F.R.S.; Lord Rayleigh, F.R.S.; W. Spottiswoode, F.R.S.; Treasurer, S. Roberts, M.A.; Hon. Secretaries, M. Jenkins, M.A., and R. Tucker, M.A. Other Members, Prof. Cayley, F.R.S., Prof. Clifford, F.R.S., T. Cotterill, M.A., J. W. L. Glaisher, F.R.S., Rev. R. Harley, F.R.S., *Prof. Henrici, F.R.S., C. W. Merrifield, F.R.S., Prof. Sylvester, F.R.S., *H. M. Taylor, M.A. (The gentlemen marked * were not on the previous Council list.)

Prof. H. W. Lloyd Tanner, M.A., Jesus College, Oxford, was elected a Member; Major J. R. Campbell, and Prof. G. M. Minchin, M.A., were proposed for election; Messrs. Jeffery and Rhodes were admitted into the Society. Visitors present were Mr. John Ball, Dr. Inglis, Mr. Tait, and the Rev. Alexander Taylor.

Prof. Sylvester gave a detailed account of results arrived at in his paper "On the Fifteen young Ladies Problem, and a General Mathematical Theory of Pure Syntax."

The following further communications were made to the Society:—Mr. J. Hammond, "On the Relation between Bernoulli's Numbers and the Binomial Coefficients;" Mr. S. Roberts, "On Three-bar Motion in Plane Space;" Mr. J. W. L. Glaisher, "Values of Certain Infinite Products, with an Application to the Summation of a Geometrical Series of the n^{th} Order as a Definite Integral;" Major J. R. Campbell, "On the Form of Cam which, acting on a Lever, shall communicate a motion such that the angular velocity-ratio of the lever and cam is a given function of the angle described by the latter" (communicated by the Secretary).

On the Relation between Bernoulli's Numbers and the Binomial Coefficients. By J. HAMMOND, B.A.

[Read November 11th, 1875.]

The expansion of $\frac{\psi(x)}{\phi(x)}$ is

$$\begin{aligned} \frac{\psi}{\phi} - \left(\frac{1}{\phi}\right)^2 \left| \begin{array}{cc} \psi & \phi \\ \psi' & \phi' \end{array} \right| x + \left(\frac{1}{\phi}\right)^3 \left| \begin{array}{ccc} \psi & \phi & 0 \\ \psi' & \phi' & \phi \\ \psi'' & \phi'' & 2\phi' \end{array} \right| \frac{x^2}{2} \\ - \left(\frac{1}{\phi}\right)^4 \left| \begin{array}{cccc} \psi & \phi & 0 & 0 \\ \psi' & \phi' & \phi & 0 \\ \psi'' & \phi'' & 2\phi' & \phi \\ \psi''' & \phi''' & 3\phi'' & 3\phi' \end{array} \right| \frac{x^3}{3} + \&c. \end{aligned}$$

And that of $\frac{1}{\phi(x)}$ is

$$\frac{1}{\phi} - \frac{\phi'}{(\phi)^2} x + \left(\frac{1}{\phi}\right)^3 \left| \begin{array}{cc} \phi' & \phi \\ \phi'' & 2\phi' \end{array} \right| \frac{x^2}{2} - \left(\frac{1}{\phi}\right)^4 \left| \begin{array}{ccc} \phi' & \phi & 0 \\ \phi'' & 2\phi' & \phi \\ \phi''' & 3\phi'' & 3\phi' \end{array} \right| \frac{x^3}{3} + \&c.$$

(Proceedings, vol. vi., p. 69.)

These give values for Bernoulli's numbers in the form of determinants of the coefficients of the Binomial Theorem.

I. Expand

$$\frac{x}{e^x - 1} = \frac{1}{1 + \frac{x}{2} + \frac{x^2}{3!} + \&c.} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_2 x^4}{4!} + \&c.,$$

by means of the second formula, and equate coefficients.

Here $\phi = 1, \phi' = \frac{1}{2}, \phi'' = \frac{1}{3} \dots\dots$

$$\begin{aligned} \text{Thus } B_n &= (-1)^{n-1} \left| \begin{array}{cccc} \frac{1}{2}, 1, 0, 0 \dots \\ \frac{1}{3}, 1, 1, 0 \dots \\ \frac{1}{4}, 1, \frac{3}{2}, 1 \dots \\ \frac{1}{5}, 1, 2, 2 \dots \\ \dots & \dots & \dots & \dots \end{array} \right| (2n \text{ rows}) \\ &= \frac{(-1)^{n-1}}{(2n+1)!} \left| \begin{array}{cccccc} 1, 2, 0, 0, 0, 0 \dots \\ 1, 3, 3, 0, 0, 0 \dots \\ 1, 4, 6, 4, 0, 0 \dots \\ 1, 5, 10, 10, 5, 0 \dots \\ 1, 6, 15, 20, 15, 6 \dots \\ 1, 7, 21, 35, 35, 21 \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{array} \right| (2n \text{ rows}) \dots (1). \end{aligned}$$

All the numbers in this determinant are coefficients of the Binomial Theorem; for the r^{th} row is

$$\phi^r, r\phi^{r-1}, \frac{r(r-1)}{2!} \phi^{r-2} \dots\dots, \text{ or } \frac{1}{r+1}, 1, \frac{r}{2} \dots\dots,$$

which, when multiplied by $(r+1)$, becomes

$$1, r+1, \frac{(r+1)r}{2} \dots\dots$$

$$\begin{aligned} \text{II. } \frac{e^x - 1}{e^x + 1} &= \frac{e^x + e^{-x} - 2}{e^x - e^{-x}} = \frac{\frac{1}{2}x + \frac{1}{4}\frac{x^3}{3} + \frac{1}{6}\frac{x^5}{5} + \&c.}{1 + \frac{1}{3}\frac{x^2}{2} + \frac{1}{5}\frac{x^4}{4} + \&c.} \\ &= (2^2 - 1)B_1 x - \frac{2^4 - 1}{2}B_2 \frac{x^3}{3} + \frac{2^6 - 1}{3}B_3 \frac{x^5}{5} - \&c. \end{aligned}$$

Expand by means of the first formula and equate the coefficients.

$$\text{Thus } (-1)^{n-1} \frac{B_n}{n} (2^{2n}-1) = - \begin{vmatrix} 0, 1, 0, 0, 0, 0 \dots \\ \frac{1}{2}, 0, 1, 0, 0, 0 \dots \\ 0, \frac{1}{3}, 0, 1, 0, 0 \dots \\ \frac{1}{4}, 0, 1, 0, 1, 0 \dots \\ 0, \frac{1}{5}, 0, 2, 0, 1 \dots \\ \frac{1}{6}, 0, 1, 0, \frac{1}{3}, 0 \dots \\ \dots \dots \dots \dots \dots \end{vmatrix} \quad (2n \text{ rows});$$

$$\begin{aligned} \text{therefore } B_n &= \frac{n}{2^{2n}-1} \begin{vmatrix} \frac{1}{2}, 1, 0 \dots \\ \frac{1}{4}, 1, 1 \dots \\ \frac{1}{8}, 1, \frac{1}{3} \dots \\ \dots \dots \dots \end{vmatrix} \quad (n \text{ rows}) \\ &= \frac{1}{(2^{2n}-1) 2^n \underline{n-1}} \begin{vmatrix} 1, 2, 0 \dots \\ 1, 4, 4 \dots \\ 1, 6, 20 \dots \\ \dots \dots \dots \end{vmatrix} \quad (n \text{ rows}) \dots (2). \end{aligned}$$

The determinant (2) may be obtained from the determinant (1) by striking out all even rows, and all odd columns except the first.

$$\begin{aligned} \text{III. } \tan x &= \frac{x - \frac{x^3}{3} + \&c.}{1 - \frac{x^2}{2} + \&c.} \\ &= \frac{2^2(2^2-1) B_1 x}{2} + \frac{2^4(2^4-1) B_3}{4} \frac{x^3}{3} + \&c. \end{aligned}$$

Expand by the first formula, and equate coefficients.

Thus

$$\begin{aligned} B_n &= - \frac{2n}{2^{2n}(2^{2n}-1)} \begin{vmatrix} 0, 1, 0, 0, 0, 0, 0, 0 \dots \\ 1, 0, 1, 0, 0, 0, 0, 0 \dots \\ 0, -1, 0, 1, 0, 0, 0, 0 \dots \\ -1, 0, -3, 0, 1, 0, 0, 0 \dots \\ 0, 1, 0, -6, 0, 1, 0, 0 \dots \\ 1, 0, 5, 0, -10, 0, 1, 0 \dots \\ 0, -1, 0, 15, 0, -15, 0, 1 \dots \\ -1, 0, -7, 0, 35, 0, -21, 0 \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \end{vmatrix} \quad (2n \text{ rows}) \\ &= \frac{2n(-1)^{n-1}}{2^{2n}(2^{2n}-1)} \begin{vmatrix} 1, 1, 0, 0 \dots \\ -1, -3, 1, 0 \dots \\ 1, 5, -10, 1 \dots \\ -1, -7, 35, -21 \dots \\ \dots \dots \dots \dots \end{vmatrix} \quad (n \text{ rows}) \end{aligned}$$

$$= \frac{2n}{2^{2n}(2^{2n}-1)} \begin{vmatrix} 1, 1, 0, 0 \dots \\ 1, 3, 1, 0 \dots \\ 1, 5, 10, 1 \dots \\ 1, 7, 35, 21 \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} (n \text{ rows}) \dots\dots\dots (3).$$

The determinant (3) may be obtained from the determinant (1) by striking out all odd rows and all odd columns, and bordering the determinant so formed with a border of units both vertically and diagonally.

$$\text{IV. } x \operatorname{cosec} x = \frac{1}{1 - \frac{x^2}{3 \lfloor 2} + \frac{x^4}{5 \lfloor 4} - \&c.}$$

$$= 1 + 2(2-1) \frac{B_1 x^2}{\lfloor 2} + 2(2^3-1) \frac{B_3 x^4}{\lfloor 4} + \&c.$$

Expand by means of the second formula, and equate coefficients.

$$\begin{aligned} \text{Thus } B_n &= \frac{1}{2(2^{2n-1}-1)} \begin{vmatrix} 0, & 1, & 0, & 0, & 0, & 0 \dots \\ -\frac{1}{3}, & 0, & 1, & 0, & 0, & 0 \dots \\ 0, & 1, & 0, & 1, & 0, & 0 \dots \\ \frac{1}{5}, & 0, & -2, & 0, & 1, & 0 \dots \\ 0, & 1, & 0, & -\frac{1}{3}, & 0, & 1 \dots \\ -\frac{1}{7}, & 0, & 3, & 0, & -5, & 0 \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} (2n \text{ rows}) \\ &= \frac{1}{2(2^{2n-1}-1)} \begin{vmatrix} \frac{1}{3}, & 1, & 0 \dots \\ \frac{1}{5}, & 2, & 1 \dots \\ \frac{1}{7}, & 3, & 5 \dots \\ \dots & \dots & \dots \end{vmatrix} (n \text{ rows}), \end{aligned}$$

which reduces to

$$B_n = \frac{2^{n-1} \lfloor n}{\lfloor 2n+1 (2^{2n-1}-1)} \begin{vmatrix} 1, & 3, & 0 \dots \\ 1, & 10, & 5 \dots \\ 1, & 21, & 35 \dots \\ \dots & \dots & \dots \end{vmatrix} (n \text{ rows}) \dots\dots\dots (4).$$

The determinant (4) may be obtained from the determinant (1) by striking out all odd rows, and all even columns.

V. The coefficients of the expansion of $\sec x$ may also be expressed as determinants of selected coefficients of the Binomial Theorem.

$$\text{For } \sec x = \frac{1}{1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \&c.} = 1 + \frac{E_2}{\lfloor 2} x^2 + \frac{E_4}{\lfloor 4} x^4 + \dots\dots$$

gives

$$E_{2n} = \begin{vmatrix} 0, & 1, & 0, & 0, & 0, & 0, & 0, & 0 \dots \\ -1, & 0, & 1, & 0, & 0, & 0, & 0, & 0 \dots \\ 0, & -3, & 0, & 1, & 0, & 0, & 0, & 0 \dots \\ 1, & 0, & -6, & 0, & 1, & 0, & 0, & 0 \dots \\ 0, & 5, & 0, & -10, & 0, & 1, & 0, & 0 \dots \\ -1, & 0, & 15, & 0, & -15, & 0, & 1, & 0 \dots \\ 0, & -7, & 0, & 35, & 0, & -21, & 0, & 1 \dots \\ 1, & 0, & -28, & 0, & 70, & 0, & -28, & 0 \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad (2n \text{ rows}).$$

$$\text{This reduces to } E_{2n} = \begin{vmatrix} 1, & 1, & 0, & 0 \dots \\ 1, & 6, & 1, & 0 \dots \\ 1, & 15, & 15, & 1 \dots \\ 1, & 28, & 70, & 28 \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad (n \text{ rows}) \dots\dots\dots (5).$$

The determinant (5) may be obtained from the determinant (1) by striking out all even rows and all even columns, and bordering the determinant so formed with a border of units diagonally.

VI. To verify the truth of these formulæ, put $n=1, 2, 3$ in them.

Then, when $n=1$, each of the first four gives $B_1 = \frac{1}{6}$.

When $n=2$, the first four formulæ give

$$(1) \dots\dots\dots B_2 = -\frac{1}{120} \begin{vmatrix} 1, & 2, & 0, & 0 \\ 1, & 3, & 3, & 0 \\ 1, & 4, & 6, & 4 \\ 1, & 5, & 10, & 10 \end{vmatrix} = -\frac{1}{120} (1 \cdot 0 - 2 \cdot 2) = \frac{1}{30}.$$

$$(2) \dots\dots\dots B_2 = \frac{1}{15 \cdot 4 \cdot 1} \begin{vmatrix} 1, & 2 \\ 1, & 4 \end{vmatrix} = \frac{1}{30}.$$

$$(3) \dots\dots\dots B_2 = \frac{4}{16 \cdot 15} \begin{vmatrix} 1, & 1 \\ 1, & 3 \end{vmatrix} = \frac{1}{30}.$$

$$(4) \dots\dots\dots B_2 = \frac{2 \cdot 2}{120 \cdot 7} \begin{vmatrix} 1, & 3 \\ 1, & 10 \end{vmatrix} = \frac{1}{30}.$$

When $n=3$,

$$(1) \dots\dots\dots B_3 = \frac{1}{7} \begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \\ 1, & 3, & 3, & 0, & 0, & 0 \\ 1, & 4, & 6, & 4, & 0, & 0 \\ 1, & 5, & 10, & 10, & 5, & 0 \\ 1, & 6, & 15, & 20, & 15, & 6 \\ 1, & 7, & 21, & 35, & 35, & 21 \end{vmatrix} = \frac{1}{7} \begin{vmatrix} 1, & 2, & 0, & 0, & 0, & 0 \\ 0, & 1, & 3, & 0, & 0, & 0 \\ 0, & 1, & 3, & 4, & 0, & 0 \\ 0, & 1, & 4, & 6, & 5, & 0 \\ 0, & 1, & 5, & 10, & 10, & 6 \\ 0, & 1, & 6, & 15, & 20, & 15 \end{vmatrix}$$

$$= \frac{1}{\underline{7}} \begin{vmatrix} 1, 2, 0, 0, 0, 0 \\ 0, 1, 3, 0, 0, 0 \\ 0, 0, 0, 4, 0, 0 \\ 0, 0, 1, 2, 5, 0 \\ 0, 0, 1, 4, 5, 6 \\ 0, 0, 1, 5, 10, 9 \end{vmatrix} = \frac{1}{\underline{7}} \begin{vmatrix} 1, 2, 0, 0, 0, 0 \\ 0, 1, 3, 0, 0, 0 \\ 0, 0, 0, 4, 0, 0 \\ 0, 0, 1, 2, 5, 0 \\ 0, 0, 0, 2, 0, 6 \\ 0, 0, 0, 1, 5, 3 \end{vmatrix}$$

$$= \frac{4 \cdot 5 \cdot 6}{\underline{7}} = \frac{1}{42}.$$

$$(2) \dots B_3 = \frac{1}{63 \cdot 8 \cdot 2} \begin{vmatrix} 1, 2, 0 \\ 1, 4, 4 \\ 1, 6, 20 \end{vmatrix} = \frac{56 - 2 \cdot 16}{63 \cdot 8 \cdot 2} = \frac{1}{42}.$$

$$(3) \dots B_3 = \frac{6}{64 \cdot 63} \begin{vmatrix} 1, 1, 0 \\ 1, 3, 1 \\ 1, 5, 10 \end{vmatrix} = \frac{(25 - 9) 6}{64 \cdot 63} = \frac{1}{42}.$$

$$(4) \dots B_3 = \frac{4 \cdot 6}{\underline{7} (32 - 1)} \begin{vmatrix} 1, 3, 0 \\ 1, 10, 5 \\ 1, 21, 35 \end{vmatrix} = \frac{4 \cdot 6 (350 - 105 - 3 \cdot 30)}{31 \underline{7}}$$

$$= \frac{4 \cdot 6 \cdot 155}{31 \underline{7}} = \frac{1}{42}.$$

The formula (5) gives

$$\sec x = 1 + \frac{x^2}{\underline{2}} + \begin{vmatrix} 1, 1 \\ 1, 6 \end{vmatrix} \frac{x^4}{\underline{4}} + \begin{vmatrix} 1, 1, 0 \\ 1, 6, 1 \\ 1, 15, 15 \end{vmatrix} \frac{x^6}{\underline{6}} + \dots$$

Or
$$\sec x = 1 + \frac{x^2}{\underline{2}} + \frac{5x^4}{\underline{4}} + \frac{61x^6}{\underline{6}} + \dots$$

This result may be verified by ordinary division, or otherwise.

On Three-bar Motion in Plane Space. By S. ROBERTS, M.A.

[Read Nov. 11th, 1875.]

1. I propose to extend briefly to general three-bar motion a discussion which I have before given of some particular cases; and fortunately I find my task rendered easy by some recent discoveries in relation to the subject.

It is known that the locus of a point rigidly attached to a link connected immediately with two others moving in a plane about two