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PARTIAL FRACTIONS ASSOCIATED WITH QUADRATIC FACTORS.

1. THE determination of the group of partial fractions corresponding to the powers of a linear factor $x+b$ in the denominator of a rational function $f(x)/\{(x+b)^n\varphi(x)\}$ is one of the comparatively rare algebraic problems whose theoretical solutions are feasible in practice. The numerator of $(x+b)^{n-r}$ is the coefficient of y^r in the expansion of $f(-b+y)/\varphi(-b+y)$, and the calculation of the group of numerators is effected by means of two Horner transformations followed by one division, which can of course be arranged synthetically; no operation need be taken further than the term in y^{n-1} , whatever the degrees of $f(x)$ and $\varphi(x)$.

The case of an unresolved quadratic factor is different, and a comparison of various processes from the standpoint of practicability is not without interest. If the function to be decomposed is $f(x)/\{X^n\varphi(x)\}$, where X denotes a quadratic function and $f(x)$, $\varphi(x)$ are polynomials prime to X , the problem is the calculation of $2n$ coefficients $A_0, B_0, A_1, B_1, \dots$ such that

$$(A) \quad \frac{f(x)}{X^n\varphi(x)} \equiv \sum \frac{A_r x + B_r}{X^{n-r}} + \frac{g(x)}{\varphi(x)};$$

the summation is for values of r from 0 to $n-1$, and $g(x)$ is a polynomial which we may or may not desire to know.

2. We remark first that the problem is not solved by the use of the irrational or complex roots of X . The substitution of $-b+y$ for x in a polynomial involves about three times as many entries for a complex value of b as for a real value. But apart from this objection, which is not conclusive, the sum of two complementary terms $C_r/(x+b)^{n-r}$, $C'_r/(x+b')^{n-r}$ is not usually of the form $(A_r x + B_r)/X^{n-r}$, and all that is actually accomplished by the use of the roots of X is the segregation of the sum $\Sigma(A_r x + B_r)/X^{n-r}$, that is, the expression of the original function by the identity

$$(B) \quad \frac{f(x)}{X^n\varphi(x)} \equiv \frac{F(x)}{X^n} + \frac{g(x)}{\varphi(x)},$$

where $F(x)$ is a polynomial of degree $2n-1$.

It is to be admitted at once that if the use of the roots of X was an economical device for finding $F(x)$ we could be satisfied to begin in this way; the resolution of $F(x)/X^n$ is comparatively simple. Unfortunately, to find $F(x)$ as

$$\Sigma X^r \{C_r(x+b)^{n-r} + C'_r(x+b')^{n-r}\}$$

is woefully extravagant if no use is to be made of this peculiar form of the polynomial.

In passing let us refer to another direct method of finding $F(x)$. By expressing $\varphi(x)/X^n$ as a simple continued fraction whose elements are polynomials and calculating the penultimate convergent, we can determine two polynomials $\psi(x)$, $G(x)$ such that identically

$$\varphi(x)G(x) - \psi(x)X^n = 1.$$

Then

$$\frac{f(x)}{X^n\varphi(x)} = \frac{f(x)G(x)}{X^n} - \frac{f(x)\psi(x)}{\varphi(x)},$$

and therefore $F(x)$ is the remainder when $f(x)G(x)$ is divided by X^n . Theoretically this process leaves nothing to be desired; it provides indeed the most elementary proof of the existence of partial fractions. Anyone who supposes that the method is tolerable in numerical examples is recommended to calculate in this way the function $F(x)$ when X^n , $\varphi(x)$ are $(1-1+3)^2$, $(2+3)^3$ and $f(x)$ is $6+9-82-316-209-141+63$.

3. In its crudest form, the method that treats $A_0, B_0, A_1, B_1, \dots$ as undetermined coefficients in the identity

$$(C) \quad \varphi(x) \Sigma(A_r x + B_r) X^r + X^n g(x) \equiv f(x)$$

is at a double disadvantage. Simple in theory, the solution of simultaneous linear equations with numerical coefficients, whether by means of determinants or by a process of successive elimination, is in practice tedious and hazardous if more than three or four variables are involved. Also, the coefficients of $g(x)$ must be introduced explicitly, although of course if they are not wanted the solution may begin with their elimination.

Perhaps for these reasons, this use of undetermined coefficients is advocated by nobody except as a last resort, and for practical purposes is entirely superseded by methods of dealing with the identity (C) which avoid the two difficulties. When $X=0$, this identity implies

$$\varphi(x)(A_0x+B_0)=f(x);$$

but if $X \equiv x^2+px+q$, the substitution $x^2 = -px-q$, which is equivalent to $X=0$, can be used to reduce $x\varphi(x)$ and $\varphi(x)$ to linear functions $\alpha x + \beta$, $\gamma x + \delta$, and $f(x)$ to a linear function $\lambda_0x + \mu_0$. The condition

$$(\alpha + \beta)A_0 + (\gamma + \delta)B_0 = \lambda_0x + \mu_0$$

can be satisfied for the *two* roots of X only if simultaneously

$$\alpha A_0 + \gamma B_0 = \lambda_0, \quad \beta A_0 + \delta B_0 = \mu_0,$$

and this pair of equations determines A_0 and B_0 . Again, because

$$f(x) - \varphi(x)(A_0x+B_0)$$

is zero for both roots of X , the function X is a factor of this difference, and when A_0 and B_0 are known, direct division gives a polynomial $f_1(x)$ such that *identically*

$$f(x) - \varphi(x)(A_0x+B_0) \equiv Xf_1(x).$$

Substituting in (C) and dividing by X , we have

$$\varphi(x)\Sigma(A_r x + B_r)X^{r-1} + X^{n-1}g(x) \equiv f_1(x),$$

where now the lowest value of r is 1, and when $X=0$,

$$\varphi(x)(A_1x+B_1)=f_1(x);$$

thus

$$\alpha A_1 + \gamma B_1 = \lambda_1', \quad \beta A_1 + \delta B_1 = \mu_1',$$

where $\alpha, \beta, \gamma, \delta$ have the same values as before and $\lambda_1'x + \mu_1'$ is the expression to which the repeated substitution of $-px-q$ for x^2 reduces $f_1(x)$. The operations may be continued, and the pairs of coefficients are found in succession by a uniform process which ignores the polynomial $g(x)$.

In simple examples this method is demonstrably effective. Complex numbers may be logically involved, but all the work is performed with real numbers, and an illustration of the value to the computer of supposing complex numbers to exist may be welcomed without reserve. The questions that remain open are of a severely practical kind. (1) Can the work be arranged economically? (2) If the numerical coefficients in the original function are integers, is the entry of fractional coefficients postponed to a late stage in the calculations? At first glance a negative answer to the second question seems inherent in the method: there is no reason why A_0 and B_0 should not be fractional, and in every step subsequent to the determination of these two coefficients fractions are to be expected. But it is easy to avoid this conclusion. On the hypothesis, $\alpha, \beta, \gamma, \delta, \lambda_0, \mu_0$ are integers, and if η denotes $|\alpha\delta - \beta\gamma|$, then A_0, B_0 , and the coefficients in $f_1(x)$ are fractions whose denominators are factors of η , A_1, B_1 , and the coefficients in $f_2(x)$ are fractions whose denominators are factors of η^2 , and so on. Hence, if we replace A_r, B_r by $A_r'/\eta^{r+1}, B_r'/\eta^{r+1}$, that is, if we deal not with (C) but with the identity (C')

$$\varphi(x)\Sigma\eta^{n-r-1}(A_r'x+B_r')X^r + \eta^n X^n g(x) \equiv \eta^n f(x),$$

fractions cannot appear.

When we turn to the question of arranging the calculations, we are led at once to present the whole argument differently. The simplest systematic process of reducing a given function to a linear form by a repeated substitution of $-px-q$ for x^2 is nothing but the synthetic division of the function by

x^2+px+q , that is, by X , and by a succession of such divisions we have the function expressed in the form $L_0+L_1X+L_2X^2+\dots$, where L_0, L_1, L_2, \dots are all linear in x . Two direct operations of this kind give the coefficients in the developments

$$(D) \quad \begin{cases} \varphi(x) = (\gamma_0x + \delta_0) + (\gamma_1x + \delta_1)X + (\gamma_2x + \delta_2)X^2 + \dots, \\ f(x) = (\lambda_0x + \mu_0) + (\lambda_1x + \mu_1)X + (\lambda_2x + \mu_2)X^2 + \dots, \end{cases}$$

and from the first of these we have

$$x\varphi(x) = (\alpha_0x + \beta_0) + (\alpha_1x + \beta_1)X + (\alpha_2x + \beta_2)X^2 + \dots,$$

where

$$\alpha_0 = \delta_0 - p\gamma_0, \quad \alpha_1 = \delta_1 - p\gamma_1, \quad \alpha_2 = \delta_2 - p\gamma_2, \dots,$$

$$\beta_0 = -q\gamma_0, \quad \beta_1 = \gamma_0 - q\gamma_1, \quad \beta_2 = \gamma_1 - q\gamma_2, \dots;$$

if these expressions are substituted in (C), each power of X from X^0 to X^{n-1} is multiplied on each side only by a linear function of x , and equating the various linear functions we have

$$(\alpha_0x + \beta_0)A_0 + (\gamma_0x + \delta_0)B_0 = \lambda_0x + \mu_0,$$

$$(\alpha_0x + \beta_0)A_1 + (\gamma_0x + \delta_0)B_1 + (\alpha_1x + \beta_1)A_0 + (\gamma_1x + \delta_1)B_0 = \lambda_1x + \mu_1,$$

$$(\alpha_0x + \beta_0)A_2 + (\gamma_0x + \delta_0)B_2 + \sum_{s=1}^2 \{(\alpha_sx + \beta_s)A_{2-s} + (\gamma_sx + \delta_s)B_{2-s}\} = \lambda_2x + \mu_2,$$

$$(\alpha_0x + \beta_0)A_3 + (\gamma_0x + \delta_0)B_3 + \sum_{s=1,2,3}^3 \{(\alpha_sx + \beta_s)A_{3-s} + (\gamma_sx + \delta_s)B_{3-s}\} = \lambda_3x + \mu_3,$$

and so on. That is, A_r, B_r are given by

$$(E) \quad \alpha A_r + \gamma B_r = \lambda_r', \quad \beta A_r + \delta B_r = \mu_r',$$

where $\alpha, \beta, \gamma, \delta, \lambda_0', \mu_0'$ coincide with $\alpha_0, \beta_0, \gamma_0, \delta_0, \lambda_0, \mu_0$, and for values of r from 1 to $n-1$,

$$(F) \quad \lambda_r' = \lambda_r - \sum (\alpha_s A_{r-s} + \gamma_s B_{r-s}), \quad \mu_r' = \mu_r - \sum (\beta_s A_{r-s} + \delta_s B_{r-s}),$$

the summations being for the values of s from 1 to r .

4. There is a method essentially more direct than that just described. For this we suppose the factor X to have the form $x^2+2bx+c$. Then substitution of a new variable y for $x+b$ gives the function the form

$$f(-b+y)/\{(y^2+k)^n \varphi(-b+y)\},$$

where k has the value $c-b^2$.

Were it not for direct evidence to the contrary in worked examples published over well-known names, we might have thought it obvious that the problem of separating from a function of the particular form

$$F(y^2)/\{(y^2+k)^n \Phi(y^2)\}$$

the partial fractions of the form $K_r/(y^2+k)^{n-r}$ is the same as that of extracting from $F(x)/\{(x+k)^n \Phi(x)\}$ the elements associated with powers of $x+k$: the numerator K_r is the coefficient of x^r in the expansion of $\bar{F}(-k+z)/\Phi(-k+z)$.

But whatever the polynomials $f(x), \varphi(x)$, the product $\varphi(-b+y)\varphi(-b-y)$ is of the form $\Phi(y^2)$, and the product $f(-b+y)\varphi(-b-y)$ is expressible by separation of the odd powers of y from the even in the form $F'(y^2)+yF''(y^2)$. That is to say, the original function needs only two Horner transformations and two multiplications to throw it into the form

$$\frac{F'(y^2)}{(y^2+k)^n \Phi(y^2)} + y \frac{F''(y^2)}{(y^2+k)^n \Phi(y^2)}.$$

Three more Horner transformations and two divisions give the coefficients in the expansions of

$$F'(-k+z)/\Phi(-k+z) \quad \text{and} \quad F''(-k+z)/\Phi(-k+z),$$

and if these are K_0', K_1', \dots and K_0'', K_1'', \dots , a typical fraction in the decomposition of

$$f(-b+y)/\{(y^2+k)^n \Phi(-b+y)\} \quad \text{is} \quad (K_r' + yK_r'')/(y^2+k)^{n-r},$$

and a typical element of the original function is

$$\{K_r''x + (K_r' + bK_r'')\}/X^{n-r}.$$

In comparison with the method of § 3, this process suffers at two points. If the quadratic factor is given in the first place as ux^2+vx+w , where u, v, w are integers, the substitution $\xi=ux$ is sufficient to replace the factor by one of the form $\xi^2+p\xi+q$, but it may be necessary to take $\xi=2ux$ if the form required is $\xi^2+2b\xi+c$. Thus the later method is liable to involve powers of 2 avoided in the earlier. Also, the three functions $F'(y^2)$, $F''(y^2)$, $\Phi(y^2)$ must be determined completely, whatever the value of n ; since the number of entries in a real Horner transformation applied to a polynomial of degree p approximates to p^2 if the transformation is complete and to $n(2p-n)$ if only the terms of degree not greater than n are wanted, it is a serious drawback to be compelled to deal at full length with $f(x)$ and $\varphi(x)$ even when n is small.

On the other hand, the method of § 4 is the more straightforward. The operations are restricted in number and of a type so familiar that we have nothing to learn as to the manner of performing them economically; it is easier to arrange intelligibly the figures of a few large operations, each of which is complete in itself, than those relating to a number of small steps, such as are involved in passing from the solution of one pair of equations of the form (E) to the calculation of the coefficients for the succeeding pair. Lastly, fractional coefficients cannot appear till the work is all but finished, and there is no temptation to forestall them by introducing multipliers that may be very large and that may in fact not be required.

5. It is possible to combine the advantages of the two methods. First, we develop $f(x)$ and $\varphi(x)$ as in (D) as far as the terms in X^{n-1} ; in this work we keep X in the form x^2+px+q , it being immaterial whether p is odd or even. Next, if p is odd, we make the substitution $\xi=2x$ to replace X by

$$\frac{1}{4}(\xi^2+2p\xi+4q);$$

the change is trivial, and to avoid accumulating symbols we will describe the remainder of the process as if this step was unnecessary, taking X as $x^2+2bx+c$, that is, as $(x+b)^2+k$, where k is $c-b^2$.

Substitution of $-b+y$ for x in the first n terms on the right of (D) gives functions $\varphi_1(X)+y\varphi_2(X)$, $f_1(X)+yf_2(X)$, where

$$\varphi_1(X)=\Sigma(\delta_r-b\gamma_r)X^r, \quad \varphi_2(X)=\Sigma\gamma_rX^r,$$

$$f_1(X)=\Sigma(\mu_r-b\lambda_r)X^r, \quad f_2(X)=\Sigma\lambda_rX^r,$$

each summation being for values of r from 0 to $n-1$. Multiplying by $\varphi_1(X)-y\varphi_2(X)$ and replacing y^2 by $X-k$, we have the function

$$\frac{\{f_1(X)\varphi_1(X)+(k-X)f_2(X)\varphi_2(X)\}+(X+b)\{f_2(X)\varphi_1(X)-f_1(X)\varphi_2(X)\}}{X^n[\{\varphi_1(X)\}^2+(k-X)\{\varphi_2(X)\}^2]},$$

which has the same elementary functions associated with powers of X as the original function; to evaluate this function completely is unnecessary, for powers of X of degree higher than $n-1$ can be omitted in every multiplication. Two synthetic divisions give the coefficients denoted in § 4 by K_r' and K_r'' , and we have as before

$$A_r=K_r'', \quad B_r=K_r'+bK_r''.$$

6. In conclusion we ought to recognise that while the method of § 5 ought to involve the minimum of labour, no estimate of numerical complexity can safely be applied to what may be called manufactured examples. If the function to be resolved has been composed by the addition of simple parts, so that the numbers A_r , B_r are small integers, the equations (E) and the formulae (F) must be abnormally simple. Even the equations that arise directly from (C) may be rendered quite manageable by some random assumption as to the magnitude of the coefficients to be determined; for example, the function suggested at the end of § 2 can be resolved quickly in this way if it is taken for granted that the coefficients are all integers numerically less than twenty!

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