LXVIII. *On the Statistical Relations of Radiant Energy.**By G. H. LIVENS*.*1. *Introduction.*

THERE still appears to be some uncertainty as to the principles on which the statistical and thermodynamical relations of radiant energy are to be based, and as to the final expression for the law of radiation which follows from these principles. The following considerations are offered in an attempt to elucidate the difficulties involved in the more usual modes of formulation of the problem.

Planck's method involves the consideration of an enclosure filled with radiation involving an entirely arbitrary succession of phases and polarizations along each ray, and also containing a system of fixed linear electric oscillators of the Hertzian type, which are taken to represent the transforming action of radiating and absorbing matter. The radiation contained in the enclosure will be passed through these oscillators over and over again, now absorbed, now radiated, and each constituent will then settle down in a unilateral or irreversible manner towards some definite intensity and composition, which is determined by the density of the energy in the æther and its distribution among the various wave-lengths. Denoting now by $E_\lambda d\lambda$ this energy density as far as it refers to the radiation with wave-lengths between λ and $\lambda + d\lambda$, then we know from the laws of Kirchhoff, Boltzmann, and Wien, that E_λ is of the form

$$E_\lambda = \frac{1}{\lambda^5} \phi(\lambda \cdot T),$$

T being the absolute temperature of the matter which is in equilibrium of exchanges of energy with the radiation. The actual form of the function ϕ can only be determined by probability considerations of the nature of those introduced by Boltzmann in gas theory. In a system of Hertzian vibrators, such as is discussed by Planck, with known properties and in given circumstances, there is a definite probability of the occurrence of each statistical distribution of energy that is finally possible when all velocities consistent with given total energy are considered to be equally likely as regards each vibrator. The distribution of greatest possible probability is then, after Boltzmann, assumed to be the state of thermal equilibrium for the system, and the probability of any other state is the usual function of the entropy

* Communicated by the Author.

of that state. By means of *a priori* considerations of this nature, Planck has been guided to an expression for E_λ of the form

$$E_\lambda = \frac{c_1}{\lambda^5} \frac{hc}{e^{\frac{hc}{\lambda kT}} - 1},$$

which corresponds exceedingly well with the empirical law deduced from the experimental data. His considerations, however, involve, as Lorentz points out, the assumption of a finite atomic structure for the energy in radiation.

In the hope of elucidating the difficulties in this mode of formulation of the problem, Rayleigh* and Jeans† have applied the same fundamental argument, but in another and more direct manner. These authors have examined the partition of energy among the various oscillatory degrees of freedom of a mass of some elastic substance enclosed in perfectly reflecting walls. After the number of possible modes of oscillation has been counted, each of them is regarded as corresponding to a possible degree of freedom in the ordinary dynamical sense, which can share equally well with all the degrees of freedom of other types in the general distribution, so that in the steady chaotic distribution of the vibrations in the medium the amount of energy associated with each period is the same. In this way a formula is obtained for ϕ which gives E_λ in the form

$$E_\lambda = CT \frac{d\lambda}{\lambda^4},$$

a result which is not only in total disagreement with experimental facts, but also with ordinary common sense.

Thus, on the one hand, we find that the result deduced from ordinary considerations is too absurd to be correct, whereas, on the other hand, a theory which is based on a very doubtful hypothesis leads to a result which if not actually correct is nevertheless so closely in agreement with experiment that it may for the present be so regarded. The following considerations, evolved in an attempt to obtain greater consistency in the general treatment of the statistical aspect of this problem, will show, however, that the two points of view are in reality not so much opposed to one

* Lord Rayleigh, "Remarks upon the law of complete radiation," *Phil. Mag.* [5] xlix. p. 539 (1900).

† J. H. Jeans, "On the partition of energy between matter and æther," *Phil. Mag.* [6] x. p. 91 (1905).

another as at first sight appears, and that if certain plausible assumptions are made the correct formula given by Planck can be obtained from both theories, and now without any implication of a fundamental difficulty such as appeared to be involved in Planck's theory.

2. *The constitution of natural radiation.*

In both the methods of Jeans and Rayleigh and of Planck, it is generally assumed that the electromagnetic field at any point traversed by a stream of steady radiation can be resolved into its harmonic constituents by a Fourier series of type

$$\sum_{n=1}^{\infty} a_n \cos \left(\frac{2\pi c t}{\lambda_n} - \theta_n \right),$$

wherein $\lambda_n = \frac{\lambda}{n}$, λ being a constant, introduced for the pur-

poses of the analysis, which is arbitrarily chosen in Planck's case, but is directly related to the dimensions of the elastic solid mass in the discussions of Jeans and Rayleigh.

An assumption of this nature implies that the complete spectrum of thermal radiation can be regarded as composed of perfectly harmonic constituents of wave-lengths

$$\frac{\lambda}{1}, \frac{\lambda}{2}, \frac{\lambda}{3}, \dots, \frac{\lambda}{n}, \dots$$

But whatever view we may take as to the conditions governing the production and distribution of thermal radiation, it seems, to me at least, very difficult to avoid the assumption of theoretically perfect continuity in its spectrum. When we consider the great variety of actions which are effective in altering to various extents the period of any monochromatic vibrator in a piece of matter, it will be difficult for us to avoid the assumption of equal opportunity for every possible wave-length. Perfect continuity is apparently a necessary hypothesis in a statistical theory of this subject.

The conclusion is therefore to be drawn that such a series of wave-lengths as that given above cannot under any circumstances be regarded as forming the complete spectrum of thermal radiation. In technical language the point involved appears to be that the complete linear set of points on a line indefinitely extended in one direction has the power of the continuum, and cannot in any way be regarded as made up of an enumerable set of points.

If the spectrum of the radiation is perfectly continuous

the energy in any enumerable infinity of perfectly monochromatic constituents must be zero. It would appear, in fact, that it is only when the aggregate of all possible wave-lengths for the system has a non-zero measure that a finite amount of energy can be associated with it in the steady state.

If we accept this conclusion we must of course deny the possibility of resolving the radiation by a Fourier analysis. But this is, after all, what we should be inclined to expect. In fact, any function can only be so resolved if d has in the interval $\left(0, \frac{\lambda}{c}\right)$ of t a finite number of singularities (discontinuities or turning values); and when we consider that a ray or filament of light even sensibly homogeneous of infinitesimal range ($\delta\lambda$) of wave-length, is not a train of uniform waves, but the aggregate of a vast complex of trains of limited lengths, coming from the various molecules or electrons that take part in the radiation, it would appear that no such condition as that mentioned can be fulfilled by any component of the electromagnetic field at a point traversed by the radiation, when examined over a finite time.

The argument does not, of course, materially affect the arguments of Gouy and Rayleigh regarding the origin of the periodicity of white light which is exhibited on dispersion. The considerations offered by these authors are in reality probably valid for each constituent pulse in the complex ray, and therefore also, by superposition, for the whole ray.

It may, however, be thought that considerations such as are offered above entirely remove the whole subject beyond the powers of our analysis. When, however, it is remembered that we are after all merely concerned with averages, and thus for the purposes of a statistical theory every possible oscillation with a wave-length in a given small range $\delta\lambda$, is practically identical with all the others, then it is seen that it will be sufficient to examine some simple system, as Planck and Jeans have in fact done, in which only one oscillation of each type occurs, one in each of the specified ranges, and then multiply by an infinitely large number N_λ the amount of energy deduced as corresponding to wave-lengths in the specified range for the simple system, a process which is practically equivalent to superposing a vast number of practically identical systems, which are, however, sufficiently different to fill out each small range $\delta\lambda$ so as to obtain a completely continuous spectrum.

3. *Equi-partition of energy amongst the oscillations.*

It has always appeared to me very difficult to see how the simple application of a direct physical principle, such as the theorem of the equi-partition of energy employed by Rayleigh and Jeans, should lead to a result which is entirely at variance with the fundamental assumptions on which the theorem is based, and in a search for the origin of the discrepancy I find that an oversight occurs in the usual proofs advanced by Rayleigh and Jeans, a consideration of which will largely remove the difficulties inherent in their method*. This oversight, as is clear from the results obtained, is mainly concerned with the convergence of the Fourier series expansions usually employed, and may be sufficiently explained by a reference to a simple illustrative problem examined by Rayleigh, viz. that of the partition of energy among the vibrations of a string of length l stretched between two points. The form of the string at any instant may be regarded as defined by the coefficients in the Fourier expansion of the displacement y at a point distance x from one end, viz.

$$y = \sum_{n=1}^{\infty} \phi_n \sin \frac{2\pi x}{\lambda_n},$$

where $\lambda_n = \frac{2l}{n}$: the kinetic and potential energies are then of the form

$$T = a \sum_{n=1}^{\infty} \dot{\phi}_n^2, \quad V = b \sum_{n=1}^{\infty} n^2 \phi_n^2,$$

a, b being definite constants, the forms of which are not however required in the present connexion. Applying then the general theorem to determine the partition of energy among the various oscillations in a steady chaotic motion of the string in equilibrium with a system at temperature T , Rayleigh is led to the result that on the average the values of $a\dot{\phi}_n^2$, $b n^2 \phi_n^2$ ($n=1, 2, \dots$) are all equal to kT , k being the usual absolute constant in gas theory. This result is, however, incompatible with finite values for T and V .

In obtaining such a result, however, no notice has been taken of the fundamentally necessary restrictions placed on the values of the coefficients $\phi_1, \phi_2, \dots, \phi_n, \dots$ by the assumption of the existence of the Fourier expansion for the

* It would appear that a similar difficulty has presented itself to Prof. Love, but he does not appear to offer any considerations in solution of it (*Phys. Zeit.* xiv. 1913, pp. 1300-1301).

certainly limited function of y , and the capability of manipulating it to secure the values given for T and V! A series of the above type is uniformly convergent in the whole range of values considered if the ϕ 's steadily decrease to zero, but this in itself is not sufficient to ensure the validity of the expressions for T and V. These expressions are only valid if they are convergent, a condition which is however secured if

$$\dot{\phi}_n^2 \leq u_n \quad n^2 \phi_n^2 \leq u_n,$$

u_n being the n th term of a convergent series

$$u_1 + u_2 + \dots + u_n + \dots$$

These conditions are also sufficient to secure the uniform convergence of the Fourier series for y and $\dot{y}$, and are therefore generally applicable. They are, however, much too general and indefinite to help us to a definite formula, and we must therefore attempt to obtain more precision. The procedure, an arbitrary one, is to apply the general tests of convergency one after the other until a satisfactory solution is found; the simplest suitable test is probably the correct one. The first test of convergence is obtained by comparison with the geometric progression, and it shows that the above conditions are satisfactorily verified if we can determine a positive number $\mu (< 1)$ such that

$$\dot{\phi}_n^2 \leq A(1 - \mu)^n, \quad n^2 \phi_n^2 \leq A(1 - \mu)^n,$$

where A is some constant number. The range of possible values of μ being $0 < \mu \leq 1$, we see that the possible range of values for $\dot{\phi}_n^2$, $n^2 \phi_n^2$, at least for all finite values of n , however large, is defined by

$$0 < \dot{\phi}_n^2 < A,$$

$$0 < n^2 \phi_n^2 < A.$$

In other words, there is no restriction except that the quantities must remain finite. There is therefore in such a case nothing to prevent the application of the theorem of the equi-partition of energy as has been done by Rayleigh and Jeans, and the result should hold good for all attainable regions of the spectrum. This test is therefore unsuitable.

The second test of convergence is obtained by comparison with the first of the logarithmic series, and it shows that the

conditions are satisfied if we can determine a positive number $\mu_1 > 0$ such that

$$\dot{\phi}_n^2 \leq \frac{A_1}{n^{1+\mu_1}}, \quad n^2 \phi_n^2 \leq \frac{A_1}{n^{1+\mu}},$$

A_1 being a finite constant. Now, however, we see that, since $0 < \mu_1 < \infty$, the possible range of values of $\dot{\phi}_n^2$ and $n^2 \phi_n^2$ for all finite values of n (*i. e.* practically in all attainable regions of the spectrum) is given by the conditions

$$0 \leq \dot{\phi}_n^2 \leq \frac{A}{n}, \quad 0 \leq n^2 \phi_n^2 \leq \frac{A}{n},$$

and this will be found absolutely sufficient for our purposes. We need not, therefore, proceed with any further tests.

Physically, various interpretations may be placed on this latter restriction in the values of the coefficients. The simplest and most obvious is that the capacities or extents of the coordinates representing the different types of oscillation are different, being in fact for any coordinate inversely proportional to the integer corresponding to it, or directly proportional therefore to the wave-length of the vibration associated with it. This would also imply that in any distribution of energy, the chance of a certain specified amount of energy getting into, say, the coordinate ϕ_n is proportional to λ_n , and this is probably the most general form which can be given to the restriction, which will be equally applicable to any more complicated system, being entirely independent of the particular conditions of the problem under immediate discussion*.

Here, then, we have the fundamental modification in the general theorem of the partition of energy applied to radiation, or in fact to any problem of the nature of those examined by Rayleigh and Jeans. But before attempting a reformulation of the theorem on this basis we must first choose a standard capacity for a known type of coordinate with which all the others can be compared. The obvious standard is provided in the type of coordinate discussed in the dynamical theory of gases: since, however, these coordinates may be regarded as corresponding to oscillations of infinitely

* This condition means that the mere chance of a specified collocation among any set of coordinates of a particular type must be properly weighted before being counted in the whole system. Thus if X_n is the number of ways in which an event can occur in connexion with the coordinates of type ϕ_n , then the proper chance of its occurring in the general system is not simply X_n but $X_n^{a\lambda_n}$.

long period, the corresponding number for their extent or capacity in the above wave-length comparison scale is infinite (∞). Thus if we are to take these ordinary molecular coordinates as of unit extent, then the extent of the coordinate ϕ_n is $\alpha\lambda_n$, α being a number to be made ultimately zero, but which will depend essentially on the choice of a scale for the statistics to be ultimately employed.

It may be thought that such a conclusion as this is untenable, inasmuch as it necessitates that the chance of a given complexion in the radiation coordinates is vanishingly small; this is so, but it is to be remembered that the number of coordinates of oscillatory type is infinitely large. In fact the two systems of coordinates, dynamical and oscillatory, are of a fundamentally different character. In gas theory we have to deal with a very large but certainly finite number of coordinates, and the chance of certain states in them is certainly calculable; but in such problems as the above we have always an infinity of types of coordinate to deal with, so that the chance of any condition among them is absolutely incalculable, unless there exists, as above, some sort of convergence of the chance of the condition in the various types to a zero value, which enables us to write off all the types beyond a certain limit as having practically no chance against the rest in any distribution. The striking success of the applications of the statistical methods in gas theory when the molecules are treated as rigid, and therefore a number of internal oscillatory degrees of freedom for each molecule entirely ignored, is almost conclusive evidence in favour of the above statement, which is here, however, derived from the analytical restrictions necessarily implied in the analysis employed.

In some respects the analytical conclusions drawn from the considerations of the present section are analogous to those required by the considerations put forward in the previous section, at least it appears possible to deduce the one conclusion from the other*. Let us consider for this purpose the wave-length spectrum plotted out on a straight line, and divide the line into a large number of equal small segments of length $\delta\lambda$. Each point on this line, and there are ultimately an infinity of such in any length $\delta\lambda$, represents a possible oscillation in the thermal radiation, and may thus be regarded as a possible receptacle of energy in the general distribution: if the ultimate dimensions of the point are zero it stands absolutely no chance of getting all the energy

* Granted the possibility of definiteness in the final result.

allotted to the element $\delta\lambda^*$; if, however, the point be assumed to be represented by a small portion of the line of length ϵ , then its chance of receiving the energy is $\frac{\epsilon}{\delta\lambda}$; in that case, however, there is only a finite number of such points in the length $\delta\lambda$, viz.

$$N_\lambda = \frac{\delta\lambda}{\epsilon},$$

so that the chance for the particular wave-length in the simpler system represented by the small length ϵ varies inversely as the number N_λ of such, which it is necessary to superpose to secure absolute continuity in the complete spectrum. But according to the above this chance is measured by $\alpha\lambda$, so that we see that

$$N_\lambda = \frac{1}{\alpha\lambda}.$$

We shall find that this result is necessary for the final determination of our formulæ†.

4. *The general theorem of partition of energy.*

The problem is now to determine the new form of the general theorem of partition of energy which will make it applicable in cases of the type of those under discussion. We must determine the most probable distribution of the energy among all the various types of coordinates, each with its own definite chance of receiving a specified amount of the energy. This problem is most easily solved in the method of Planck, as a problem in molecular statistics by dividing the energy into small differential elements ϵ (to be made ultimately zero) and then determining by direct counting the most probable distribution of these elements in the various coordinates, divided, say, into $(r+1)$ sets of specified types containing numbers $n_1 n_2 \dots n_r$ —with respective extents $1, \alpha_1, \alpha_2, \dots \alpha_r$. We have to find $p, p_1, p_2, \dots p_r$, which are the numbers of the elements of energy that, in the ultimate state of statistical equilibrium, will reside in these groups of similar coordinates respectively subject to the conservation of its total amount E . The chance of the distribution just expressed being made up of independent

* When it acts as a representative point in the simpler system.

† It is in fact the condition implied in the existence of a definite formula which requires that some relation of this sort must hold.

chances, each weighted, however, in a manner corresponding to the particular set of coordinates to which it refers, is proportional to the number of ways X , in which this statistical distribution can be realized, which is of the form *

$$X = f(n, p) \cdot f_1(n_1, p_1) \cdot \dots \cdot f_r(n_r, p_r),$$

wherein each separate number is weighted properly, so that

$$f_s(n_s, p_s) = [f(n_s, p_s)] \quad (s=1, 2, 3, \dots, r) \dagger.$$

In the natural distribution this must be a maximum subject to the restriction

$$p + p_1 + p_2 + \dots + p_r = \frac{E}{\epsilon},$$

which requires

$$\frac{f'_s(n_s, p)}{f(n_s, p)} = \frac{f'_s(n_s, p_s)}{f(n_s, p_s)} = \frac{\epsilon}{\theta} \quad f'_s = \frac{\partial f_s}{\partial p_s},$$

for all values of s , where θ is an undetermined quantity, the same for all sets of coordinates. The usual argument derived from ordinary considerations of gas theory shows that if the first type of coordinate is the ordinary molecular type then $\theta = kT$, T being the absolute temperature and k the usual absolute constant.

The value of $f(n, p)$ is the number of modes of distribution of p like objects in n like compartments and, following Planck, this is

$$\frac{(n+p-1)!}{(n-1)! p!},$$

which when n and p are very large numbers, by application of Stirling's approximation, assumes the form

$$f(n, p) = \frac{(n+p)^{n+p}}{n^n p^p},$$

so that

$$\log \frac{n+p}{p} = \alpha_s \log \frac{n_s + p_s}{p_s} = \frac{\epsilon}{kT},$$

or

$$p\epsilon = \frac{n\epsilon}{l^{\frac{\epsilon}{kT}} - 1}, \quad p_s\epsilon = \frac{n_s\epsilon}{l^{\frac{\epsilon}{kT}} - 1}.$$

* See J. Larmor, Proc. Roy. Soc. A. lxxxiii. p. 82 (1909).

† In applications where the number r is infinite no meaning can be attached to the expression for X unless the infinite product on the right is in some manner convergent. This implies convergence of the series $\sum \alpha_s$ which leads to the conclusions adopted above.

The average amount of energy in the coordinate of the first type is thus

$$\frac{\frac{\epsilon}{\epsilon}}{l^{\frac{\epsilon}{kT}} - 1},$$

and for those of the other types it is

$$\frac{\frac{\epsilon}{\epsilon}}{l^{\alpha_s kT} - 1}. \quad (s=1, 2, \dots r)$$

In the limit when ϵ is very small the average amount in the first type of coordinate is correctly kT ; it will be $\alpha_s kT^*$ in the other types unless α_s is very small, in which case the amount per coordinate is very small but of the form

$$\frac{\frac{\epsilon}{\eta_s}}{l^{\frac{\epsilon}{kT}} - 1}, \quad (s=1, 2, \dots r)$$

wherein

$$\eta_s = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\alpha_s}.$$

This is the modified form of the general theorem of the partition of energy in the steady thermal state, which must replace the theorem quoted by Jeans and Rayleigh, when the application is to problems of the nature of those sketched above. Equi-partition no longer exists, and however ideal the system, could never possibly be attained, not however for any peculiar physical reason, but just simply because of the mathematical (and physical) impossibility of such a state.

Jeans contends† that, since equi-partition of energy does not exist among all the coordinates, including those specifying the oscillations, a steady state of thermal equilibrium is never reached, a doctrine which, as he admits, would entirely remove the foundations of the application of thermodynamic principles to this subject. It must, however, be noticed that equi-partition is a test of thermal equilibrium only when all possible types of coordinate are equally probable. If, as suggested above, the various types of coordinates ultimately

* This confirms our general interpretation of the extent or limitation of a coordinate imposed by the analytical conditions.

† See his papers "Temperature radiation and the partition of energy in continuous media," *Phil. Mag.* [6] xvii. p. 219 (1909); "The motion of electrons in solids," *ibid.* [6] xvii. p. 773 (1909), & xviii. p. 209 (1909).

involved in the discussion have different extents or opportunities, then equi-partition can no longer exist in the steady state of thermal equilibrium.

5. Applications to the problem in radiation.

We can now apply the results just deduced to the fundamental problem of radiation, either in the method of Planck or of Jeans and Rayleigh.

The analysis given by Planck* still remains valid for determining the energy distribution in an elementary constituent of the radiation capable of being resolved by a Fourier analysis, so that the energy in it corresponding to the oscillations with wave-lengths between

$$\left(\lambda_s - \frac{\delta\lambda_s}{2}, \lambda_s + \frac{\delta\lambda_s}{2} \right) \quad \bullet$$

is, after him, given by

$$\frac{c\delta\lambda_s}{\lambda_s^4} \frac{\epsilon}{\eta_s} \frac{1}{l^{kT} - 1}.$$

The energy in this part of the total radiation is, as above explained, to be obtained by the superposition of a large number N_{λ_s} of such elementary amounts, and is, therefore,

$$E_{\lambda_s} \delta\lambda_s = \frac{c\delta\lambda_s}{\lambda_s^4} \frac{N_{\lambda_s} \epsilon}{\eta_s} \frac{1}{l^{kT} - 1},$$

which, in the limit, since $N_{\lambda_s} = \frac{1}{\alpha_s}$ is of the form

$$E_{\lambda_s} \delta\lambda_s = \frac{cd\lambda_s}{\lambda_s^4} \frac{\eta_s}{\eta_s} \frac{1}{l^{kT} - 1},$$

which is Planck's formula for natural radiation if

$$\eta_s = \frac{hc}{\lambda_s},$$

a form which is in fact suggested by the considerations

* See his book 'Vorlesungen über die Theorie der Wärmestrahlung' (Leipzig, 1906).

offered in section 3, since according to the views there expressed,

$$\alpha_s = \alpha \lambda_s,$$

α being ultimately zero: so that, however,

$$\eta_s = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\alpha} = \frac{hc}{\lambda_s}.$$

It thus appears that a suitable modification of Planck's theory of radiation, involving a suitable weighting of the probabilities of specified complexions among the degrees of freedom of the various types considered, which is suggested by considerations of the applicability of the Fourier series analysis, entirely removes all difficulties in the formulation of the problem given by Planck. Moreover, since in the above analysis ϵ must be very small, the essential condition is satisfied for a final approach to a continuous analysis, so that apparently no implication is now contained as to a finite atomic structure for the elements of energy forming the basis of the statistics*.

A difficulty may be felt as to the implied relation between ϵ and N_{λ_s} , viz.

$$N_{\lambda_s} \epsilon = \frac{hc}{\lambda_s},$$

but such a relation is necessarily involved in a statistical theory of the present type, since we must choose N_{λ_s} (ultimately infinite, of course) so that the extents or capacities of the particular type of cell are such that ϵ is the smallest amount, other than zero, they can accommodate. Of course, in a theory of the present type, where no less than three of the quantities involved are dependent on the scale of the statistics, some such relations as those indicated here and in the previous section must necessarily hold, otherwise it would be hopeless to proceed further: this latitude of indefiniteness is however probably the saving factor of the analysis, indicating that in reality no conclusions can be drawn as to a definite limit of subdivision such as are involved in Planck's exposition.

A similar modification also applies to the Rayleigh-Jeans theory, and it is shown to lead also to the same general formula for the distribution of the energy. Of course the actual system examined by these authors cannot possibly be

* Nor as to the finite limit to the fineness of structure in the spectrum which is involved in Planck's theory.

regarded as a model applicable to thermal radiation, owing to the discontinuity in the spectrum of the vibrations obtained in it: in fact, the number of vibrations with a wave-length greater than any specified length is finite, although directly related to the dimensions of the space in which they take place. To obtain a more analogous system we must probably dispose of the perfection in the ideal conditions assumed (absolute rigidity and perfect reflexion), or in some other way introduce a means of securing thermal equilibrium directly*. But then the number of oscillations possible in a volume V with wave-lengths between λ_s and $\lambda_s + d\lambda_s$ is no longer simply the number counted by Jeans and Rayleigh, viz.

$$CV \frac{d\lambda_s}{\lambda_s^4},$$

but is to be obtained by the insertion of a very large number N_{λ_s} of vibrations between each of those counted, N_{λ_s} being ultimately infinite to secure perfect continuity. But then the total amount of energy associated with these wave-lengths is

$$CV \frac{d\lambda_s}{\lambda_s^4} \frac{N_{\lambda_s} \epsilon}{\frac{\epsilon}{l^{\alpha_s k T}} - 1},$$

which is again Planck's formula if, as above,

$$\lim_{\epsilon=0} N_{\lambda_s} \epsilon = \lim_{\epsilon=0} \frac{\epsilon}{\alpha_s} = \frac{hc}{\lambda_s}.$$

It thus appears that the Rayleigh-Jeans theory of radiation is also consistent with the present scheme, and the result to be deduced from it is identical with Planck's formula. The assumption of finite units of energy or of any definite limit in the scale of the statistics is, however, no longer involved in the considerations on which it is based.

* It is interesting to notice how essential are the imperfections on the ideal conditions usually assumed in any such system, both to secure thermal equilibrium and also complete continuity in the spectrum. Another example of this principle is afforded in Planck's method of discussion. As Larmor points out, such a system of fixed linear vibrators of the type assumed by Planck are not effective for the transformation of the relative intensities of the various types of radiation to those corresponding to a common temperature. There would be equilibrium established only between the mean internal vibratory energy in the vibrators of each period and the density of the radiation of that period. In order to obtain a means of interchanging energy between vibrators of different types we must probably do away with their absolute rigidity or else introduce more complex vibrators.

6. *Larmor's formulation of the fundamental problem**.

An interesting attempt has been made by Prof. Larmor to generalize the basis of the ordinary theory of the connexion between probability and entropy, and in applying this generalization to the radiation problem as above he concludes that the formula of Planck, although not necessarily requiring an atomic structure for the energy itself, nevertheless implies a definite new quality in the radiation which he interprets as a specification of a limiting value for the ratio of the element of energy to extent of the cell regarded as the receptacle of the particular type of disturbance corresponding to the wave-length. The generalization he makes involves the fact that in the usual formula

$$S = k \log W,$$

connecting the entropy S and probability W of any state of a system, the constant k , undoubtedly a function of the scale of the statistics, must be of a different order of magnitude for the molecular as opposed to the oscillatory degrees of freedom. In fact, if we take k to be the usual constant for the molecular coordinates, but of the form αk for the oscillatory coordinates (α being ultimately zero), then we obtain a formula for the average energy per unit volume in the oscillations corresponding to the wave-lengths λ_s of the form

$$E_{\lambda_s} d\lambda_s = \frac{Cd\lambda_s}{\lambda_s^4} \frac{N_{\lambda_s} \epsilon}{\epsilon^{1/\alpha} kT - 1},$$

which is again Planck's law if

$$\lim_{\epsilon=0} N_{\lambda_s} \epsilon = \lim_{\epsilon=0} \frac{\epsilon}{\alpha} = \frac{hc}{\lambda_s}.$$

If, as Larmor assumes, α is constant, then N is also a constant, and thus it is now ϵ that is to be taken as different for each type of oscillation (the opportunities or extents of each type being the same), although there is still no limit to its subdivision involved in the final form of the result.

This view is in some respects analogous to that adopted above, but is distinguished from it by the different physical interpretation placed on the necessary restrictions implied in the analytical discussion. On the present view it appears that the various cells representing receptacles of energy are of different opportunities for the occurrence of a specified

* Bakerian Lecture, Proc. Roy. Soc. A. lxxxiii. pp. 82-95 (1909).

collocation, whereas Larmor's cells representing receptacles of elements of disturbance are of equal opportunity among themselves, but are differentiated from each other by the relative size of the energy element which becomes attached to the element of disturbance when it gets into the cell. The two points of view may ultimately be the same, but the one I have adopted appears to be more consistent with the preliminary discussions given in the former part of this paper.

The one difficulty involved in Larmor's scheme is that the fundamental constant k is to be taken different for the molecular and oscillatory coordinates; but this difficulty is probably unavoidable until a complete and unassailable specification of the mode of counting the probability applicable to every part of the system is obtained. In the above mode of formulation I preferred to retain the one constant k , and have therefore thrown the existing doubtfulness over on to the calculus of the probability, an ultimately hypothetical procedure, which is, however, to a certain extent justified by my preliminary discussions.

7. *Conclusion.*

In conclusion it may be of interest to again notice how it appears that the special characteristics of radiation in its present aspects are governed entirely by the necessary analytical restrictions involved in the possibility of resolving the irregularities in steady thermal radiation into its harmonic constituents by a Fourier series, and are in no way connected with any special or peculiar physical structure of the molecule, electron, or æther. It may be considered difficult to understand how it is that the indefinite fact of convergence could lead to any definite result such as is actually obtained. It must, however, be insisted that nowhere is this asserted to be possible. All I have attempted to do is to show that a free interpretation of the simplest suitable test of convergency can be shown to lead to a formula which is identical with Planck's in probably all attainable regions of the spectrum. After all, the simplest test is probably the correct one to apply. It is, however, interesting to notice how the simplest known test of convergence, when applied as above, corresponds to the law of equi-partition for the finite parts of the system. This test, therefore, fails in the case of radiation probably because there are an infinite number of different coordinates in any finite region of the spectrum: and we

have therefore to proceed to the next test, as explained above, to obtain a more consistent theory.

The considerations here offered are necessarily of a provisory nature, and many of the really fundamental difficulties involved in them have necessarily been merely cursorily reviewed. They are, however, put forward in their present form in the hope of stimulating further discussion of these difficulties, with the object of ultimately obtaining a consistent formulation of the central problem. It is held that, in reality, no difficulties of a fundamental nature underlie the theory, and that no implications are contained in it of any new physical basis for the phenomena of radiation.

The University, Sheffield,
June 21, 1914.

LXIX. *Note on the Form assumed by the Red Corpuscles of the Blood, or by the Suspended Particles in a Lecithin Emulsion.* By J. RICE, M.A., Lecturer in Physics, Liverpool University*.

IT is well known that on examination by a microscope, a small drop of blood is seen to be composed of a multitude of corpuscles floating and rolling over and over in the minute eddies of a fluid—the plasma. The great majority of these corpuscles are coloured, and on close examination these are found to have the shape of a disk cupped on both surfaces—a biconcave disk with rounded edges. If lecithin is shaken up with water, in the emulsion thus formed will also be found minute particles having a similar shape. Measurement reveals the fact that the diameters of these disks are from 5 to 10 μ ; their thickness at the centre eludes accurate measurement owing to diffraction patterns, but is undoubtedly very small. When viewed edge-on, a “diametral section” of these little floating bodies would have the appearance of a dumbbell, and it is striking that they should preserve this shape, and not adopt the spherical shape, which is that commonly assumed by drops of one liquid floating in another of the same density. If water be added to the blood, water enters the red corpuscles and causes them to become globular and then burst; if a strong solution of salt be added, water leaves the corpuscles, which become shrivelled; these experiments indicate that each corpuscle is surrounded by a film. Now it is a natural assumption that the extreme thinness of the little disks at their centre produces such a change

* Communicated by Prof. L. R. Wilberforce.