cdot D_{bb}(25).$$

But $(\mathfrak{J} \mid 23)(\mathfrak{J} \mid 35) = s(\mathfrak{J}_3 \mid 25) - D_3(25).$

Hence $s_3 = D_{ab}, \quad D_3 = D_{aa} \cdot D_{bb} \dots\dots\dots (17),$

as may be easily verified.

If, however, we multiply the equation $(a \mid 12) + (a \mid 21) = -s(12)$ by $(a \mid 12)$, we obtain

$$(a \mid 12)(a \mid 21) = 2D_a - s_a^2;$$

hence $(\mathfrak{J} \mid 12)(\mathfrak{J} \mid 21) = 2D_3 - s_3^2 = 2D_a D_b - D_{ab}^2 \dots\dots\dots (18),$

and consequently

$$\{(\mathfrak{J} \mid 12) - (\mathfrak{J} \mid 21)\}^2 = -4D_a D_b - 2(2D_a D_b - D_{ab}^2) \} \dots\dots (19). \\ = -2(4D_a D_b - D_{ab}^2)$$

The formula $s_3 = D_{ab}$ is important as showing that $\mathfrak{J}$ is not made symmetrical by making a and b symmetrical. Hence, in passing from these invariants and covariants to those of symmetrical forms, we are obliged to use the symmetrised form of $\mathfrak{J}$, namely $(\mathfrak{J} \mid 12) - (\mathfrak{J} \mid 21)$. Thus, to adapt equation (16) to symmetrical forms, we have

$$(\mathfrak{J} \mid 23)(a \mid 43) = D_{ab}(a \mid 42) - D_{aa}(b \mid 42), \\ (\mathfrak{J} \mid 23)(a \mid 43) = (a \mid 13)(b \mid 12)(a \mid 43) \\ = D_a(14)(b \mid 12) = -D_a(b \mid 42).$$

In general, we write $\bar{a}$ for the mean value of a , i.e., if

$$2(\bar{a} \mid 12) = (a \mid 12) - (a \mid 21),$$

we shall have $(\bar{a} \mid 12) = (a \mid 12) + \frac{1}{2}s(12).$

Hence, since $(12)^2 = -2$,

$$(\bar{a} \mid 12)^2 = (a \mid 12)^2 + s^2 - \frac{1}{2}s^2 \dots\dots\dots (20),$$

and

$$\bar{D} = D - \frac{1}{4}s^2,$$

where $\bar{D}$ is the discriminant of the symmetrical function.

Similarly

$$\mathfrak{J}_{ab} = (\bar{a} \mid 12)(\bar{b} \mid 13) = \{(a \mid 12) + \frac{1}{2}s_a(12)\} \{(b \mid 13) + \frac{1}{2}s_b(13)\} \\ = \mathfrak{J}_{ab} - \frac{1}{2}\{s_a(b \mid 23) + s_b(a \mid 23)\} + \frac{1}{4}s_a s_b(23),$$

and therefore $2\bar{\mathfrak{J}}_{ab} = (\mathfrak{J}_{ab} \mid 23) - (\mathfrak{J}_{ab} \mid 32) - s_a s_b(23) \} \dots\dots\dots (21). \\ = 2(\mathfrak{J}_{ab} \mid 23) - s_3(23) - s_a s_b(23)$

If we multiply this into $(a \mid 43)$, we have

$$2(\mathfrak{J}_{ab} \mid 23)(a \mid 43) = 2(\mathfrak{J} \mid 23)(a \mid 43) - D_{ab}(a \mid 42) - s_a s_b(a \mid 42) \\ = D_{ab}(a \mid 42) - 2D_a D_b(b \mid 42) - s_a s_b(a \mid 42).$$

Putting $s_a = 0, s_b = 0$, we get the formula for symmetric functions.

May 8th, 1879.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

Messrs. A. J. C. Allen and Edwyn Anthony were elected Members; and Messrs. Allen, Hargreaves, and McAlister were admitted into the Society.

Dr. Hirst, F.R.S., presented a carte-de-visite likeness of M. Chasles to the Society's Album; and Mr. Tucker, on behalf of Mrs. Clifford, presented a likeness of the late Prof. Clifford to take the place of the earlier likeness in the Album.

The following communications were made:—

"Note on the Complexes generated by two Correlative Planes:" Dr. Hirst, F.R.S.

"On a Theorem relating to Conformable Figures:" by Prof. Cayley, F.R.S.

"Notes on some Definite Integrals:" by late Prof. Clifford (communicated by Mr. Tucker).

"A Method of Constructing by Pure Analysis Functions X , Y , &c., which possess the property that $\int XY d\sigma = 0$, and such that any given function can be expanded in the form $\alpha X + \beta Y + \gamma Z + \dots$:" by Mr. Routh, F.R.S.

"The Numerical Calculation of a Class of Determinants, and a Continued Fraction:" by Mr. J. D. H. Dickson, M.A.

"The Nephroid, Heptagon, &c.:" by Rev. Dr. Freeth.

The following presents were received:—

(Second Memoir) "On the Equations of Circles:" by Dr. Casey, F.R.S. ("Trans. of Royal Irish Academy," Vol. xxvi.): from the Author.

"Educational Times," May, 1879.

"Monatsbericht," Januar, Februar, 1879.

"Proceedings of Royal Society," Vol. xxviii., No. 195.

"Journal of Institute of Actuaries," Vol. xxi., Pt. iv., No. cxiv., Jan., 1879.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome ii., Dec. 1878.—"Table des Matières, et Noms d'Auteurs," Tome ii. (xiii. de la collection); Tome iii. Janv. 1879 (xiv. de la collection).

"Atti della R. Accademia dei Lincei—Transunti," Vol. iii., Fasc. 5°, Apr. 1879.

"Archiv for Mathematik og Naturvidenskab," Første Bind, Fjerde Hefte; Andet Bind, Første, Andet, Tredie, Fjerde Hefte; Tredie Bind, Første, Andet, Tredie Hefte.