"On Systems of Porismatic Equations, Algebraical and Trigonometrical;" "Note on Epicycloids and Hypocycloids;" "Locus of Point of Concourse of Perpendicular Tangents to a Cardioid;" and "Elliptic Motion under Acceleration constant in direction:" Prof. Wolstenholme.

"On the Theory of a System of Electrified Conductors;" "On the Focal Lines of a Refracted Pencil:" Prof. J. Clerk Maxwell.

The following presents were received:—

"Sur les trajectoires des points d'une droite mobile dans l'espace," and "Démonstration géométrique d'une proposition due à M. Bernard," by M. Mannheim: from the Author.

On the Calculation of the Value of the Theoretical Unit-Angle to a Great Number of Decimal-Places. By J. W. L. Glaisher, B.A., Fellow of Trinity College, Cambridge.

[*Read April 10th*, 1873.]

The number of degrees in the theoretical unit of angular measure (viz., the angle the length of whose arc is equal to the radius) is $\frac{180}{\pi}$, so that merely by a division this angle can be determined to about as many decimals of a degree as there are decimals of π known. More than a year ago I had need of the value of the unit-angle to a much greater degree of accuracy than it was given to in any place that I was acquainted with, and to make certain of having an abundant number of decimals, I had the division performed, taking as the divisor the first fifty-four figures of π (this being a good point to break off at, as the fifty-fifth figure is a cipher), and shortly afterwards I had the same work performed by another computer, nearly independently. Having in the course of the year examined a great number of tables, collections of constants, &c., and never having met with the angle in question given to even a moderate number of places, I thought it would be desirable to communicate the result of the calculation to the Society. I have accordingly examined one of the calculations myself, and have also had the division performed a third time, so that the present results are submitted with confidence.

The theoretical unit-angle to 52 decimals of a degree is

57°·29577	95130	82320	87679	81548
14105	17033	24054	72466	56432
15 . . .				

which, expressed in degrees, minutes, and seconds, is, to 47 decimals of a second,

$$57^\circ\ 17'\ 44''\cdot 80624 \quad 70963 \quad 55156 \quad 47335 \quad 73307$$
$$78613 \quad 19665 \quad 97008 \quad 79631 \quad 55\ldots$$

The same, expressed in grades or centesimal degrees (400 to the circumference), is, to 51 decimals of a grade

$$63^g\cdot 66197 \quad 72367 \quad 58134 \quad 30755 \quad 35053$$
$$49005 \quad 74481 \quad 37838 \quad 58296 \quad 18257$$
$$9\ldots$$

Although a division of 50 figures by 50 figures to 50 figures quotient is neither a very elaborate nor a very laborious piece of work, still it is sufficiently troublesome to render the publication of the result valuable, as the difficulty of attainment of a numerical quantity is measured, not by the time required for the performance of the arithmetical work, but by this amount plus the time required to ensure certainty that what has been done is *free from error.*

It is a matter of surprise that the unit-angle is not more generally regarded as one of the mathematical constants, along with π, e, γ, &c., for it is one with which the mathematical computer is much concerned, chiefly in semi-convergent series of the form

$$\cos x\left(\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}\ldots\right)+\sin x\left(\frac{B_1}{x}+\frac{B_2}{x^2}+\frac{B_3}{x^3}+\ldots\right).$$

In tabulating a function from such a series (very frequently the only one available when x is large), the process is to multiply the unit-angle by x, reduce to degrees, minutes, and seconds, and then use the ordinary trigonometrical canon. It thus appears that there is some reason for giving the unit-angle to a fair number of decimal places, as it is likely in practice to be required after multiplication by large quantities; thus for $x=1{,}000{,}000$ the angle is $57295779^\circ\cdot 513082\ldots$, and leaving out multiples of 360°, and reducing to minutes and seconds, $\sin x$ is equal to the sine of $339^\circ\ 30'\ 47''\cdot 096$; and the log sine of this angle (the sine being treated as positive) is taken out from the ordinary tables as $9\cdot 5440600$.

I had written so far, and, as I thought, completed this communication, when I remembered to have seen the value of $\frac{1}{\pi}$ (from which the unit-angle can of course be easily deduced) given (together with other constants) to a great many places in an early volume of Grunert's "Archiv." On examination, I found the paper in question, which is by Dr. G. Paucker, in vol. i., pp. 9—11 (1841), where $\frac{1}{\pi}$ is given to 140 decimal places.

Taking this value of $\frac{1}{\pi}$, we have, for the number of grades,

$63^g\cdot$66197	72367	58134	30755	35053
49005	74481	37838	58296	18257
94990	66937	62355	87190	53690
61403	60455	21106	50123	43824
29137	09070	31832	14757	16473
84458	31460	55 ... ;		

for the number of degrees,

$57^\circ\cdot$29577	95130	82320	87679	81548
14105	17033	24054	72466	56432
15491	60243	86120	28471	48321
55263	24409	68995	85111	09441
86223	38163	28648	93281	44826
46012	48314	49 ... ;		

for the number of degrees, minutes, and seconds,

$57^\circ\ 17'\ 44''\cdot$80624	70963	55156	47335	73307
78613	19665	97008	79631	55757
69768	77900	33024	97339	57589
47678	74883	85063	99939	90704
04173	87831	36158	13213	75256
44939	32194 ... ;			

agreeing exactly with my own calculations, as far as the latter extend. But I must here point out that the last four (and perhaps the last fourteen) figures of the value of π used by Paucker are erroneous, so that about the last four figures of the three results last written are necessarily erroneous, and perhaps as many as the last fourteen figures. The cause of this uncertainty is that Paucker says that he took Vega's value of π ("Hier folgt diese Zahl auf 140 Stellen, nach dem Vega-schen Werthe von π berechnet"), but gives no further information.

Now Vega gave two values of π, both to 140 places, the one correct to 126 places, and the other to 136; and there is nothing to indicate distinctly which of the two was the value that Paucker used.

The first was published in the "Nova Acta Petropolitana" for 1790 (t. ix., p. 41), and the last twenty figures (viz., those from the 121st to the 140th both inclusive) are

09384 44767 21386 11733

There are also three more figures added, viz. 138, but these are printed with bars through them, to indicate that they are not to be relied upon; so that the value may be described as extending to 140 places.

Vega's second value was published in his "Thesaurus Logarithmorum Completus," fol. Leipzig, 1794 (p. 633), where the same twenty figures appear as

09384 46095 50582 26136,

the correct figures being

09384 46095 50582 23172.

It is much to be regretted that Vega, in giving his second value, did not call attention to the fact that he had previously published one that was erroneous; and this omission is very surprising, when it is remembered how much Vega did for the detection of errors in the table of logarithms of numbers, and in the cause of accuracy generally.

The formula he used in his first calculation was equivalent to

$$\pi = 4 \tan^{-1}\frac{1}{7} + 8 \tan^{-1}\frac{1}{3},$$

while the second was derived from

$$\pi = 20 \tan^{-1}\frac{1}{7} + 8 \tan^{-1}\frac{3}{79};$$

and in his "Thesaurus" he gives the values of the series depending on $\tan^{-1}\frac{1}{7}$ and $\tan^{-1}\frac{3}{79}$ to more than 140 places, while that depending on $\tan^{-1}\frac{1}{3}$ is only given to 128; and he remarks that a verification to 126 places is thus obtained. It appears therefore that in the first calculation an error must have been made in the 127th place (or thereabouts) of $\tan^{-1}\frac{1}{3}$, and it is a pity he did not allude to his former calculation, and state definitely that such was the case.

It is unfortunate, too, that although Paucker has given the values of $\sqrt{\pi}$ to 140 places, yet the only two quantities, viz. $\frac{1}{4}\pi$ and $\frac{1}{6}\pi$, from which the value of π he used could be easily determined, are only given by him to 51 places. I am inclined to think, for several reasons, that it was the second of Vega's values that Paucker referred to; and if so, only the last four figures of the quantities written above, and deduced from it, are necessarily erroneous.

Paucker remarks that Euler gave the value ("Introd. in Anal. Infin.," Lausannæ, 1748, tom. i., § 198, p. 160) to 36 places, but that the 9 in the 25th place should be a 5; and this is so. In Lambert's "Supplementa Tabularum" (Olisipone, 1798, p. 139) $\frac{1}{\pi}$ is given correctly to 20 places; and on the same page the value of the unit-angle is given as $57^{\circ}. 17^{i}. 44^{ii}. 48^{iii}. 22^{iv}. 29^{v}. 21^{vi}$. This is not quite correct in the last figure, as the sixths and sevenths are $22^{vi}. 22^{vii}$....

It will have been seen that the present triple calculation of the angle, confirmed by Paucker, leaves not a shadow of a doubt of the accuracy of the values of the unit-angle here given to as many places as the former extends to.

On Systems of Porismatic Equations, Algebraical and Trigonometrical. By Prof. Wolstenholme.

[*Read April* 10*th*, 1873.]

The system of algebraical equations

$$\left.\begin{aligned}\frac{a}{yz}+byz+c+a'(y+z)+b'\left(\frac{1}{y}+\frac{1}{z}\right)+c'\left(\frac{y}{z}+\frac{z}{y}\right)&=0\\ \frac{a}{zx}+bzx+c+a'(z+x)+b'\left(\frac{1}{z}+\frac{1}{x}\right)+c'\left(\frac{z}{x}+\frac{x}{z}\right)&=0\\ \frac{a}{xy}+bxy+c+a'(x+y)+b'\left(\frac{1}{x}+\frac{1}{y}\right)+c'\left(\frac{x}{y}+\frac{y}{x}\right)&=0\end{aligned}\right\}\text{(A)},$$

where x, y, z are unequal quantities, is porismatic; that is, there is either no solution at all; or if a certain relation between the coefficients be satisfied, there is an infinite number of solutions, any one of the equations being then deducible from the other two.

The same proposition is true for a system of any number of such equations, as also of course for the system of trigonometrical equations of which the type is

$$a\cos\beta\cos\gamma+b\sin\beta\sin\gamma+c$$
$$+a'(\sin\beta+\sin\gamma)+b'(\cos\beta+\cos\gamma)+c'\sin(\beta+\gamma).$$

I propose to investigate these propositions directly for systems of three, four, and five algebraical equations, and for systems of three and four trigonometrical equations.

I. Suppose we have the system (A). The second and third equations prove that y, z are the two roots of the quadratic equation in u,

$$u^2(bx^2+a'x+c')+u(a'x^2+cx+b')+(c'x^2+b'x+a)=0,$$

so that we have

$$\frac{1}{bx^2+a'x+c'}=\frac{-(y+z)}{a'x^2+cx+b'}=\frac{yz}{c'x^2+b'x+a}\;\cdots\cdots\cdots\;(1);$$

or $bx^2+a'x+c'=\lambda$, $\quad a'x^2+cx+b'=-\lambda(y+z)$, $\quad c'x^2+b'x+a=\lambda yz$.