equations. We have a set of the type

$$\Sigma \pm [0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2s+1)] \frac{\partial (a, a')}{\partial (x_p, x_q)} = 0, \quad (14)$$

and a set of the type

$$\Sigma \pm [0123 \dots (p-1)(p+1) \dots (q-1)(q+1) \dots (2s+1)] \frac{\partial (\beta, a)}{\partial (x_p, x_q)} = 0. \quad (15)$$

This accomplishes our present purpose, which is the determination of the forms of certain equations known to exist. The discussion opens up some interesting questions with regard to the general properties of the derived functions of a Pfaffian expression, but these must be left for future investigation.

On Ampère's Equation $Rr + 2Ss + Tt + U(rt - s^2) = V$. By A. C.

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1. The methods given hitherto for solving the above equation do not apply in general, but only if certain conditions are satisfied by the coefficients R, S, T, U, V . I propose in this article to investigate a modification of the ordinary Monge and Ampère method which applies to all cases.

It must be remembered that no method can be given which will in all cases give an integral in finite terms. It seems almost certain further that no method will ever be found for reducing the solution of Ampère's equation in general to that of equations of the first order with one dependent variable. Even if such a method were found, it would not enable us to solve the equation in all cases, for there is no general method of solving such equations of the first order, except by continued approximation. The ordinary methods, those of Lagrange, Charpit, and Jacobi, depend on inspection. The

method of continued approximation, as by infinite series, applies to equations of all orders, and not only to those of the first. The statement, often made, that a single function satisfying given partial differential equations of the first order may be considered known is therefore, in the present connexion at least, not well founded. No doubt much more is known of the relations among different solutions in the case of an equation of the first order, but the solutions themselves are equally well known for all orders.

The methods of Monge and Ampère depend for their success on the combination of certain differential expressions so as to form a complete differential. When this can be done by inspection, the methods are of practical use, but hardly otherwise. The method I now propose depends on the combination of the same differential expressions in a particular way, and it will not be of much use in practice unless this can be done by inspection. Its advantage is that it applies universally; that is, that for any equation of Ampère's form any solution, involving three arbitrary constants in such a way that their elimination does not lead to an equation of the first order, satisfies the conditions on which the method depends, and is therefore discoverable by a sufficiently keen inspection.

If the two sets of characteristics coincide, the statement of the method has to be modified, and in this case the practical difficulty is specially marked.

Suppose the equation

$$Rr + 2Ss + Tt + U(rt - s^2) = V \quad (1)$$

$$\text{to have a solution} \quad \phi(x, y, z, c_1, c_2, c_3) = 0 \quad (2)$$

involving three arbitrary constants. Then, by taking the first derivatives of this, we form two other equations

$$\frac{\partial \phi}{\partial x} + p \frac{\partial \phi}{\partial z} = 0, \quad \frac{\partial \phi}{\partial y} + q \frac{\partial \phi}{\partial z} = 0, \quad (3)$$

and these three may be supposed solved for c_1, c_2, c_3 , so as to give equations

$$u_1 = c_1, \quad u_2 = c_2, \quad u_3 = c_3, \quad (4)$$

unless it is possible to eliminate c_1, c_2, c_3 from (2) and (3), in which case (2) will be a primitive of a differential equation of the first order only, involving no arbitrary constant. This case we shall suppose excluded.

We shall consider the solution rather in the form (4) than (2). If it is a complete primitive depending on five arbitrary constants, then u_1, u_2, u_3 must be supposed to involve two constants as well as x, y, z, p, q .

2. Now, if two of the functions u_1, u_2, u_3 , say u_1, u_2 , have been found by any means, the third is found without any theoretical difficulty. For the equations

$$u_1 = c_1, \quad u_2 = c_2$$

may be solved for p, q ,* and the resulting values substituted in the equation

$$dz = p dx + q dy.$$

The integration of this must lead to (2), which, when written

$$\psi(x, y, z, u_1, u_2, c_3) = 0,$$

must be equivalent to

$$u_3 = c_3.$$

The problem is therefore reduced to the finding of two functions u_1, u_2 such that the values of p, q given by solving the equations

$$u_1 = c_1, \quad u_2 = c_2$$

may satisfy the conditions

$$Rr + 2Ss + Tt + U(rt - s^2) = V, \quad (1)$$

that is,

$$\begin{aligned} R \left(\frac{\partial p}{\partial x} + p \frac{\partial p}{\partial z} \right) + 2S \left(\frac{\partial q}{\partial x} + p \frac{\partial q}{\partial z} \right) + T \left(\frac{\partial q}{\partial y} + q \frac{\partial q}{\partial z} \right) \\ + U \left\{ \frac{\partial (p, q)}{\partial (x, y)} + p \frac{\partial (p, q)}{\partial (z, y)} + q \frac{\partial (p, q)}{\partial (x, z)} \right\} = V \end{aligned} \quad (5)$$

and

$$\frac{\partial q}{\partial x} + p \frac{\partial q}{\partial z} = \frac{\partial p}{\partial y} + q \frac{\partial p}{\partial z}. \quad (6)$$

3. Now (1) may be written (unless $U = 0$)

$$\begin{vmatrix} Ur + T, & Us - S_1 \\ Us - S_2, & Ut + R \end{vmatrix} = 0, \quad (7)$$

* Otherwise p, q could be eliminated, and thus there would be a relation among x, y, z, c_1, c_2 not involving c_3 . This cannot be since (2) involves c_3 .

where S_1, S_2 are such expressions that

$$S_1 + S_2 = 2S, \quad S_1 S_2 = RT + UV.$$

The equation (7) may be replaced by

$$\left. \begin{aligned} Ur + T + \lambda (Us - S_1) &= 0 \\ Us - S_2 + \lambda (Ut + R) &= 0 \end{aligned} \right\}, \quad (8)$$

or by
$$\left. \begin{aligned} Ur + T + \mu (Us - S_2) &= 0 \\ Us - S_1 + \mu (Ut + R) &= 0 \end{aligned} \right\}, \quad (9)$$

λ, μ being proper multipliers.

Taking first the equations (8), let us suppose quantities A_1, A_2 so chosen that

$$A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p} = U, \quad A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q} = U\lambda.$$

Then $\lambda S_1 - T = Ur + U\lambda s = A_1 \left(r \frac{\partial u_1}{\partial p} + s \frac{\partial u_1}{\partial q} \right) + A_2 \left(r \frac{\partial u_2}{\partial p} + s \frac{\partial u_2}{\partial q} \right),$

so that
$$T - \lambda S_1 = A_1 \left(\frac{\partial u_1}{\partial x} + p \frac{\partial u_1}{\partial z} \right) + A_2 \left(\frac{\partial u_2}{\partial x} + p \frac{\partial u_2}{\partial z} \right),$$

and in like manner, from the second equation (8),

$$\lambda R - S_2 = A_1 \left(\frac{\partial u_1}{\partial y} + q \frac{\partial u_1}{\partial z} \right) + A_2 \left(\frac{\partial u_2}{\partial y} + q \frac{\partial u_2}{\partial z} \right).$$

Thus

$$\begin{aligned} A_1 du_1 + A_2 du_2 &= Udp + U\lambda dq + (T - \lambda S_1) dx + (\lambda R - S_2) dy \\ &\quad + \left(A_1 \frac{\partial u_1}{\partial z} + A_2 \frac{\partial u_2}{\partial z} \right) (dz - p dx - q dy) \\ &= (Udp + Tdx - S_2 dy) + \lambda (Udq - S_1 dx + R dy) \\ &\quad + \left(A_1 \frac{\partial u_1}{\partial z} + A_2 \frac{\partial u_2}{\partial z} \right) (dz - p dx - q dy). \end{aligned}$$

Thus a linear combination of du_1, du_2 can be expressed as the sum of multiples of the expressions

$$Udp + Tdx - S_2 dy, \quad Udq - S_1 dx + R dy, \quad dz - p dx - q dy.$$

To this list of three may be added the following, combinations of the first two, for use when U vanishes :—

$$S_1 dp + T dq - V dy, \quad R dp + S_2 dq - V dx.$$

In like manner it follows from (9) that a linear combination of du_1, du_2 can be expressed as the sum of multiples of

$$U dp + T dx - S_1 dy, \quad U dq - S_2 dx + R dy, \quad dz - p dx - q dy, \\ S_2 dp + T dq - V dy, \quad R dp + S_1 dq - V dx.$$

It is easily seen that these combinations are different unless $S_1 = S_2$.

4. Conversely, when S_1, S_2 are unequal, if two linear combinations of du_1, du_2 can be expressed as just explained, the equations

$$u_1 = c_1, \quad u_2 = c_2,$$

if they can be solved for p, q , will furnish a solution. For we must have

$$U(A_1 du_1 + A_2 du_2) = \left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}\right) (U dp + T dx - S_1 dy) \\ + \left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}\right) (U dq - S_2 dx + R dy) \\ + U \left(A_1 \frac{\partial u_1}{\partial z} + A_2 \frac{\partial u_2}{\partial z}\right) (dz - p dx - q dy) \quad (10)$$

$$\text{and } U(B_1 du_1 + B_2 du_2) = \left(B_1 \frac{\partial u_1}{\partial p} + B_2 \frac{\partial u_2}{\partial p}\right) (U dp + T dx - S_1 dy) \\ + \left(B_1 \frac{\partial u_1}{\partial q} + B_2 \frac{\partial u_2}{\partial q}\right) (U dq - S_2 dx + R dy) \\ + U \left(B_1 \frac{\partial u_1}{\partial z} + B_2 \frac{\partial u_2}{\partial z}\right) (dz - p dx - q dy). \quad (11)$$

Now let r, s, σ, t denote

$$\frac{\partial p}{\partial x} + p \frac{\partial p}{\partial z}, \quad \frac{\partial p}{\partial y} + q \frac{\partial p}{\partial z}, \quad \frac{\partial q}{\partial x} + p \frac{\partial q}{\partial z}, \quad \frac{\partial q}{\partial y} + q \frac{\partial q}{\partial z},$$

the derivatives being formed on the supposition that p, q are defined in terms of x, y, z by the equations

$$u_1 = c_1, \quad u_2 = c_2.$$

Then, the expressions on the left in (10), (11) being put equal to zero, we have, from (10),

$$\left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}\right)(Ur + T) + \left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}\right)(U\sigma - S_1) = 0, \quad (13)$$

$$\left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}\right)(Us - S_2) + \left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}\right)(Ut + R) = 0. \quad (14)$$

Now $A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}$, $A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}$ cannot both vanish, since $\frac{\partial(u_1, u_2)}{\partial(p, q)}$ is supposed not to be zero; we must therefore have

$$\begin{vmatrix} Ur + T, & U\sigma - S_1 \\ Us - S_2, & Ut + R \end{vmatrix} = 0$$

or $U(rt - s\sigma) + Rr + S_1s + S_2\sigma + Tt = V. \quad (15)$

In the same way, from (11), we have

$$U(rt - s\sigma) + Rr + S_2s + S_1\sigma + Tt = V. \quad (16)$$

By subtraction, $(S_1 - S_2)(s - \sigma) = 0,$

so that, as S_1, S_2 are supposed unequal,

$$s = \sigma.$$

Thus (15), (16) become

$$U(rt - s^2) + Rr + 2Ss + Tt = V; \quad (1)$$

that is, the values of p, q given by the equations

$$u_1 = c_1, \quad u_2 = c_2$$

make the equation $dz = p dx + q dy$

integrable, and also cause the condition (1) to be satisfied.

The above proof can easily be modified so as to apply when U vanishes; for the equations (13), (14) may be supplemented by the following two which are consequences of them unless U vanishes:—

$$\left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}\right)(S_2r + Ts) + \left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}\right)(S_2\sigma + Tt - V) = 0,$$

$$\left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}\right)(Rr + S_1s - V) + \left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}\right)(R\sigma + S_1t) = 0.$$

Thus, when S_1, S_2 are unequal, we have the following method of solution. Find, by inspection, a combination of the expressions

$$Udp + Tdx - S_2dy, \quad Udq - S_1dx + Rdy, \quad S_1dp + Tdq - Vdy, \\ Rdp + S_2dq - Vdx, \quad dz - pdx - qdy,$$

which can be written in the form

$$A_1 du_1 + A_2 du_2,$$

and a combination of the same expressions with S_1, S_2 interchanged that can be written in the form $B_1 du_1 + B_2 du_2$; the differentials du_1, du_2 are to be the same in both cases, but the coefficients A_1, A_2, B_1, B_2 are not restricted, and in particular need not be such that either combination shall be a complete differential. Then, if the equations

$$u_1 = c_1, \quad u_2 = c_2$$

can be solved for p, q , a solution of the equation (1) will be given by the integral of

$$dz = pdx + qdy.$$

Although the theory of the method is general, its practical application is, of course, limited to solutions in finite terms, and even when such exist it may not be at all easy to find them by this or any other method. This difficulty is shared, though in a less degree, by the ordinary methods for solving equations of the first order.

5. The equations (8) may be discussed in another way. They may be written

$$U \frac{\partial p}{\partial x} + U\lambda \frac{\partial p}{\partial y} + U(p + \lambda q) \frac{\partial p}{\partial z} = \lambda S_1 - T, \quad (17)$$

$$U \frac{\partial q}{\partial x} + U\lambda \frac{\partial q}{\partial y} + U(p + \lambda q) \frac{\partial q}{\partial z} = S_2 - \lambda R. \quad (18)$$

Multiply (17) by $\frac{\partial u_1}{\partial p}$, (18) by $\frac{\partial u_1}{\partial q}$, and add: the result is

$$U \frac{\partial u_1}{\partial x} + U\lambda \frac{\partial u_1}{\partial y} + U(p + \lambda q) \frac{\partial u_1}{\partial z} + (\lambda S_1 - T) \frac{\partial u_1}{\partial p} + (S_2 - \lambda R) \frac{\partial u_1}{\partial q} = 0, \quad (19)$$

whence it follows that du_1 is a linear combination of the expressions

$$\left\| \begin{array}{ccccc} dx, & dy, & dz, & dp, & dq \\ U, & U\lambda, & U(p+\lambda q), & \lambda S_1-T, & S_2-\lambda R \end{array} \right\|, \quad (20)$$

that is, of

$$\begin{aligned} & dy - \lambda dx, \\ & dz - p dx - q dy, \\ & U dp + T dx - S_1 dy, \\ & U dq - S_2 dx + R dy. \end{aligned}$$

Since u_2 must also satisfy (19), du_2 must also be a combination of these expressions, and thus B_1, B_2 can be so chosen that $B_1 du_1 + B_2 du_2$ is a combination of the last three, a conclusion to which we were before led by considering (9).

6. The matrix (20) may be written in another form by the use of the known value of λ , namely,

$$\left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q} \right) \div \left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p} \right).$$

For

$$U = A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p},$$

$$T - \lambda S_1 = A_1 \frac{\partial u_1}{\partial x} + A_2 \frac{\partial u_2}{\partial x} + p \left(A_1 \frac{\partial u_1}{\partial z} + A_2 \frac{\partial u_2}{\partial z} \right),$$

$$\lambda R - S_2 = A_1 \frac{\partial u_1}{\partial y} + A_2 \frac{\partial u_2}{\partial y} + q \left(A_1 \frac{\partial u_1}{\partial z} + A_2 \frac{\partial u_2}{\partial z} \right).$$

Thus, if we use the symbol* $d\Omega$ for $A_1 du_1 + A_2 du_2$, we may write (20) thus

$$\left\| \begin{array}{ccccc} dx, & dy, & dz, & dp, & dq \\ \frac{\partial \Omega}{\partial p}, & \frac{\partial \Omega}{\partial q}, & p \frac{\partial \Omega}{\partial p} + q \frac{\partial \Omega}{\partial q}, & -\frac{\partial \Omega}{\partial x} - p \frac{\partial \Omega}{\partial z}, & -\frac{\partial \Omega}{\partial y} - q \frac{\partial \Omega}{\partial z} \end{array} \right\|, \quad (21)$$

in which form it has a striking analogy to the auxiliary matrix used in the theory of partial differential equations of the first order.

* It is not meant that $A_1 du_1 + A_2 du_2$ is a complete differential.

The condition (19) may be written symbolically

$$\frac{\partial(\Omega, u_1)}{\partial(x, p)} + p \frac{\partial(\Omega, u_1)}{\partial(z, p)} + \frac{\partial(\Omega, u_1)}{\partial(y, q)} + q \frac{\partial(\Omega, u_1)}{\partial(z, q)} = 0, \quad (22)$$

and thus reduces to

$$\frac{\partial(u_1, u_2)}{\partial(x, p)} + p \frac{\partial(u_1, u_2)}{\partial(z, p)} + \frac{\partial(u_1, u_2)}{\partial(y, q)} + q \frac{\partial(u_1, u_2)}{\partial(z, q)} = 0, \quad (23)$$

which, as is known, leads directly to $s = \sigma$.

Now, in the argument of § 4, only one of the identities (10), (11) is necessary, if we know by any means that s, σ are equal.

Hence we have modified forms of the original theorem; in fact, $u_1 = c_1, u_2 = c_2$ will furnish a solution if multipliers A_1, A_2 can be found such that $A_1 du_1 + A_2 du_2$ is a linear combination of the expressions in the following list:—

$$Udp + Tdx - S_2 dy, \quad Udq - S_1 dx + Rdy,$$

$$S_1 dp + Tdq - Vdy, \quad Rdp + S_2 dq - Vdx, \quad dz - p dx - q dy;$$

and if any one of the following further conditions is satisfied:—

- (a) du_1 (or du_2) is a linear combination of the expressions got by interchanging S_1, S_2 in the list just given, together with

$$\left(A_1 \frac{\partial u_1}{\partial q} + A_2 \frac{\partial u_2}{\partial q}\right) dx - \left(A_1 \frac{\partial u_1}{\partial p} + A_2 \frac{\partial u_2}{\partial p}\right) dy.$$

- (b) du_1 (or du_2) is a linear combination of the determinants (21).

- (c) The expressions given for p, q by the equations $u_1 = c_1, u_2 = c_2$ are such that $s = \sigma$.

- (d) The compatibility condition (23).

- (e) u_1, u_2 are two of the new principal variables in a tangential transformation. (The third principal variable will be u_3 .)

In any of these modified forms the theorem applies equally well when $S_1 = S_2$. There is no difficulty in the verification, for we have seen that the above ways of stating the condition are equivalent; the equation (15) is proved as before, and we have now $s = \sigma$.

When the two sets of characteristics coincide, it is then necessary first to find by inspection, or otherwise, two functions u_1, u_2 satisfying the conditions just specified. The solution may then be completed as before, but it is doubtful whether the method in this form is of any practical use.

7. To illustrate the method, let us use it to find the most general solution of the equation

$$rt-s^2=0.$$

Here $A_1 du_1 + A_2 du_2$ must be a combination of dp, dq and $dz - p dx - q dy$, for

$$U=1, \quad R=S_1=S_2=T=V=0.$$

Writing u for $px+qy-z$, we may say that $A_1 du_1 + A_2 du_2$ must be a combination of dp, dq, du , and therefore u_1, u_2, p, q, u must be connected by at least one relation. Since u_1, u_2 are equal to constants, this shows that the equations of the solution yield at least one relation of the form

$$\phi(p, q, u)=0,$$

and we have further $A_1 du_1 + A_2 du_2 \equiv d\phi$,

it being understood that u_1, u_2 are substituted for c_1, c_2 in $d\phi$ after formation. Thus du_1, du_2 are linear combinations of the determinants

$$\left\| \begin{array}{cccc} dx, & dy, & dz, & dp, \quad dq \\ \frac{\partial \phi}{\partial p} + x \frac{\partial \phi}{\partial u}, & \frac{\partial \phi}{\partial q} + y \frac{\partial \phi}{\partial u}, & p \frac{\partial \phi}{\partial p} + q \frac{\partial \phi}{\partial q} + (px+qy) \frac{\partial \phi}{\partial u}, & 0, \quad 0 \end{array} \right\|.$$

Thus u_1, u_2 must be functions of

$$p, q, u, \quad \text{and} \quad \left(\frac{\partial \phi}{\partial p} + x \frac{\partial \phi}{\partial u} \right) \div \left(\frac{\partial \phi}{\partial q} + y \frac{\partial \phi}{\partial u} \right),$$

and the two equations $u_1 = c_1, \quad u_2 = c_2$

are equivalent to two of the form

$$u = f(p, q), \quad \left(x - \frac{\partial f}{\partial p} \right) \bigg/ \left(y - \frac{\partial f}{\partial q} \right) = F(p, q), \quad (\alpha)$$

or of the form

$$f(p, q) = 0, \quad x \frac{\partial f}{\partial q} - y \frac{\partial f}{\partial p} = F(p, q, u), \quad (\beta)$$

or to two relations connecting p, q, u . (\gamma)

The third equation of the solution is to be found by integration from

$$dz = p dx + q dy,$$

or, what is the same thing,

$$du = x dp + y dq.$$

Thus γ gives p, q, u all constant.

From (a),
$$\left(x - \frac{\partial f}{\partial p}\right) dp + \left(y - \frac{\partial f}{\partial q}\right) dq = 0,$$

that is,

$$F(p, q) dp + dq = 0.$$

Since the form of F is arbitrary, this gives an arbitrary relation between p , and q ; thus, p, q, u may be taken as arbitrary functions of one variable only, and the solution comes out as the result of eliminating v from the equations

$$\begin{cases} z = xF_1(v) + yF_2(v) + F_3(v), \\ 0 = xF'_1(v) + yF'_2(v) + F'_3(v). \end{cases}$$

This form includes the solution derived from (γ), and (β) leads to the same. This, then, is the most general solution involving three or more arbitrary constants.

The above example was chosen on account of its familiarity, so as to give an opportunity of comparing the present method with others. Let us now consider the following, which does not, I believe, fall under the rules of Monge and Ampère,

$$2xqr + (yp - 2xq)s + 2pyt + 2xy(rt - s^2) = -3pq.$$

Here

$$S_1 + S_2 = yp - 2xq,$$

$$S_1 S_2 = 4pqxy - 6pqxy = -2pqxy.$$

Put then

$$S_1 = yp, \quad S_2 = -2xq.$$

Hence $A_1 du_1 + A_2 du_2$ is to be a combination of

$$2xydp + 2pydx + 2xqdy, \quad 2xydq - ypdx + 2xqdy, \quad dz - pdx - qdy,$$

that is, of

$$dp + p \frac{dx}{x} + \frac{q}{y} dy, \quad 2dq - p \frac{dx}{x} + 2q \frac{dy}{y}, \quad dz - pdx - qdy.$$

Also $B_1 du_1 + B_2 du_2$ is to be a combination of

$$2xydp + 2pydx - ypdy, \quad 2xydq + 2xqdx + 2xqdy, \quad dz - pdx - qdy,$$

that is, of

$$2x \frac{dp}{p} + 2dx - dy, \quad y \frac{dq}{q} + dx + dy, \quad dz - pdx - qdy.$$

Compare, for instance,

$$\left\{ dp + p \frac{dx}{x} + \frac{q}{y} dy \right\} + \lambda \left\{ 2dq - p \frac{dx}{x} + 2q \frac{dy}{y} \right\}$$

with $\left\{ 2x \frac{dp}{p} + 2dx - dy \right\} + \mu \left\{ y \frac{dq}{q} + dx + dy \right\}.$

These expressions may be written

$$\left\{ \frac{dp}{p} + (1 - \lambda) \frac{dx}{x} \right\} p + \left\{ 2\lambda \frac{dq}{q} + (2\lambda + 1) \frac{dy}{y} \right\} q$$

and $\left\{ 2x \frac{dp}{p} + (2 + \mu) \frac{dx}{x} \right\} x + \left\{ \mu \frac{dq}{q} + (\mu - 1) \frac{dy}{y} \right\} y.$

If, then, we put $\mu = -2\lambda,$

we find as admissible forms these

$$u_1 = px^{1-\lambda},$$

$$u_2 = q^{2\lambda} y^{2\lambda+1},$$

λ having any constant value.

From the equations $u_1 = c_1, \quad u_2 = c_2,$
we derive $p = c_1 x^{\lambda-1}, \quad q = c_2^{1/2\lambda} y^{-(1/2\lambda)-1},$
and the solution thus found is

$$z = ax^{\lambda} + by^{-(1/2\lambda)} + c,$$

containing four arbitrary constants, $a, b, c, \lambda.$

Of course this is not the most general integral; but it may be the most general one in finite terms.

8. If in any particular case one of the multipliers $A_1, A_2, B_1, B_2,$ say $A_2,$ vanishes, we have substantially the methods of Monge and Ampère. We may take $A_1 = 1,$ so that du_1 is a linear combination of

$$Udp + Tdx - S_2 dy, \quad Udq - S_1 dx + Rdy, \quad S_1 dp + Tdq - Vdy,$$

$$Rdp + S_2 dq - Vdx, \quad dz - p dx - q dy,$$

and the process of finding $u_2, u_3,$ as above indicated, is then exactly that of Lagrange and Charpit for the integration of the equation

$$u_1 = c_1.$$

The solution derived will involve three arbitrary constants, or one arbitrary constant and an arbitrary function.

9. The following is another method of finding u_1, u_2 :—Since p, q are given in terms of x, y, z by

$$u_1 = c_1, \quad u_2 = c_2,$$

we have
$$r = \left\{ \frac{\partial(u_1, u_2)}{\partial(q, x)} + p \frac{\partial(u_1, u_2)}{\partial(q, z)} \right\} \div \frac{\partial(u_1, u_2)}{\partial(p, q)},$$

and so on.

$$rt - s^2 = \left\{ \frac{\partial(u_1, u_2)}{\partial(x, y)} + p \frac{\partial(u_1, u_2)}{\partial(z, y)} + q \frac{\partial(u_1, u_2)}{\partial(x, z)} \right\} \div \frac{\partial(u_1, u_2)}{\partial(p, q)}.$$

Hence the equation (1) gives

$$\begin{aligned} & U \left\{ \frac{\partial(u_1, u_2)}{\partial(x, y)} + p \frac{\partial(u_1, u_2)}{\partial(z, y)} + q \frac{\partial(u_1, u_2)}{\partial(x, z)} \right\} \\ & + R \left\{ \frac{\partial(u_1, u_2)}{\partial(q, x)} + p \frac{\partial(u_1, u_2)}{\partial(q, z)} \right\} - 2S \left\{ \frac{\partial(u_1, u_2)}{\partial(p, x)} + p \frac{\partial(u_1, u_2)}{\partial(p, z)} \right\} \\ & + T \left\{ \frac{\partial(u_1, u_2)}{\partial(y, p)} + q \frac{\partial(u_1, u_2)}{\partial(z, p)} \right\} - V \frac{\partial(u_1, u_2)}{\partial(p, q)} = 0. \end{aligned} \quad (24)$$

We have also the compatibility condition

$$\frac{\partial(u_1, u_2)}{\partial(p, x)} + p \frac{\partial(u_1, u_2)}{\partial(p, z)} + \frac{\partial(u_1, u_2)}{\partial(q, y)} + q \frac{\partial(u_1, u_2)}{\partial(q, z)} = 0. \quad (25)$$

Now, if we denote by $d(u_1, u_2)$ the double differential element $du_1 du_2$, as it occurs, for instance, in a double integral, such as $\iint du_1 du_2$, we have

$$\begin{aligned} d(u_1, u_2) &= \frac{\partial(u_1, u_2)}{\partial(x, y)} d(x, y) + \frac{\partial(u_1, u_2)}{\partial(y, z)} d(y, z) + \frac{\partial(u_1, u_2)}{\partial(z, x)} d(z, x) \\ &+ \frac{\partial(u_1, u_2)}{\partial(x, p)} d(x, p) + \frac{\partial(u_1, u_2)}{\partial(y, p)} d(y, p) + \frac{\partial(u_1, u_2)}{\partial(z, p)} d(z, p) \\ &+ \frac{\partial(u_1, u_2)}{\partial(x, q)} d(x, q) + \frac{\partial(u_1, u_2)}{\partial(y, q)} d(y, q) \\ &+ \frac{\partial(u_1, u_2)}{\partial(z, q)} d(z, q) + \frac{\partial(u_1, u_2)}{\partial(p, q)} d(p, q). \end{aligned}$$

Thus, by reason of the relations just found among the Jacobians, it follows that $d(u_1, u_2)$ is a linear combination of the determinants

$$\left\| \begin{array}{ccccccc} d(x, y), & d(y, z), & d(z, x), & d(x, p), & d(y, p), & d(z, p), & d(x, q), \\ U, & -pU, & -qU, & 2S, & T, & 2pS+qT, & -R, \\ 0, & 0, & 0, & -1, & 0, & -p, & 0, \\ & & & & & d(y, q), & d(z, q), & d(p, q) \\ & & & & & 0, & -pR, & -V \\ & & & & & -1, & -q, & 0 \end{array} \right\|,$$

that is, of

$$\left\| \begin{array}{ccccccc} d(x, y), & d(z, y), & d(x, z), & d(x, p)+d(q, y), & d(y, p), & d(z, p)-pd(x, p)-qd(y, p), \\ U, & pU, & qU, & 2S, & T, & 0, \\ & & & d(x, q), & d(z, q)-pd(x, q)-qd(y, q), & d(p, q) \\ & & & -R, & 0, & -V \end{array} \right\|.$$

Also, if u_1, u_2 are any two functions of x, y, z, p, q such that $d(u_1, u_2)$ is a linear combination of the determinants of this matrix, then

$$u_1 = c_1, \quad u_2 = c_2,$$

if they can be solved for p, q , will furnish a solution.

10. Since $d(u_1, u_3)$ and $d(u_2, u_3)$ must also be linear combinations of the same expressions, it follows that a solution of the form (2) gives three such combinations, independent of each other. A solution given by Ampère's method leads to five independent combinations, for, if $u_1 = c_1$ is the intermediary integral, we have

$$dz - p dx - q dy \equiv \lambda du_1 + \mu (du_2 - u_4 du_3),$$

the other integral equations being, in the case of the general integral,

$$u_2 = f(u_3),$$

$$u_4 = f'(u_3).$$

We have then two complete primitives, among others,

$$u_1 = c_1, \quad u_2 = c_2, \quad u_3 = c_3;$$

$$u_1 = c_1, \quad u_4 = c_2, \quad u_2 - u_4 u_3 = c_3,$$

and the five combinations are

$$d(u_1, u_2), \quad d(u_1, u_3), \quad d(u_1, u_4), \quad d(u_3, u_3), \\ d(u_2, u_4) - u_4 d(u_3, u_4).$$

On this subject compare a paper by the writer presented this year to the Royal Society.

10. A point of interest in the theory is that an equation of Ampère's form (1) is changed into another of the same form by any tangential transformation. (Goursat, *Equations aux dérivées partielles du second ordre*, Vol. I., chapter ii.)

Suppose that x, y, z, p, q are expressed in terms of five other variables x', y', z', p', q' , in such a way that

$$dz - p dx - q dy \equiv \rho (dz' - p' dx' - q' dy').$$

Then we have r, s, t given by the equations

$$r \frac{dx}{dx'} + s \frac{dy}{dx'} = \frac{dp}{dx'}, \quad r \frac{dx}{dy'} + s \frac{dy}{dy'} = \frac{dp}{dy'}; \\ s \frac{dx}{dx'} + t \frac{dy}{dx'} = \frac{dq}{dx'}, \quad s \frac{dx}{dy'} + t \frac{dy}{dy'} = \frac{dq}{dy'};$$

where $\frac{d}{dx'}$, $\frac{d}{dy'}$ stand for

$$\frac{\partial}{\partial x'} + p' \frac{\partial}{\partial z'} + r' \frac{\partial}{\partial p'} + s' \frac{\partial}{\partial q'}$$

and

$$\frac{\partial}{\partial y'} + q' \frac{\partial}{\partial z'} + s' \frac{\partial}{\partial p'} + t' \frac{\partial}{\partial q'}.$$

Thus r, s, t and $rt - s^2$ are given in terms of $x', y', z', p', q', r', s', t'$ as fractions having a common denominator $\frac{d(x, y)}{d(x', y')}$, the numerators and denominator being all linear in $r', s', t', r't' - s'^2$. Thus Ampère's equation (1) is changed into another of the same form.

The new coefficients R', S', T', U', V' are easily found to be as follows:—

$$U' = U \frac{\partial(p, q)}{\partial(p', q')} + R \frac{\partial(p, y)}{\partial(p', q')} + 2S \frac{\partial(x, p)}{\partial(p', q')} + T \frac{\partial(x, q)}{\partial(p', q')} - V \frac{\partial(x, y)}{\partial(p', q')} \\ = [p', q'], \text{ say,}$$

and, with a like notation,

$$R' = [p', y'] + q' [p', z'],$$

$$2S' = [q', y'] + q' [q', z'] + [x', p'] + p' [z', p'],$$

$$T' = [x', q'] + p' [z', q'],$$

$$-V' = [x', y'] + p' [z', y'] + q' [x', z'].$$

It is natural to expect that $RT + UV - S^2$ will be an invariant for this transformation, and I find, in fact, that

$$R'T' + U'V' - S'^2 = \rho^2 (RT + UV - S^2).$$

12. We have seen that, if we have a solution

$$u_1 = c_1, \quad u_2 = c_2, \quad u_3 = c_3$$

of Ampère's equation (1), the three expressions u_1, u_2, u_3 in terms of x, y, z, p, q are the principal variables in a tangential transformation. The question then arises: What will be the result of applying this tangential transformation to the equation? Let us then suppose that in the transformation just discussed x', y', z' are such that the equations

$$x' = c_1, \quad y' = c_2, \quad z' = c_3$$

would give a solution. Any two of the three functions x', y', z' must satisfy the equations (24), (25), which are clearly covariant, so that U' is the coefficient of $\frac{\partial(u_1, u_2)}{\partial(x', y')}$ in (24) after transformation, and therefore, on this supposition,

$$U' = 0.$$

We thus have the following theorem:—If a solution of Ampère's equation has been found involving three arbitrary constants, and if other solutions are sought by replacing these constants by parameters which are taken as a new set of principal variables, then the resulting differential equation is of Monge's form

$$Rr + 2Ss + Tt = V.$$

This is Imschenetsky's result, given in Forsyth's *Treatise on Differential Equations*, § 271.

13. The two equations

$$\alpha_{13}(p_1q_2 - p_2q_1) + \alpha_{41}p_1 + \alpha_{13}q_1 + \alpha_{42}p_2 + \alpha_{23}q_2 + \alpha_{34} = 0,$$

$$\beta_{12}(p_1q_2 - p_2q_1) + \beta_{41}p_1 + \beta_{13}q_1 + \beta_{42}p_2 + \beta_{23}q_2 + \beta_{34} = 0,$$

where the coefficients α, β are functions of x_1, x_2, x_3, x_4 , and

$$p_1 = \frac{\partial x_3}{\partial x_1}, \quad q_1 = \frac{\partial x_4}{\partial x_1},$$

$$p_2 = \frac{\partial x_3}{\partial x_2}, \quad q_2 = \frac{\partial x_4}{\partial x_2},$$

may be treated in somewhat the same way as Ampère's equation. We may suppose without loss of generality that*

$$\alpha_{12}\alpha_{34} + \alpha_{31}\alpha_{24} + \alpha_{23}\alpha_{14} = 0,$$

and that either

$$\text{I. } \beta_{12}\beta_{34} + \beta_{31}\beta_{24} + \beta_{23}\beta_{14} = 0$$

$$\text{or II. } \alpha_{12}\beta_{34} + \alpha_{34}\beta_{12} + \alpha_{31}\beta_{24} + \alpha_{24}\beta_{31} + \alpha_{23}\beta_{14} + \alpha_{14}\beta_{23} = 0,$$

since the given equations may be combined linearly in any way. Then the first of the two equations may be written

$$\begin{vmatrix} \alpha_{12}p_1 - \alpha_{33}, & \alpha_{12}p_2 - \alpha_{13} \\ \alpha_{12}q_1 - \alpha_{43}, & \alpha_{12}q_2 - \alpha_{14} \end{vmatrix} = 0,$$

and replaced by the following pair:—

$$\alpha_{12}(p_1 + \lambda q_1) = \alpha_{33} + \lambda \alpha_{43},$$

$$\alpha_{12}(p_2 + \lambda q_2) = \alpha_{13} + \lambda \alpha_{14}.$$

Hence, if

$$u_1 = c_1, \quad u_2 = c_2$$

constitute a solution, and, if we choose A_1, A_2 so that

$$A_1 \frac{\partial u_1}{\partial x_3} + A_2 \frac{\partial u_2}{\partial x_3} = \alpha_{13},$$

$$A_1 \frac{\partial u_1}{\partial x_4} + A_2 \frac{\partial u_2}{\partial x_4} = \lambda \alpha_{12},$$

we have

$$A_1 \frac{\partial u_1}{\partial x_1} + A_2 \frac{\partial u_2}{\partial x_1} = \alpha_{23} + \lambda \alpha_{24},$$

$$A_1 \frac{\partial u_1}{\partial x_2} + A_2 \frac{\partial u_2}{\partial x_2} = \alpha_{31} + \lambda \alpha_{41},$$

$$A_1 du_1 + A_2 du_2 = (\alpha_{23}dx_1 + \alpha_{31}dx_2 + \alpha_{12}dx_3) + \lambda (\alpha_{24}dx_1 + \alpha_{41}dx_2 + \alpha_{12}dx_3).$$

* It is convenient to assume that two coefficients α (or β) with the same suffixes in different orders are equal with opposite signs. Thus $\alpha_{11}, \beta_{11}, \alpha_{22}, \dots$ are zero.

In case I. the second equation may be treated in the same way as the first, and the converse theorem may be easily proved by reversing the argument as in the case of Ampère's equation (see § 4).

In either case, I. or II., we deduce from the second equation that

$$\sum_{i,j} \beta_{ij} \frac{\partial (u_1, u_2)}{\partial (x_i, x_j)} = 0,$$

which may be written *symbolically*

$$\sum_{i,j} \beta_{ij} \frac{\partial (\Omega, u)}{\partial (x_i, x_j)} = 0,$$

where u is u_1 or u_2 , and $d\Omega$ denotes $A_1 du_1 + A_2 du_2$.

Thus du is a linear combination of the determinants of the matrix of four columns

$$\left\| \begin{array}{ccc} \dots, & dx_j, & \dots \\ \dots, & \sum_i \beta_{ij} \frac{\partial \Omega}{\partial x_i}, & \dots \end{array} \right\|.$$

In this matrix write the columns as rows, substituting the known expressions for $\frac{\partial \Omega}{\partial x_1}$, ..., in terms of λ . It thus becomes

$$\left\| \begin{array}{l} dx_1, \quad \beta_{21}(a_{31} + \lambda a_{41}) + \beta_{31}a_{12} + \beta_{41}\lambda a_{12} \\ dx_2, \quad \beta_{12}(a_{23} + \lambda a_{24}) + \beta_{32}a_{12} + \beta_{42}\lambda a_{12} \\ dx_3, \quad \beta_{13}(a_{23} + \lambda a_{24}) + \beta_{23}(a_{31} + \lambda a_{41}) + \beta_{43}\lambda a_{12} \\ dx_4, \quad \beta_{14}(a_{23} + \lambda a_{24}) + \beta_{24}(a_{31} + \lambda a_{41}) + \beta_{34}a_{12} \end{array} \right\|,$$

and the following rows may be added, being merely combinations of those just written :—

$$\begin{aligned} & \beta_{33}dx_1 + \beta_{31}dx_2 + \beta_{13}dx_3, \quad \lambda a_{12}(\beta_{23}\beta_{41} + \beta_{31}\beta_{42} + \beta_{12}\beta_{43}), \\ & \beta_{24}dx_1 + \beta_{41}dx_2 + \beta_{12}dx_3, \quad a_{12}(\beta_{23}\beta_{14} + \beta_{31}\beta_{24} + \beta_{12}\beta_{34}), \\ & a_{23}dx_1 + a_{31}dx_2 + a_{12}dx_3, \quad \lambda a_{12}(a_{23}\beta_{41} + a_{41}\beta_{23} + a_{31}\beta_{42} + a_{42}\beta_{31} + a_{13}\beta_{43} + a_{43}\beta_{12}), \\ & a_{24}dx_1 + a_{41}dx_2 + a_{12}dx_3, \quad a_{12}(a_{23}\beta_{14} + a_{14}\beta_{23} + a_{31}\beta_{24} + a_{24}\beta_{31} + a_{12}\beta_{34} + a_{34}\beta_{12}). \end{aligned}$$

Thus in case I. we are led to the same result as before ; in case II. we find that du_1 and du_2 are linear combinations of the expressions

$$\begin{aligned} & a_{23}dx_1 + a_{31}dx_2 + a_{12}dx_3, \\ & a_{24}dx_1 + a_{41}dx_2 + a_{12}dx_3, \\ & (\beta_{23}dx_1 + \beta_{31}dx_2 + \beta_{12}dx_3) + \lambda(\beta_{21}dx_1 + \beta_{41}dx_2 + \beta_{12}dx_3), \end{aligned}$$

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λ being the ratio of the coefficients of the first two expressions in that combination of them which is equal to a linear combination of du_1, du_2 .

The resulting methods are then as follows :—

CASE I.—Find by inspection two functions u_1, u_2 such that one linear combination of their differentials is equal to a linear combination of

$$a_{23}dx_1 + a_{31}dx_2 + a_{12}dx_3, \quad a_{24}dx_1 + a_{41}dx_2 + a_{12}dx_4,$$

and another linear combination of their differentials to a linear combination of

$$\beta_{23}dx_1 + \beta_{31}dx_2 + \beta_{12}dx_3, \quad \beta_{24}dx_1 + \beta_{41}dx_2 + \beta_{12}dx_4;$$

then

$$u_1 = c_1, \quad u_2 = c_2$$

will constitute a solution.

CASE II.—Find by inspection two functions u_1, u_2 such that one linear combination of their differentials is equal to a linear combination of

$$a_{23}dx_1 + a_{31}dx_2 + a_{12}dx_3, \quad a_{24}dx_1 + a_{41}dx_2 + a_{12}dx_4,$$

say to $a_{23}dx_1 + a_{31}dx_2 + a_{12}dx_3 + \lambda(a_{24}dx_1 + a_{41}dx_2 + a_{12}dx_4)$,

and that another linear combination of their differentials is equal to a linear combination of these same two expressions together with

$$\beta_{23}dx_1 + \beta_{31}dx_2 + \beta_{12}dx_3 + \lambda(\beta_{24}dx_1 + \beta_{41}dx_2 + \beta_{12}dx_4);$$

then

$$u_1 = c_1, \quad u_2 = c_2$$

will constitute a solution.

In either case all solutions of the form

$$u_1 = c_1, \quad u_2 = c_2$$

must satisfy the condition here specified.