251. Some Properties of the Conic Treated by Other Methods

Author(s): P. N. Dutt

Source: *The Mathematical Gazette*, Vol. 4, No. 69 (Jan., 1908), pp. 210-211

Published by: [Mathematical Association](#)

Stable URL: <http://www.jstor.org/stable/3602963>

Accessed: 11-02-2016 14:24 UTC

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Mathematical Association is collaborating with JSTOR to digitize, preserve and extend access to *The Mathematical Gazette*.

<http://www.jstor.org>

When the modulus is a power of 2, then, generally, if $r = 2^q - 1$,
 $r^2 = (2^q - 1)^2 \equiv +1 \pmod{2^q \text{ and } 2^{q+1}}$; and $r < 2^q$.

This appears to be the only case for which a general rule is known (under the above limitation).

ALLAN CUNNINGHAM, Lt.-Col. R.E.

251. [L₁. 19. d.] *Some properties of the Conic treated by other methods.*

1. If we represent the distance from a fixed point by r_1 and that from another fixed point by r_2 , the equation of the central conic may be written in the form $r_1 \pm r_2 = 2a$, where the two fixed points are the foci of the conic.

Differentiating, we get $\frac{dr_1}{ds} \pm \frac{dr_2}{ds} = 0$, but $\frac{dr_1}{ds}$ and $-\frac{dr_2}{ds}$ are the cosines of the angles which the tangent makes with the focal distances respectively, and, therefore, the tangent bisects the angle between the focal distances, it being the external bisector in the case of the ellipse and the internal bisector in the case of the hyperbola, and we have, therefore, two confocals intersecting at right angles.

2. Suppose we have two confocals $r_1 + r_2 = 2a$ and $r_1 - r_2 = 2b$. At points of intersection $r_1 = a + b$ and $r_2 = a - b$, which represent two circles with the foci as centres. If $2c$ be the distance between the foci, the condition that the two circles should intersect in real points is that $a > c > b$, which is, therefore, also the condition that the two confocals should intersect in real points. If we suppose the two points $r_1 = a + b$ and $r_2 = a - b$ as fixed, we have only two confocals, viz. $r_1 + r_2 = 2a$ and $r_1 - r_2 = a - b$ passing through them.

3. The following propositions on two confocals passing through P are easy deductions from the principle indicated (S, S' as usual representing the two foci).

- (i) Rectangle $SP \cdot S'P$ = difference of squares on the semi-axes of the confocals.
- (ii) $SP^2 - S'P^2$ = rectangle contained by the axes of the two confocals.
- (iii) $SP^2 + S'P^2$ = twice the sum of the squares on the semi-axes of the two confocals.

Joining (i) to a known property, we have the square on the semi-conjugate diameter = difference of squares on the semi-axes of the confocals.

Also from (iii) we can show that the sum of the squares on the semi-axes of the confocals = $CP^2 + CS^2$, where C is the centre.

Hence as CS is constant, the locus of P is a circle if the sum of the squares of the axes of the confocals through P is given.

4. The equation of a conic may also be written in the form $r = ex$, where r denotes the distance from a fixed point and x the distance from a fixed straight line.

The fixed point is the focus, and the fixed straight line is the directrix. When $e = 1$ the equation represents a parabola; for the ellipse $e < 1$; and for the hyperbola $e > 1$. We may if we choose let r represent the distance from a fixed circle, then if a denote the radius of the circle, writing the equation in the form $r + a = e\left(x + \frac{a}{e}\right)$, we see that the equation represents a conic, the focus being at the centre of the circle and the directrix parallel to the line from which the distances are measured, and at a distance $\frac{a}{e}$.

5. Differentiating, we have $\frac{dr}{ds} = e \frac{dx}{ds}$, and therefore the angle which the focal distance makes with the tangent is $> =$ or $<$, the angle which the tangent makes with the axis according as $e < = > 1$. If for ds we write dt , we get a similar proposition about velocities. Also if G be the point where the normal at P meets the axis, $\sin SPG = e \sin SGP$ or $\frac{SG}{SP} = e$.

6. Let P, P' be two consecutive points on the conic, then

$$GG' - e^2 dx = e^2 PP' \sin SGP.$$

Also $\frac{PP'}{\rho} = \frac{GG' \sin SGP}{\rho - PG};$

$$\therefore \frac{1}{\rho} = \frac{e^2 \sin^2 SGP}{\rho - PG},$$

$$\begin{aligned} \text{or } \rho &= \frac{PG}{1 - e^2 \sin^2 SGP} = \frac{PG}{1 - e^2 \sin^2 PSG} \cdot \frac{SP^2}{PG^2} \\ &= \frac{PG^3}{PG^2 - e^2 SP^2 \sin^2 PSG} \\ &= \frac{PG^3}{SP^2 + SG^2 - 2SP \cdot SG \cos PSG - e^2 \sin^2 PSG \cdot SP^2} \\ &= \frac{PG^3}{SP^2(1 + e^2 - 2e \cos PSG - e^2 \sin^2 PSG)} \\ &= \frac{PG^3}{SP^2(1 - e \cos PSG)^2} = \frac{PG^3}{l^2}, \end{aligned}$$

where l is the semi latus rectum.

P. N. DUTT.

252. [L. 17. e.] *A Case of Double Contact.*

The equation of any conic having double contact with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the chord of contact being $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$, is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = \lambda \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 \right)^2$; when λ is so chosen that the terms of the highest degree form a perfect square and the point $x'y'$ is on the ellipse, the above represents the parabola of closest contact at $x'y'$. The condition for this is that

$$\begin{aligned} \left(\frac{1}{a^2} - \frac{\lambda x'^2}{a^4} \right) \left(\frac{1}{b^2} - \frac{\lambda y'^2}{b^4} \right) &= \frac{\lambda^2 x'^2 y'^2}{a^4 b^4}, \\ \text{or } \frac{1}{a^2 b^2} &= \frac{\lambda}{a^2 b^2} \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} \right); \therefore \lambda = 1. \end{aligned}$$

Extending the above result to the case of the conic given by the general equation, we see that the equation of the parabola of closest contact is given by $SA + CT^2 = 0$ (Mathematical Tripos, Cambridge, 1906), where S is the equation of the conic, A the discriminant, C its first minor with respect to C , and T is the equation of the tangent at a point. For, proceeding as in the previous example, we have

$$\begin{aligned} &\{a - \lambda(ax' + hy' + g)^2\} \{b - \lambda(hx' + by' + f)^2\} \\ &= \{h - \lambda(ax' + hy' + g)(hx' + by' + f)\}^2, \\ \text{or } ab - h^2 &= \lambda \{a(hx' + by' + f)^2 + b(ax' + hy' + g)^2 \\ &\quad - 2h(ax' + hy' + g)(hx' + by' + f)\} \\ &= \lambda \{[ab - h^2][ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy'] \\ &\quad + af^2 + bg^2 - 2fgh\} \\ &= \lambda \{(ab - h^2)(ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c) \\ &\quad - c(ab - h^2) + af^2 + bg^2 - 2fgh\} \\ &= -\lambda \Delta; \therefore \lambda = -\frac{C}{\Delta}. \end{aligned}$$

P. N. DUTT.