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TRANSMISSION CHARACTERISTICS OF THE SUBMARINE CABLE.*

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I.

THE transmission characteristics of a conducting system, such as a submarine cable circuit, are determined by its propagation constant, Γ, and characteristic impedance, K, which may be calculated for the frequency $p/2\pi$ from the formulas:

$$\Gamma=\sqrt{(R+ipL)\ (G+ipC)}\,, \qquad (1)$$
$$K=\sqrt{\frac{R+ipL}{G+ipC}}\,,$$

where R, L, G and C are the four fundamental line parameters, resistance, inductance, leakance, and capacity, all per unit length. These formulas are rigorous for all types of transmission systems; but the determination of the line parameters is not always possible by elementary methods, and may indeed be a matter of considerable complexity and involve rather difficult analysis. In the case of the submarine cable, exact formulas are available for calculating the capacity and leakage and the core impedance. Considerable uncertainty is introduced into the theory, however, on

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account of the lack of a method of determining the "return impedance," that is, the contribution of the "sea return" (sea water, armor wires, etc.) to the effective resistance and inductance of the circuit. An investigation of this problem was undertaken by the writers in connection with the research program of the American Telephone and Telegraph Company and the Western Electric Company.

The purpose of the present paper is to discuss transmission over the submarine cable, and, more particularly, to develop rigorous formulas for the calculation of the impedance of the return conductor of the cable. The results of theoretical calculations are then compared with actual experimental data; and the agreement between theory and experiment is so satisfactory as to indicate that the former is a reliable guide in the design and predetermination of the cable.

Besides providing a method for accurately calculating the transmission characteristics of a submarine cable, the present analysis leads to the following general conclusions:

(1) Contrary to usual assumption, the "sea return" impedance is by no means negligible. Even at quite moderate frequencies there is a considerable crowding of the return current into the immediate neighborhood of the cable, with a consequent rapid increase of the resistance and a corresponding decrease of the inductance of the circuit. Except at the lowest frequencies, therefore, the impedance of the "sea return" is a very important factor.

(2) The armor wires which surround the cable, and which are necessary for mechanical protection, have a very pronounced effect on the impedance of the sea return, and even at moderate frequencies may become the controlling factor. Their action is to screen the current from the sea water itself, and, as the frequency increases, to carry more and more of the return current, until it is almost entirely confined to the armor wires and excluded from the sea water.

(3) The rapid increase in the impedance of the armor wires with frequency, and their pronounced and even controlling effect on transmission makes a thorough-going study of their role in the electrical system a matter of first-class importance. Heretofore they appear to have been regarded only as a mechanical protection, and their effect on transmission has been ignored. The accurate

method of calculating their impedance which is developed in the following pages is believed to have considerable value in this connection.

(4) At relatively high frequencies, the return impedance, and hence the attenuation and the distortion, may be very greatly decreased by a correctly designed thin metallic sheath concentric with the core, and in electrical contact with the armor wires. The very important action of such a sheath, even when extremely thin, does not appear to have been adequately recognized or studied. It is suggested that the introduction of such a sheath affords a means of greatly increasing the range of frequencies which the cable can transmit.

The general problem of determining the transmission characteristics of a system consisting of an insulated conductor surrounded by a concentric ring of armor wires immersed in sea water is of considerable difficulty, since in this case the propagated wave must be represented as a set of component waves centered upon or diverging from the axes of the core and of the individual armor wires. The problem was first simplified by replacing the ring of armor wires by a cylindrical sheath, thus giving circular symmetry to the the structure. The analysis of this case, however, showed that the effect of the iron sheath replacing the armor wires was so pronounced as to make this simplifying assumption of doubtful validity. The general problem was therefore attacked, and rigorous methods developed for calculating the effect of the armor wires upon transmission. The results in this case differ markedly from those obtained for the case of a continuous iron sheath, which indicates that great caution must be used in making assumptions regarding the physical structure of the armoring.

The present paper follows rather closely the course of the writers' investigation. In Section II is analyzed the problem of transmission over a system consisting of n coaxial cylindrical conductors, which may be either in electrical contact at their adjacent surfaces or separated from each other by dielectric spaces. The outermost conductor, consisting of the sea water, is assumed to extend to infinity. This analysis is then applied, in Section III, to the case of a submarine cable which is armored with a continuous iron sheath. This problem is not only of interest in itself, but serves as a first approximation to the case of an actual cable, and gives a clear qualitative idea of the effect of the various factors on transmission. In Section IV the problem of the submarine cable armored with a ring of iron wires is attacked and solved by rigorous methods, and the theoretical results are then compared with experimental data.

II.

The solution of the problem of transmission of periodic currents over a system comprising n coaxial cylindrical conductors consists in finding the particular solution of Maxwell's equations which satisfies the boundary conditions—continuity of tangential and magnetic forces at the surfaces of the conductors. Let the common axis of the conductors coincide with the Z axis of a system of polar coördinates, R, Φ, Z, and let the electric and magnetic variables involve the common factor $exp(-\Gamma z+ipt)$, Γ is therefore the propagation factor characterizing transmission, and p is 2π times the frequency. This factor will not be explicitly written in any of the work that follows, but it will be assumed to be incorporated in each of the electric variables so that

$$\frac{\partial^n}{\partial z^n} = (-\Gamma)^n, \qquad \frac{\partial^n}{\partial t^n} = (ip)^n.$$

From symmetry, it is evident that the component of electric field intensity in the direction of ϕ vanishes, and that the magnetic lines of force are circles lying in planes perpendicular to the axis of the system, and centered on that axis. Also, the axial and radial electric forces are independent of ϕ. It can be shown that the radial component of electric field intensity *in the conductors* is negligibly small compared with the axial component. The latter, for a given conductor, is of the form $E\ exp(-\Gamma z + ipt)$, where E is a solution of the differential equation

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} + (\Gamma^2 - 4\pi\lambda\mu i p)E = O. \qquad (2)$$

Here λ and μ are the electrical conductivity and the magnetic permeability of the particular conductor, measured in absolute electromagnetic units, and E is a function of r alone.

For the frequencies in which we are interested it may be show that $\Gamma^2/4\pi\lambda\mu p$ is exceedingly small, so that (2) may be written

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} - 4\pi\lambda\mu i p E = 0. \tag{3}$$

We will designate by the subscript j all quantities pertaining to the j^{th} conductor, counting from the axis. The solution of (3) for this conductor may then be written

$$E_j = A_j J_o(\rho_j) + B_j K_o(\rho_j), \tag{4}$$

where J_o and K_o are Bessel functions of zero order, A_j and B_j are arbitrary constants and

$$\rho_j = ri\sqrt{4\pi\lambda_j\mu_j p i} = r\alpha_j .$$

The magnetic field intensity can then be obtained from the curl law,

$$\mu \frac{dH}{dt} = \frac{dE}{dr},$$

which gives

$$H_j = \frac{\alpha_j}{\mu_j i p}\left[A_j J_o'(\rho_j) + B_j K_o'(\rho_j)\right], \tag{5}$$

where the prime indicates differentiation with respect to ρ_j. Taking the line integral of both sides of (5) around circular paths in conductor j lying close to the inner and outer surfaces of the cylinder we obtain

$$A_j J_o'(y_j) + B_j K_o'(y_j) = \frac{2\mu_j i p}{y_j}(I_1 + I_2 + \cdots + I_{j-1}), \tag{6}$$

$$A_j J_o'(x_j) + B_j K_o'(x_j) = \frac{2\mu_j i p}{x_j}(I_1 + I_2 + \cdots + I_j),$$

in which

I_j = current in the jth conductor,

$x_j = \alpha_j a_j$,

$y_j = \alpha_j b_j$,

a_j = external radius of jth conductor,

b_j = internal radius of jth conductor.

The values of the electric field intensity at the inner and outer surfaces of the j^{th} conductor can be written, from (4)

$$E_j' = A_j J_o(y_j) + B_j K_o(y_j),$$
$$E_j'' = A_j J_o(x_j) + B_j K_o(x_j).$$

Combining, in turn, each of these equations with relations (6) to eliminate A_j and B_j, we obtain

$$
\begin{aligned}
E'_j &= Z'_{j1}I_1 + Z'_{j2}I_2 + \cdots + Z'_{jj}I_j \\
E''_j &= Z''_{j1}I_1 + Z''_{j2}I_2 + \cdots + Z''_{jj}I_j,
\end{aligned}
\tag{7}
$$

in which

$$
\begin{aligned}
Z''_{jk} &= 2\mu_j ip\left[\frac{1}{x_j}\frac{J_0(x_j)K'_0(y_j) - J'_0(y_j)K_0(x_j)}{J'_0(x_j)K'_0(y_j) - J'(y_j)K'_0(x_j)}\right. \\
&\qquad \left. - \frac{1}{y_j}\frac{J_0(x_j)K'_0(x_j) - J'_0(x_j)K_0(x_j)}{J'_0(x_j)K'_0(y_j) - J'_0(y_j)K'_0(x_j)}\right], \quad k \neq j \\
Z''_{jj} &= \frac{2\mu_j ip}{x_j}\left[\frac{J_0(x_j)K'_0(y_j) - J'_0(y_j)K_0(x_j)}{J'_0(x_j)K'_0(y_j) - J'_0(y_j)K'_0(x_j)}\right], \\
Z'_{jk} &= 2\mu_j ip\left[\frac{1}{x_j}\frac{J_0(y_j)K'_0(y_j) - J'_0(y_j)K_0(y_j)}{J'_0(x_j)K'_0(y_j) - J'_0(y_j)K'_0(x_j)}\right. \\
&\qquad \left. - \frac{1}{y_j}\frac{J_0(y_j)K'_0(x_j) - J'_0(x_j)K_0(y_j)}{J'_0(x_j)K'_0(y_j) - J'_0(y_j)K'_0(x_j)}\right], \quad k \neq j \\
Z'_{jj} &= \frac{2\mu_j ip}{x_j}\left[\frac{J_0(y_j)K'_0(y_j) - J'_0(y_j)K_0(y_j)}{J'_0(x_j)K'_0(y_j) - J'_0(y_j)K'_0(x_j)}\right].
\end{aligned}
\tag{8}
$$

We have now succeeded in expressing the electric forces in the conductors as linear functions of the currents $I_1 \ldots I_n$, the coefficients being of the nature of impedances, by a method which is simply an application of the principle of continuity of magnetic field intensity. The remaining boundary condition, continuity of the tangential component of electrical field intensity gives, where two consecutive cylinders are in electrical contact,

$$
E'_{j+1} - E''_j = \left[Z'_{j+1,1} - Z''_{j,1}\right]I_1 + \cdots + \left[Z'_{j+1,j} - Z''_{j,j}\right]I_j + Z'_{j+1,j+1}I_{j+1} = 0. \tag{9}
$$

This gives m relations between the n currents of the system, m being the number of contacts between successive cylinders. In the case where the j and $(j+1)$st conductors are separated by a layer of dielectric material, a relation between the boundary values of electric field intensity may be obtained as follows:

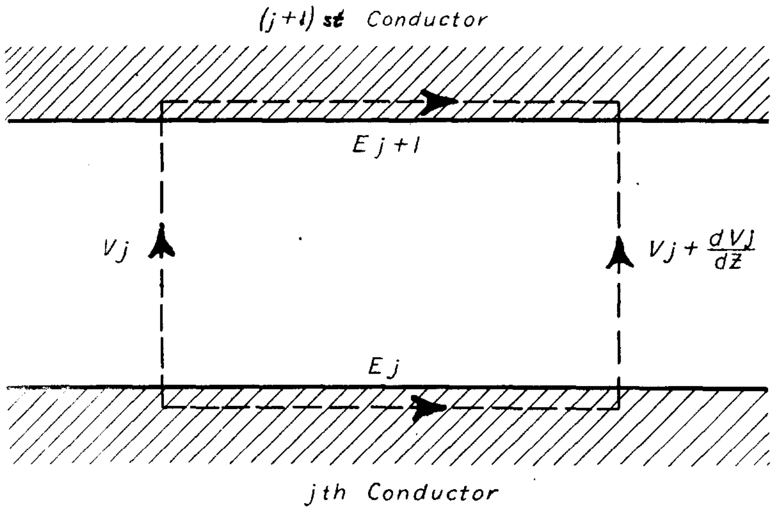
If E_r is the radial electric field intensity in the dielectric, then

$$
V_j = \int_{a_j}^{b_{j+1}} E_r\, dr
$$

is the potential difference between the j and $(j+1)$st conductors, in the sense employed in ordinary circuit theory. If we now apply the law

$$\text{curl } E = -\mu \frac{dH}{dt}$$

FIG. 1.



to the elementary contour shown in Fig. 1 we get

$$-\frac{\partial V_j}{\partial z} + E'_{j+1} - E''_j = \mu i p \Phi_j, \tag{10}$$

or

$$\Gamma V_j + E'_{j+1} - E''_j = \mu i p \Phi \tag{11}$$

where Φ_j is the magnetic flux threading the contour and is given by

$$\Phi_j = 2(I_1 + I_2 + \cdots + I_j)\log\frac{b_{j+1}}{a_j}.$$

From the law

$$\text{div } kE = 4\pi Q,$$

$$E_r = \frac{2}{k_j r}(Q_1 + Q_2 + \cdots + Q_j)$$

where Q_j is the charge on the j^{th} conductor and k_j is the dielectric constant of the medium, whence,

$$V_j=\frac{2}{k_j}\log\frac{b_{j+1}}{a_j}(Q_1+Q_2+\cdots+Q_j). \tag{12}$$

Furthermore, the rate of gain of charge is

$$\frac{\partial}{\partial t}(Q_1+Q_2+\cdots+Q_j)=-\frac{\partial}{\partial z}(I_1+I_2+\cdots+I_j)-4\pi(Q_1+Q_2+\cdots+Q_j)g_j/k_j, \tag{13}$$

where the last term represents the leakage current, g_j being the specific conductivity of the dielectric.

From (13) we have

$$\frac{1}{k_j}(4\pi g_j+ipk_j)(Q_1+Q_2+\cdots+Q_j)=\Gamma(I_1+I_2+\cdots+I_j,)$$

and substituting this value of $(Q_1+Q_2+\ldots+Q_j)$ in (12) gives

$$V_j=2(I_1+I_2+\cdots+I_j)\frac{\Gamma}{4\pi g_j+ipk_j}\log\frac{b_{j+1}}{a_j}; \tag{14}$$

and from this and (11)

$$-\left[\frac{\Gamma^2}{G_j+ipC_j}-ipL_j\right](I_1+I_2+\cdots+I_j)=E'_{j+1}-E''_j \tag{15}$$

where $G_j=\dfrac{4\pi g_j}{2\log\dfrac{b_{j+1}}{a_j}}$ $C_j=\dfrac{k_j}{2\log\dfrac{b_{j+1}}{a_j}}$ $L_j=2\mu_j\log\dfrac{b_{j+1}}{a_j}.$

Subtituting the values of E''_j and E'_{j+1} from (7) in (15) gives

$$-\left[\frac{\Gamma^2}{G_j+ipC_j}-ipL_j\right](I_1+I_2+\cdots+I_j)=(Z'_{j+1,1}-Z''_{j,1})I_1+\cdots$$
$$+Z'_{j+1,j+1}I_{j+1}. \tag{16}$$

An equation of this sort may be obtained for each layer of dielectric, and these combined with equations (9) and the condition that the electric field intensity in the sea water must vanish at infinity,

$$E'_n=Z''_{n1}I_1+\cdots+Z''_{nn}I_n=0,$$

given n relations between $I_1 \ldots I_n$. In order that these shall be consistent, the determinant of the coefficients must vanish.

$$\begin{vmatrix} Z'_{21}-Z''_{11}, & Z'_{22}, & 0 & - - - & 0 \\ Z'_{31}-Z''_{21}, & Z'_{32}-Z''_{22}, & Z'_{33}, & - - - & 0 \\ - & - & - & - - - & - \\ Z'_{j+1,1}-Z''_{j,1}+Z_j, & Z'_{j+1,2}-Z''_{j,2}+Z_j, & - & - - - & - \\ - & - & - & - - - & - \\ Z''_{n1}, & Z''_{n2}, & - & - - - & Z''_{nn} \end{vmatrix} = 0 \qquad (17)$$

where

$$Z_j = \frac{\Gamma^2}{G_j + ipC_j} - ipL_j\,.$$

This is an equation in Γ^2 of degree equal to the number of dielectric layers, consequently, there are as many independent modes of propagation in the system as there are branches in the network of conductors.

From this point the method of determining the behavior of the system depends upon conditions in the particular problem. For the case where there are k dielectric layers separating the conductors into $k+1$ groups the current on the j^{th} group may be written in the form

$$I_j = A_{j1}\,exp\,(-\Gamma_1 z + ipt) + \cdots + A_{jk}\,exp\,(-\Gamma_k z + ipt)$$
$$+ B_{j1}\,exp(\Gamma_1 z + ipt) + \cdots + B_{jk}\,exp(\Gamma_k z + ipt),$$

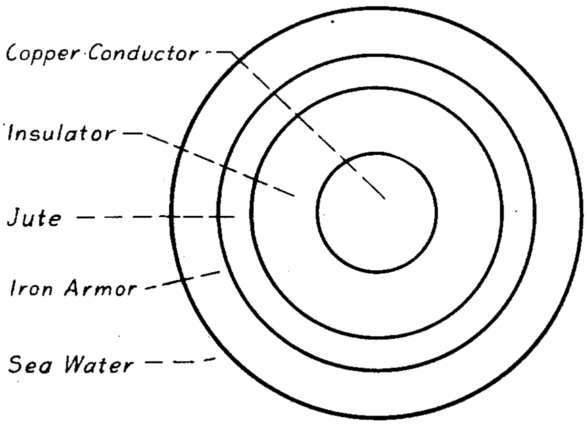
where $\Gamma_1^2 \ldots \Gamma_k^2$ are the k roots of the determinant (17) and A_{j1} $\ldots A_{jk}$, $B_{j1} \ldots B_{jk}$ are constants. These constants are not all independent, however, since, for each value of Γ, Γ_1 for instance, there exist k relations of the form (16) which the corresponding set of constants A_{11}, A_{21}, ... A_{k1} must satisfy. The remaining $2k$ independent constants can then be determined from a knowledge of the conditions at the terminals of the conductors.

It is important to observe that the transmission characteristics of a system of coaxial conductors are influenced to a great extent by the manner of connecting the various members of the system. Anomalies in the impedance of a complicated network such as a submarine cable with several conducting sheaths in the return path, may often be traced to lack of proper connections between the sheaths, or to faulty joints.

III.

The submarine cable armored with a continuous coaxial sheath, as shown in Fig. 2, is a particular case of the foregoing, and one which presents a clearer idea of the physical significance of the various steps in the general theory. There are only two

FIG. 2.



groups of conductors, the first consisting of the core conductor, and the second comprising the iron sheath and the sea water, the two groups being separated by the insulating material and the layer of jute. Consequently, there is only one mode of propagation, and the analysis is considerably simplified.

The jute is assumed to contain sufficient sea water so that although it conducts practically no current axially, it maintains equality of potential between the outer surface of the gutta percha and the inner surface of the iron sheath. Consequently equation (10) may be written

$$\frac{\partial V}{\partial z} - E'_2 + E''_1 = -\mu i p \Phi = -ipL_{12}I_1, \qquad (18)$$

where E''_1 and E'_2 are the values of electric field intensity at the outer surface of the core conductor and the inner surface of the iron, respectively, V is the potential difference between these two surfaces, and Φ is the magnetic flux threading unit length of the gutta percha and jute. Also, from (14)

$$-\frac{\partial V}{\partial z}=\frac{\Gamma^2}{G+ipC}I_1 \tag{19}$$

in which I_1 is the current in the core and

$$G=\frac{4\pi g_{12}}{2\log\frac{b}{a_1}}, \qquad C=\frac{k_{12}}{2\log\frac{b}{a_1}}, \tag{20}$$

where g_{12} and k_{12} are the electrical constants of the gutta percha, and b is the external radius of the core. It is evident, that G and C are respectively the leakage and capacity of unit length of the cable. Therefore, from (1),

$$\frac{\Gamma^2}{G+ipC}=R+ipL=Z, \tag{21}$$

where R and L are the resistance and induction of unit length of the cable, including the sea return. Equation (18) may then be written

$$Z\,I_1=E''_1-E'_2+ipL_{12}I_1. \tag{22}$$

To determine Z we must express E''_1 and E'_2 as functions of I_1.

We have seen that

$$E''_1=Z_1 I_1, \tag{23}$$

where Z_1 may be termed the "internal impedance" per unit length of this conductor. In fact, when we place $y_1=o$ in (8) we obtain

$$Z''_{11}=\frac{2\mu_1 ip}{x_1}\frac{J_o(x_1)}{J'_o(x_1)}, \tag{24}$$

which is the usual formula for the internal impedance of a cylindrical conductor.

Similarly

$$E'_2=-Z_2 I_1 \tag{25}$$

where Z_2 is the internal impedance of the return conductor, the minus sign being due to the fact that the current in the return is in the negative direction of z.

Inserting (23) and (25) in (22) gives

$$Z=Z_1+Z_2+ipL_{12}.$$

The quantity Z_2 may be determined in the following manner. From (7) we have

$$E'_2 = Z'_{21} I_1 + Z'_{22} I_2, \tag{26}$$

where I_2 is the current in the iron sheath. The value of this current can be found by applying the condition of continuity of electric field intensity at the common surface of the iron and the sea water, as in equation (9). This gives

$$Z''_{21} I_1 + Z''_{22} I_2 = Z'_{31} I_1 + Z'_{32} I_2 + Z'_{33} I_3,$$

in which I_3 is the current in the sea water. From (8) it can be seen that $Z_{33} = O$, since $x_3 = \infty$, therefore

$$I_2 = \frac{Z'_{31} - Z''_{21}}{Z''_{22} - Z'_{32}} I_1. \tag{27}$$

Substituting (27) in (26) gives

$$E'_2 = \left[Z'_{21} + \frac{Z'_{31} - Z''_{21}}{Z''_{22} - Z'_{32}} Z'_{22} \right] I_1,$$

and by comparison with (25) we have

$$Z_2 = -Z'_{21} - \frac{Z'_{31} - Z''_{21}}{Z''_{22} - Z'_{32}} Z'_{22} \tag{28}$$

as the internal impedance of the return conductor. The resistance and reactance per unit length of this portion of the circuit are then represented by the real and imaginary parts of (28) respectively.

We may then determine R and L from the formula

$$Z = R + ipL = Z_1 + Z_2 + ipL_{12}, \tag{29}$$

where Z_1 and Z_2 are calculated from (23) and (28) and

$$L_{12} = 2 \log \frac{b_2}{a_1},$$

b_2 and a_1 being the inner radius of the iron and the outer radius of the core conductor, respectively.

For purposes of comparison, the return impedance is calculated for the case where the iron armoring is absent, the return current being conducted by the sea water alone. As in the preceding case,

$$Z_1 = \frac{2\mu_1 ip}{x_1} \frac{J_o(x_1)}{J_o'(x_1)}.$$

The expression for Z_2 simplifies considerably. The electric field intensity in the sea water may be written, from (4),

$$E_2 = B_2 K_o(\rho_2), \tag{30}$$

the term in J_o being absent in order to permit E_2 to vanish at infinity.

Also, from (6),

$$B_2 K_o'(y_2) = \frac{2\mu_2 ip}{y_2} I_1. \tag{31}$$

From (30) and (31) we have

$$E_2' = \frac{2\mu_2 ip}{y_2} \frac{K_o(y_2)}{K_o'(y_2)} I_1. \tag{32}$$

from which the return impedance can be written,

$$Z_2 = -\frac{2\mu_2 ip}{y_2} \frac{K_o(y_2)}{K_o'(y_2)}. \tag{33}$$

We have then

$$Z = R + ipL = \frac{2\mu_1 ip}{x_1} \frac{J_o(x_1)}{J_o'(x_1)} - \frac{2\mu_2 ip}{y_2} \frac{K_o(y_2)}{K_o'(y_2)} + ipL_{12}.$$

The resistance and inductance of the sea return of a submarine cable were calculated from formula (28), employing the following values for the constants:

$$\text{Copper} \quad \begin{cases} a_1 = .226 \text{ cm.} \\ b_1 = 0 \\ \mu_1 = 1 \\ \lambda_1 = 6.06 \times 10^{-4} \end{cases}$$

$$\text{Iron} \quad \begin{cases} a_2 = .990 \text{ cm.} \\ b_2 = .737 \text{ cm.} \\ \mu_2 = 100 \\ \lambda_2 = 8 \times 10^{-5} \end{cases}$$

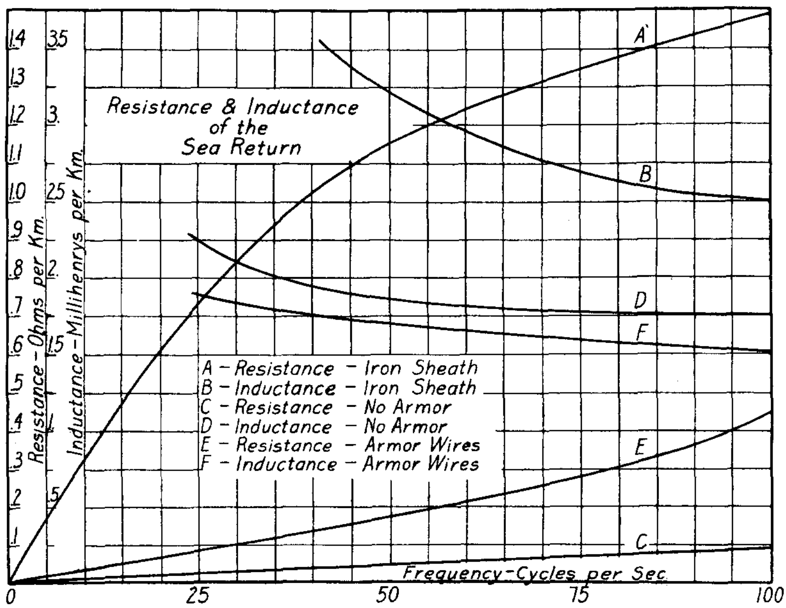
$$\text{Sea Water} \quad \begin{cases} a_3 = \infty \\ b_3 = .990 \text{ cm.} \\ \mu_3 = 1 \\ \lambda_3 = 5 \times 10^{-11} \end{cases}$$

The armoring was then assumed to be replaced by sea water, and the resistance and inductance of the cable were calculated from (33).

The results of the calculations are shown in the curves of Fig. 3.

It is evident from these curves that the effect of the iron armoring is to increase considerably the impedance of the return

FIG. 3.

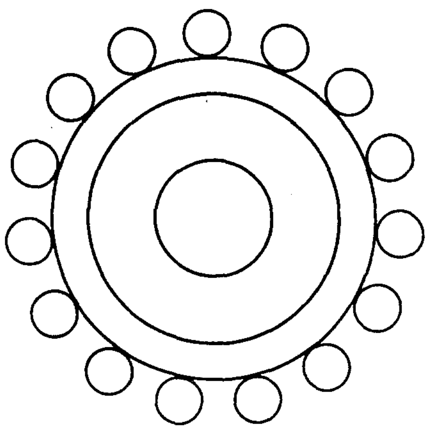


path. The physical explanation of this fact is that the iron acts as a shield to screen from the sea water the electromagnetic effects of the current flowing in the cable conductor. Energy is dissipated in the armoring and is prevented from spreading out through the surrounding medium. The assumption that the armor wires could be replaced by a solid cylinder of iron is, therefore, subject to question, since it is possible that the larger surface area of the assemblage of armor wires, and the gaps between these wires may be effective in diminishing the energy dissipated in the armoring and consequently diminishing the screening effect. This problem is investigated in the following section.

IV.

The physical system under consideration is shown schematically in cross-section in Fig. 4, and consists of an insulated conductor and protective covering of jute, surrounded by a ring of N armor wires immersed in sea water. The method of solution is essentially similar to that given in the preceding pages, and consists in determining the values of electric field intensity at the outer surface of the core conductor and the inner surface of the return conductor, from which the internal impedances of the two conductors can be found.

FIG. 4.



The main difficulty in the analysis is caused by the lack of uniaxial symmetry in the return conductor. This was overcome by employing a method developed by one of the authors [1] in a study of transmission in parallel wires.

The electric field intensity in the sea water satisfies the differential equation

$$\frac{\partial^2 E}{\partial r^2}+\frac{1}{r}\frac{\partial E}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E}{\partial \phi^2}-4\pi\lambda\mu piE=0,$$

the solution of which is a Fourier-Bessel expansion,

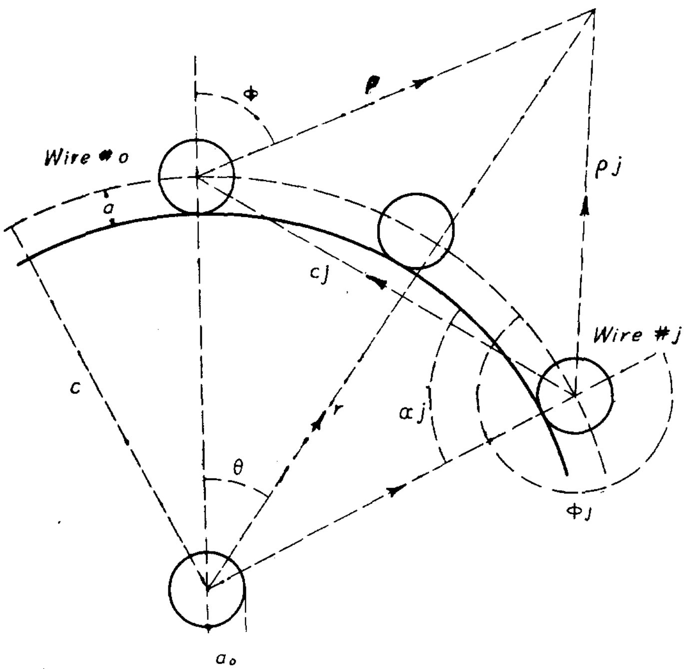
$$E=A_0K_0(r\alpha)+A_1K_1(r\alpha)\cos\phi+A_2K_2(r\alpha)\cos 2\phi+\cdots+,$$

r and ϕ being referred to the axis of the particular wire.

[1] "Wave Propagation over Parallel Wires; The Proximity Effect." John R. Carson, *Phil. Mag.*, vol. xli, p. 607 (1921).

Assuming that the current distribution in the core conductor is independent of the angle ϕ, that is, neglecting the individual character of the armor wires only in their effect on the current distribution in the core, the effect due to the current in the core is

FIG. 5.



represented by the first term of such a series, and the total field intensity may be written

$$E = AK_0(r\alpha) + \sum_{j=0}^{N-1} \sum_{s=0}^{\infty} B_s K_s(\alpha\rho_j) \cos s\phi_j, \tag{34}$$

ρ_j and ϕ_j being referred to the axis of wire j, as shown in Fig. 5. That is, the resultant field is expressible as a set of waves centered on the axis of the cable and the axes of the N armor wires.

In the neighborhood of the armor wires the arguments of the Bessel functions are sufficiently small [2] to permit of the approximations

[2] See Note I at end of paper.

$$K_0(\alpha\rho) = K - \log \rho,$$

where

$$K = 0.11593 \log \frac{1}{\alpha},$$

and

$$K_s(\alpha\rho) = \frac{1}{(-\alpha\rho)^s}.$$

The series (34) can, therefore, be written

$$E = A(K - \log r) + B_0(NK - \sum_{j=0}^{N-1} \log \rho_j) + \sum_{j=0}^{N-1} \sum_{s=1}^{\infty} B_s \frac{\cos(s\phi_j)}{\rho_j^s}, \tag{35}$$

in which B_s has absorbed the constant quantities. From this, the magnetic intensity in the sea water can be obtained by differentiation.

Inside any armor wire, at the surface, the field intensities are

$$E = C_0 J_0(\xi) + C_1 J_1(\xi) \cos\phi + \cdots + C_n J_n(\xi) \cos n\phi + \cdots +, \tag{36}$$

$$H_\Phi = \frac{1}{auip}\left[C_0 J_0'(\xi) + C_1 J_1'(\xi)\cos\phi + \cdots + C_n J_n'(\xi)\cos n\phi + \cdots +\right], \tag{37}$$

where $\xi = ai\sqrt{4\pi\lambda\mu pi}$,

λ and μ being the electrical conductivity and the magnetic permeability, respectively, of the material of the armor wire. The quantities a and ϕ are centered on the axis of the wire.

In order to determine the coefficients A, B_0, B_1, –, C_0, C_1, – we make use of the fact that the electric and the magnetic field intensities are continuous at the surface of the wire. It is obvious, however, that nothing can be learned by equating (35) and (36) since they are formally dissimilar. We therefore transform[8] the various terms of (35) to a common axis which coincides with the axis of one of the armor wires, hereafter called wire "zero," and the electric field intensity in the sea water, close to the surface of the armor wire, is

$$E = (A + NB_0)K - A\log c - B_0\log(a.c_1.c_2\ldots c_{n-1}) - \Sigma_0$$

[8] See Note II.

$$+\left[q_1/\zeta-\zeta\,(A+S_{11}B_o)+\frac{\zeta}{1!}\Sigma_1\right]\cos\phi$$

$$+\left[q_2/\zeta^2+\frac{\zeta^2}{2}(A+S_{22}B_o)+\frac{\zeta^2}{2!}\Sigma_2\right]\cos 2\phi \tag{38}$$

$$- - - - - - - - -$$

$$+\left[q_n/\zeta^n+\frac{(-\zeta)^n}{n}(A+S_{nn}B_o)-\frac{(-\zeta)^n}{n!}\Sigma_n\right]\cos n\phi,$$

where

$$\begin{aligned}
\Sigma_0 &= S_{11}q_1 - S_{22}q_2 + S_{33}q_3 \cdots, \\
\Sigma_1 &= S_{02}q_1 - 2S_{13}q_2 + 3S_{24}q_3 \cdots, \\
\Sigma_2 &= 1.2S_{13}q_1 - 2.3S_{04}q_2 + 3.4S_{15}q_3 \cdots, \\
\Sigma_3 &= 1.2.3S_{24}q_1 - 2.3.4S_{15}q_2 + 3.4.5S_{06}q_3 \cdots,
\end{aligned} \tag{39}$$

$$- - - - - - - - -$$

$$\begin{aligned}
S_{pq} &= \sum_{j=1}^{N-1}\frac{\cos pa_j}{(2\sin\frac{j\pi}{n})q}, \\
q_n &= B_n/c^n, \\
\zeta &= \frac{a}{c}.
\end{aligned} \tag{40}$$

The quantities ϕ and a have the same significance as in equations (36) and (37).

The tangential magnetic field intensity in the sea water at the surface of wire "zero" is, therefore,

$$H_\Phi=\frac{1}{ip}\,\frac{dE_z}{da}=\frac{1}{ipa}\Bigg[-B_o+\cos\phi\left[-q_1/\zeta-\zeta(A+S_{11}B_o)+\frac{\zeta}{1!}\Sigma_1\right] \tag{41}$$

$$+2\cos 2\phi\left[-q_2/\zeta^2+\frac{\zeta^2}{2}(A+S_{22}B_o)-\frac{\zeta^2}{2!}\Sigma_2\right]$$

$$- - - - - - - - - -$$

$$+n\cos n\phi\left[-q_n/\zeta^n+\frac{(-\zeta)^n}{n}(A+S_{nn}B_o)-\frac{(-\zeta)^n}{n!}\Sigma_n\right]- - -\Bigg].$$

To satisfy the condition of continuity of electric and magnetic field intensities at the surface of the armor wire it is necessary that the coefficients of the corresponding terms of (36) and (38) and of (37) and (41) to be equal. This gives

$$C_o J_o(\xi) = (A + NB_o)K - A \log c - B_o \log (a.c_1 \ldots c_n) - \Sigma_o \,, \tag{42}$$

$$C_n J_n(\xi) = q_n/\zeta^n + \frac{(-\zeta)^n}{n}(A + S_{nn}B_o) - \frac{(-\zeta)^n}{n!}\Sigma_n \,, \quad n = 1 - - \infty \tag{43}$$

$$\xi J'_o(\xi) C_o = -\mu B_o \,, \tag{44}$$

$$\xi J'_n(\xi) C_n = -n\mu \left[q_n/\zeta^n - \frac{(-\zeta)^n}{n}(A + S_{nn}B_o) - \frac{(-\zeta)^n}{n!}\Sigma_n \right] . \; n = 1 - - \infty \tag{45}$$

From these expressions the quantities $B_1 \ldots C_1 \ldots$ can be determined. Multiplying (43) by $n\mu$ and subtracting (45) gives

$$C_n = \frac{2n\mu q_n}{\zeta^n} \, \frac{1}{n\mu J_n(\xi) - \xi J'_n(\xi)} \,, \tag{46}$$

which expresses C_n in terms of q_n. Multiplying (43) by $\zeta J'_n(\zeta)$ and (45) by $J_n(\zeta)$, and subtracting gives

$$q_n = (-1)^n \lambda_n \zeta^{2n} \left[\frac{1}{n}(A + S_{nn}B_o) - \frac{1}{n!}\Sigma_n \right] , \; n = 1 - - \infty \tag{47}$$

where

$$\lambda_n = \frac{n\mu J_n(\xi) - \xi J'_n(\xi)}{n\mu J_n(\xi) + \xi J'_n(\xi)} \,. \tag{48}$$

From the infinite set of simultaneous equations (47) the infinitely many varieties q_n may be determined in terms of A and B_o.[4]

We have thus determined the arbitrary constants $C_o \ldots C_n$ and $q_1 \ldots q_n$ (or $B_1 \ldots B_n$) as functions of A and B_o. It remains to express the latter quantities in terms of physical quantities. If I_1 is the

[4] See Note III.

current in the armor then $\frac{I_1}{N}$ is the current in a single wire. Integrating (41) completely around the armor wire "zero" gives, therefore,

$$2pi\frac{I_1}{N} = -B_o . \tag{49}$$

Similarly, if I_o is the current in the core conductor, we find

$$2piI_o = -A . \tag{50}$$

We can, therefore, express all the arbitrary constants as linear, homogeneous functions of I_o and I_1.

To determine the relation between these currents, we have from (42) and (44),

$$CJ_o(\xi) = \frac{ZI_1}{N} , \tag{51}$$

where

$$Z = \frac{2\mu ip}{\xi}\frac{J_o(\xi)}{J_o'(\xi)} .$$

Substituting (49), (50) and (51) in (42) gives

$$\frac{Z}{N}I_1 = -2ip\,(I_o+I_1)K + 2ipI_o\log c + 2ip\frac{I_1}{N}\log(a.c_1 \ldots c_n) \tag{52}$$

$$-(S_{11}q_1 - S_{22}q_2 + S_{33}q_3 - \ldots),$$

from which, since $q_1 \ldots q_n$ are functions of I_1 and I_o, the ratio I_o/I_1 can be obtained.

Having shown that the constants A, B_o ... of the series (35) are proportional to I_o, we can express the electric field intensity at the inner surface of the return conductor in the form

$$E_2 = -Z_2 I_o .$$

The computation of Z_2 is facilitated by transforming the terms of (35) to the axis of the core conductor[5] and placing $r = c - a$. We thus obtain

$$E_2 = -Z_2 I_o = (A + NB_o)K - A\log(c-a) - NB_o\log c - N(q_1 - q_2 + q_3 - -)$$

$$+ (\text{terms containing } \cos\theta, \cos 2\theta, \text{etc., as factors}). \tag{53}$$

[5] See Note II.

We have, by applying the curl law to an elementary contour which links the core conductor and the return,

$$\frac{\partial V}{\partial z} - E_1 + E_2 = -ip\Phi_{12}, \tag{54}$$

where

$$E_1 = Z_1 I_0 = \frac{2\mu_0 ip}{\xi_0} \frac{J_0(\xi_0)}{J_0'(\xi_0)} I_0,$$

$$\Phi_{12} = L_{12} I_0 = 2 I_0 \log \frac{c-a}{a_0}, \tag{55}$$

and

$$\xi_0 = a_0 i \sqrt{4\pi\lambda_0 \mu_0 ip},$$

λ_0 and μ_0 being the electrical constants of the core conductor and a_0 its radius. The value given above for Φ_{12} holds only for the contour on which E_2 is independent of the angle θ, that is, when the terms of (53) that contain $\cos\theta$, $\cos 2\theta$, etc., vanish. The value of Z_2 to be used in (54) is therefore determined from

$$E_2 = -Z_2 I_0 = (A + NB_0)K - A \log(c-a) - NB_0 \log c - N(q_1 - q_2 +). \tag{56}$$

As before,

$$-\frac{\partial V}{\partial z} = (R + ipL) I_0, \tag{57}$$

where R and L are the resistance and inductance per unit length of the cable, including the sea return.

We have then from (54),

$$R + ipL = Z_1 + Z_2 + ipL_{12}, \tag{58}$$

from which R and L can be determined.

The process of calculating the resistance and inductance of a submarine cable by the method just described may be summarized as follows:

(1) Determine from (47) the quantities $q_1 \ldots q_n$ in terms of A and B_0, and then in terms of I_1 and I_0 by (49) and (50).

(2) Substitute these values of $q_1 \ldots q_n$ in (52) and obtain the ratio I_0/I_1.

(3) Substitute for A, B_0 and $q_1 \ldots q_n$ in (56) their values in terms of I_0 and I_1.

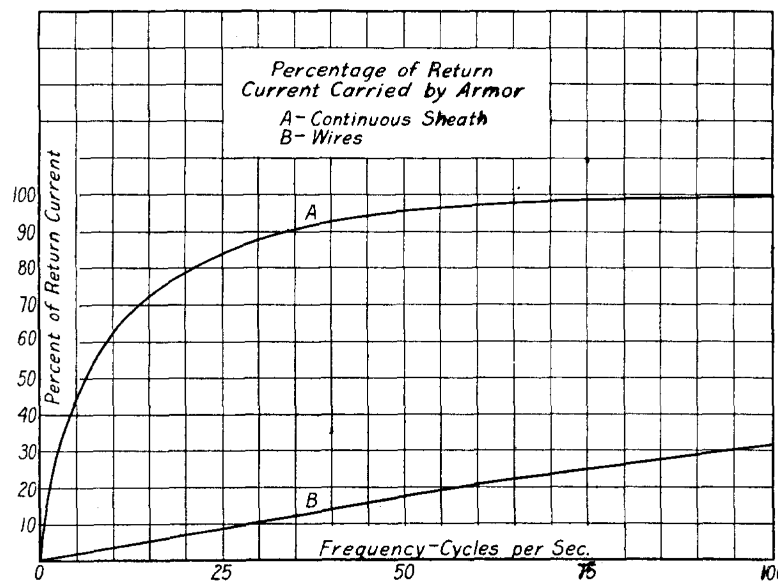
(4) Eliminate I_1 from these two relations, thus obtaining E_2 in terms of I_o. Then $Z_2 = -E_2/I_o$.

(5) Substitute this value of Z_2 and the value of Z_1 calculated from (55) in equation (58).

(6) The resistance and the inductance per unit length of the cable may then be determined from the real and imaginary parts of the latter equation.

The resistance and inductance of a cable of cross-section shown in Fig. 4 were computed by the method just described, the results being given by curves E and F of Fig. 3. The cable in this case is identical with that shown in Fig. 2 previously described,

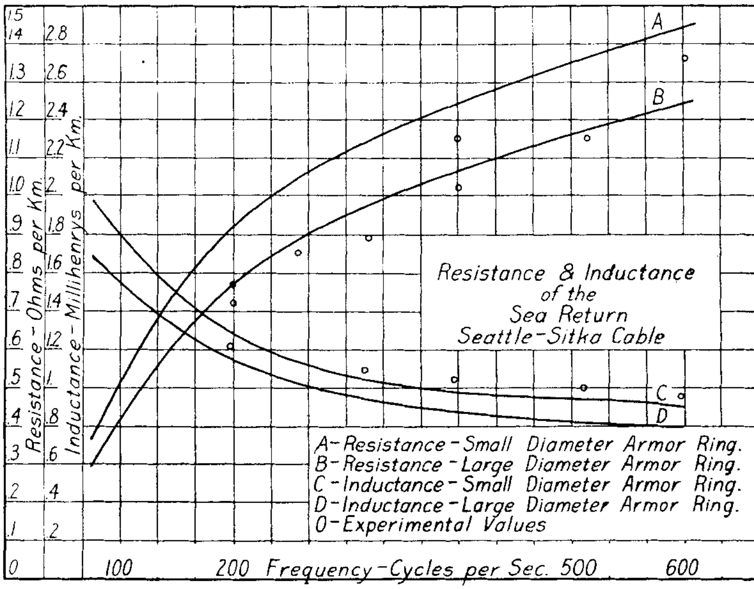
FIG. 6.



except that the continuous iron sheath has been replaced by fifteen wires. The effect of the presence of the iron upon the resistance of the return conductor is still noticeable, although it is much less than in the case of the continuous iron sheath. The reason for this is evident after inspection of the curves of Fig. 6, which show the percentage of return current carried by the armor in the two cases. Especially at the lower frequencies, the return current is much more confined by the continuous sheath than it is by the wires.

As a check of the method, the resistance and inductance of the Seattle-Sitka cable of the United States Signal Corps were calculater for frequencies in the range 50 to 600 cycles per second,

FIG. 7.



and the values so obtained were then compared with the results of measurements recently made upon this cable.[6] The constants used in the calculations were as follows:

Conductor	
Diameter	.216 cm.
Resistance per nautical mile	9 ohms
Rubber Insulation	
Outside diameter	.718 cm.
Capacity per nautical mile	.38 mf.
Armoring	
16 wires	each .242 cm. diameter
Outside Diameter of Cable	2.06 cm.

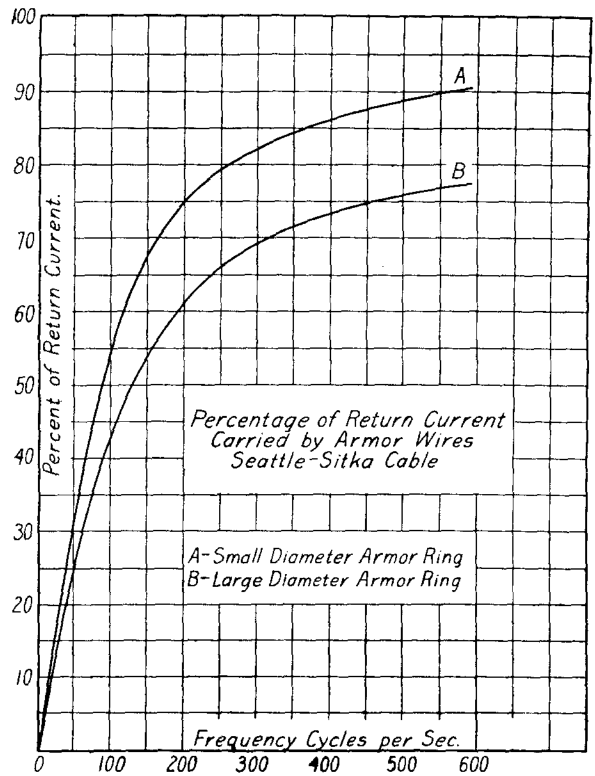
Owing to lack of information concerning the mean radius of the ring of armor wires, two sets of data were computed em-

[6] "The Use of Alternating Currents for Submarine Cable Transmission," Frederick E. Pernot, JOUR. OF THE FRANKLIN INSTITUTE, vol. 190, p. 323, 1920.

ploying the values $c=0.6148$ and $c=0.920$, which correspond, respectively, to zero and maximum separation of the armor wires.

The results of the calculations are shown in Fig. 7. The experimental values are indicated by small circles, and agree well with the theoretical values throughout the range of frequencies.

FIG. 8.



The resistance of the sea return increases most rapidly in the region of frequencies used in ordinary telegraphy, 0 to 100 cycles per second. In this range the inductance of the cable also has its greatest values, and these two effects have considerable influence in determining the transmission characteristics of the cable.

The percentage of the return current that is carried by the armor wires is shown in Fig. 8.

CONCLUSIONS.

As was previously pointed out, the effect of the shielding action of the iron armor of a submarine cable is to diminish the electromagnetic field which is propagated through the sea water, and which gives rise to the return current. Combined with this effect is the shielding action of the sea water adjacent to the cable, upon the distant portions. The total shielding effect increases with the frequency until a point is reached where practically the whole of the return current is carried by the armor wires.

Several remedies have been suggested for diminishing the damping effect of the armor wires. It can be proved, for example, that for a given size of core and weight of armor, the number and size of armor wires can be chosen so as to give a minimum value of return impedance. A proper choice of the electrical constants of the material of which the armor is constructed would also be of advantage, since the return impedance is somewhat larger for iron than it is for material of higher or lower conductivity.

Another method of diminishing the return impedance, which has been used in practice, is to wrap the cable core with a number of concentric layers of conducting tape before it is covered with jute. The return current, as it crowds in toward the core with increasing frequency, will then have a path of comparatively low impedance, and at the higher fequencies only a small portion of the current will be carried by the armor wires and the sea water. The impedance of the return path can be calculated for this case by the methods given in the preceding pages. The following table compares the values of the resistance of the return conductor calculated by three different methods, and determined experimentally, for a cable provided with a brass tape 5 mils in thickness.

Resistance of Return Conductor—OHMS per Statute Mile.

Frequency Cycles per Sec.	Approximate Method	Approx. Method* Corrected by Factor $\frac{2}{\pi}$	Exact Method	Experimental
3000	4.00	3.15	2.87	2.92
10000	4.90	4.25	4.45	4.60

*This is an empirical formula which has been found to be fairly close in most cases. The correction factor suggested itself in that it takes care of the increased surface of the armor wires, as compared with the corresponding continuous sheath.

The experimental values are the results of a series of measurements made by the Department of Development and Research of the American Telephone and Telegraph Company upon the Victoria-Vancouver submarine cable. The calculated values were obtained by both the approximate and the exact methods, discussed in the preceding pages, in which the armor of the cable is treated, respectively, as a continuous sheath and as a ring of wires. The modifications which must be introduced to include the effect of the conducting tape are outlined in the discussion of the general theory. The agreement between the calculated and the measured values of return resistance proves that the method developed in the present paper is accurate even at the highest frequencies employed in telephony.

NOTE I.

NOTE ON BESSEL FUNCTIONS.

The Bessel Functions of zero order of the first and second kinds, $J_0(\rho)$ and $K_0(\rho)$, used in the preceding work are all to a complex argument $\rho = iq\sqrt{i}$ where q is a real number and $i = \sqrt{-1}$. The following formulas may be used for determining the values of these functions:

$q < 0.1$

$$J_0(\rho) = 1 \qquad J_0'(\rho) = -\frac{1}{2}\rho$$

$$K_0(\rho) = \log_e \frac{2}{\gamma\rho} = .11593 - \log_e q - \frac{\pi i^*}{4}$$

$$K_0'(\rho) = -\frac{1}{\rho}$$

(*Jahnke u. Emde*, "Funktionentafeln," pp. 97, 98.)

$0.1 < q < 10$

The report of the British Association for 1912 and 1915 give the values in this range of the functions ber q, ber$'q$, bei q, bei$'q$, ker q, ker$'q$, kei q, kei$'q$ which are defined by the relations

*It is to be noted that this approximation for K_0 (ρ) differs from the expression used by J. J. Thomson, "Recent Researches in Electricity and Magnetism," p. 263. Thomson's formula (2), from which his approximation was derived, contains a number of errors and should read

$$K_0(x) = (-C + \log 2i - \log x) J_0(x) - 2 J_2(x) - \frac{1}{2} J_4(x) + \frac{1}{3} J_6(x) \ldots \ldots,$$

where $C = .5772 = \log \gamma$.

$$J_0(iq\sqrt{i}) = \operatorname{ber} q + i \operatorname{bei} q \;,$$

$$i\sqrt{i}\; J_0'(iq\sqrt{i}) = \operatorname{ber}' q + i \operatorname{bei}' q \;,$$

$$K_0(iq\sqrt{i}) = \operatorname{ker} q + i \operatorname{kei} q \;,$$

$$i\sqrt{i}\, K_0'(iq\sqrt{i}) = \operatorname{ker}' q + i \operatorname{kei}' q \;.$$

$q > 10$

$$J_0(q\sqrt{-i}) = \frac{\varepsilon^{q/\sqrt{2}}}{\sqrt{2\pi q}}\left[\cos\left(\frac{q}{\sqrt{2}} - \frac{\pi}{8}\right) - i\sin\left(\frac{q}{\sqrt{2}} - \frac{\pi}{8}\right)\right],$$

$$J_0'(q\sqrt{-i}) = -\,i\, J_0(q\sqrt{-i}),$$

$$K_0(q\sqrt{-i} = \frac{\varepsilon^{-q/\sqrt{2}}}{\sqrt{\frac{1}{2}\pi q}}\left[\sin\left(\frac{q}{\sqrt{2}} + \frac{\pi}{8}\right) + i\cos\left(\frac{q}{\sqrt{2}} + \frac{\pi}{8}\right)\right],$$

$$K_0'(q\sqrt{-i}) = i\, K_0(q\sqrt{-i}).$$

(*Jahnke u. Emde*, pp. 101, 102.)

NOTE II.

TRANSFORMATION OF FOURIER-BESSEL EXPANSION.

In problems involving Fourier-Bessel expansions it is sometimes necessary to transform quantities of the form

$$\frac{\cos s\phi_j}{\rho_j^s},\ \frac{\sin s\phi_j}{\rho_j^s},\ \log \rho_j\,,$$

from the system of coördinates ρ_j, ϕ_j to the systems ρ, ϕ or r, θ which are related as shown in Fig. 5.

The necessary formula may be derived as follows. We have

$$\frac{\cos s\phi_j + i \sin s\phi_j}{\rho_j} = \frac{\varepsilon^{is\phi_j}}{\rho_j^s} = \left(\frac{\varepsilon^{i\phi}}{\rho_j}\right)^s = \frac{1}{Z_j^s},$$

where Z_j is the conjugate of the vector $Z'_j = \rho_j \varepsilon^{i\phi_j}$. Similarly we may write

$$Z = \rho\varepsilon^{i(\phi - \pi + 2a_j)},$$

$$C_j = c_j\varepsilon^{i(\pi + a_j)}$$

The vectors Z_j, Z and C, as may be seen from Fig. 5, have the lengths ρ_j, ρ and c, respectively, and the directions indicated by the arrows.

By vector addition,

$$Z'_j = Z + C_j$$

whence

$$Z_j = Z' + C'_j,$$

where Z' and C'_j are the conjugates of Z and C_j respectively.

By expansion

$$\frac{1}{Z_j^s} = \frac{1}{(Z' + C'_j)^s} = \frac{1}{C_j'^s}\left[1 - \frac{s}{1}\frac{Z'}{C'_j} + \frac{s(s+1)}{1.2}\frac{Z'^2}{C_j'^2} - \frac{s(s+1)(s+2)}{1.2.3}\frac{Z'^3}{C_j'^3} + --\right].$$

We have

$$\frac{1}{C_j'^s} = \frac{\varepsilon^{is(\pi + a_j)}}{C_j^s} = (-1)^s \frac{\varepsilon^{isa_j}}{C_j^s},$$

and

$$\frac{Z'^n}{C_j'^n} = \frac{\rho^n}{C_j^n}\varepsilon^{in(2\pi - \phi - a_j)} = \frac{\rho^n}{C_j^n}\varepsilon^{-in(\phi + a_j)}$$

Therefore

$$\frac{\varepsilon^{si\phi_j}}{\rho_j^s} = \frac{1}{Z_j^s} = \frac{(-1)^s}{C_j^s}\left[\varepsilon^{isa_j} - \frac{s}{1}\frac{\rho}{C_j}\varepsilon^{-i(\phi - a_j(s-1))} + \frac{s(s+1)}{1.2}\frac{\rho^2}{C_j^2}\varepsilon^{-i(2\phi - a_j(s-2))} - \right]$$

Equating the real and imaginary parts gives

$$\frac{\cos s\phi_j}{\rho_j^s} = \frac{(-1)^s}{C_j^s}\Bigg[\cos sa_j - \frac{s}{1}\frac{\rho}{C_j}\cos(\phi - a_j[s-1]) +$$

$$\frac{s(s+1)}{1.2}\frac{\rho^2}{C_j^2}\cos(2\phi - a_j[s-2]) - - - - - \Bigg]$$

$$\frac{\sin s\phi_j}{\rho_j^s} = \frac{(-1)^s}{C_j^s}\Bigg[\sin sa_j + \frac{s}{1}\frac{\rho}{C_j}\sin(\phi - a_j[s-1]) +$$

$$\frac{s(s+1)}{1.2}\frac{\rho^2}{C_j^2}\sin(2\phi - a_j[s-2]) - - - - - - \Bigg].$$

Similarly

$$\log Z_j = \log (C'_j + Z')$$

$$= \log C_j + \frac{Z'}{C'} - \frac{1}{2}\frac{Z'^2}{C'^2} + \frac{1}{3}\frac{Z'^3}{C'^3} - - - - - - - -$$

$$= \log C_j + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{\rho^n}{C_j^n} \varepsilon^{in(\phi - a_j)}.$$

Equating real and imaginary parts we have

$$\log \rho_j = \log C_j + \frac{\rho}{C_j}\cos(\phi - a_j) - \frac{1}{2}\frac{\rho^2}{C_j^2}\cos 2(\phi - a_j) + \cdots + ,$$

$$\phi_j = \frac{\rho}{C_j}\sin(\phi - a_j) - \frac{1}{2}\frac{\rho^2}{C_j^2}\sin 2(\phi - a_j) + \cdots + .$$

The following formulas may be derived in a similar manner:

$$\frac{\cos s\phi_j}{\rho_j^s} = \frac{(-1)^s}{C^s}\left[1 + \frac{s}{1}\frac{r}{c}\cos(\theta - \gamma_j) + \frac{s(s+1)}{1.2}\frac{r^2}{c^2}\cos 2(\theta - \gamma_j) + \cdots + \right],$$

$$\frac{\sin s\phi_j}{\rho_j^s} = \frac{(-1)^{s+1}}{C^s}\left[\frac{s}{1}\frac{r}{c}\sin(\theta - \gamma_j) + \frac{s(s+1)}{1.2}\frac{r^2}{c^2}\sin 2(\theta - \gamma_j) + \cdots + \right],$$

$$\log \rho_j = \log c - \frac{r}{c}\cos(\theta - \gamma_j) - \frac{1}{2}\frac{r^2}{c^2}\cos 2(\theta - \gamma_j) - - - - - .$$

NOTE III.

DETERMINATION OF q_1, q_2, ETC.

The equations (47),

$$q_n = (-1)^n \lambda_n \frac{\zeta^{2n}}{n}(A + S_{nn} B_o) - (-1)^n \lambda_n \frac{\zeta^{2n}}{n!}\Sigma_n, \quad n = 1 - - \infty$$

are linear in the variables q_1, q_2 –, since

$$\Sigma_n = n! S_{n-1,n+1} q_1 - \frac{n!}{1!} S_{n-2,n+2} q_2 - - - .$$

The values q_1, q_2 ... may be determined by a method of approximations, q_n being the limit of the sequence

$$q_n^{(0)}, q_n^{(1)}, q_n^{(2)} - - - - - ,$$

the successive terms of which are defined by the expressions

$$q_n^{(0)} = (-1)^n \lambda_n \frac{\zeta^{2n}}{n} (A + S_{nn} B_0),$$

$$- - - - - - -$$

$$q_n^{(j+1)} = (-1)^n \frac{\zeta^{2n}}{n!} (A + S_{nn} B_0) - (-1)^n \frac{\zeta^{2n}}{n!} \Sigma_n (q^{(j)}),$$

where $\Sigma_n(q^{(j)})$ is the value of Σ_n when q_1, q_2 – are replaced by

$q_1^{(j)}$, $q_2^{(j)}$ – – – .

This method, however, while formally simple and direct is not usually well adapted for numerical solution. For all sizes of armor wire and for frequencies of practical importance the argument ζ in the expression (48) is small compared with μ and the quantities,

$$\lambda_1, \lambda_2 ---$$

are all nearly unity. This suggests the use of the following method of solution of equations (47).

The solution of the auxiliary set of equations

$$p_1 = -\zeta^2 (A + S_{11} B_0) + \frac{\zeta^2}{1!} \Sigma_1(p),$$

$$- - - - - - - -$$

$$p_n = (-1)^n \frac{\zeta^{2n}}{n} (A + S_{11} B_0) - (-1)^n \frac{\zeta^{2n}}{n!} \Sigma_n(p)$$

in the auxiliary variables p_1, p_2 – may be written,

$$p_1 = -\zeta^2 C_{11} (A + S_{11} B_0) + \frac{\zeta^4}{2} C_{12} (A + S_{22} B_0) + \cdots + ,$$

$$- - - - - - - - - - - -$$

$$p_n = -\zeta^2 C_{n1} (A + S_{11} B_0) + \frac{\zeta^4}{2} C_{n2} (A + S_{22} B_0) + \cdots + ,$$

in which C_{11}, etc., are numerics. This solution is effected by retaining a finite number of equations and an equal number of variables and solving by the usual methods. It will be found that except in extreme cases, a very good approximation can be gotten by ignoring all the p's except the first four. The q's may then be obtained by the relation

$$q_n = p_n + d_n$$

d_n being defined by

$$d_n = (\lambda_n - 1)p_n - (-1)^n \lambda_n \frac{\zeta^{2n}}{n!} \Sigma_n(d).$$

This system is easily adapted to solution by successive approximations,

$$d_n = d_n^{(0)} + d_n^{(1)} + d_n^{(2)} + - - -$$

in which

$$d_n^{(0)} = (1 - \frac{1}{\lambda_1})C_{n1}p_1 + \cdots + (1 - \frac{1}{\gamma_n})C_{nn}p_n ,$$

$$d_n^{(j+1)} = (1 - \frac{1}{\lambda_1}) C_{n1}d_1^{(j)} + \cdots + (1 - \frac{1}{\lambda_n})C_{nn}d_n^{(j)} ,$$

C_{n1}, etc., being the numerical coefficients which appear in the expressions for p_1, p_2–.

A very good approximation which holds in most cases is

$$d_n = (\lambda_1 - 1) C_{n1} p_1 + (\lambda_2 - 1)C_{n2} p_2 + \cdots + (\lambda_n - 1) C_{nn} p_n .$$

FEBRUARY 15, 1921.

Compressibility of Aromatic Hydrocarbons. THEODORE W. RICHARDS, EDWARD P. BARTLETT, and JAMES H. HODGES, of Harvard University (*Jour. Am. Chem. Soc.*, 1921, xliii, 1539–1542), find that *solid* benzene at a temperature of 0° C. has a compressibility of 0.0000305 over the range of 100 to 500 megabars, and that *liquid* benzene at a temperature of 20° C. has a compressibility of 0.00007207 over the same range. *Liquid* toluene at a temperature of 0° C. has a compressibility of 0.0000618 over the range stated.

J. S. H.