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LXXXV. *Note on the Higher Derivative of a Function, the variable of which is a Function of an independent variable.*
By I. J. SCHWATT*.

IN E. R. Hedrick's translation of Goursat's work 'A Course in Mathematical Analysis' appears the following problem (p. 32, 6.):—

Show that the n th derivative of a function $y = \phi(u)$, where u is a function of the independent variable x , may be written in the form

$$(a) \quad \frac{d^n y}{dx^n} = A_1 \phi'(u) + \frac{A_2}{1.2} \phi''(u) + \dots + \frac{A_n}{1.2.3\dots n} \phi^n(u),$$

where

$$(b) \quad A_\kappa = \frac{d^\kappa u}{dx^\kappa} - \frac{\kappa}{1} u \frac{d^{\kappa-1} u}{dx^{\kappa-1}} + \frac{\kappa(\kappa-1)}{1.2} u^2 \frac{d^{\kappa-2} u}{dx^{\kappa-2}} + \dots \\ + (-1)^{\kappa-1} \kappa u^{\kappa-1} \frac{d^n u}{dx^n} (\kappa = 1, 2, \dots, n).$$

[First notice that the n th derivative may be written in the form (a), where the coefficients $A_1, A_2, \dots, A_n$ are independent of the form of the function $\phi(u)$. To find their values, set $\phi(u)$ equal to $u, u^2, \dots, u^n$ successively, and solve the resulting equations for $A_1, A_2, \dots, A_n$. The result is the form (b)].

I have quoted the problem and the suggestions in full, and shall now proceed to give several proofs for it, in the hope that these proofs will illustrate certain operations with series which might be useful in similar work.

I. Let $y = \phi(u)$, wherein u is a function of x .

Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \phi'(u) \frac{du}{dx}.$$

$$\frac{d^2 y}{dx^2} = \phi'(u) \frac{d^2 u}{dx^2} + \phi''(u) \left(\frac{du}{dx} \right)^2.$$

$$\frac{d^3 y}{dx^3} = \phi'(u) \frac{d^3 u}{dx^3} + 3\phi''(u) \frac{du}{dx} \frac{d^2 u}{dx^2} + \phi'''(u) \left(\frac{du}{dx} \right)^3.$$

* Communicated by the Author.

We now assume

$$\frac{d^n y}{dx^n} = \frac{A_1}{1!} \phi'(u) + \frac{A_2}{2!} \phi''(u) + \frac{A_3}{3!} \phi'''(u) + \dots + \frac{A_n}{n!} \phi^n(u), \quad (1)$$

wherein the A 's are functions of u and x , but independent of $\phi(u)$.

To show that (1) is true, we must show that it holds also for the $n+1$ st derivative.

Differentiating (1) with respect to x , we have

$$\begin{aligned} \frac{d^{n+1}y}{dx^{n+1}} &= \phi'(u) \frac{d}{dx} \frac{A_1}{1!} + \phi''(u) \left[\frac{A_1}{1!} \frac{du}{dx} + \frac{d}{dx} \frac{A_2}{2!} \right] \\ &+ \phi'''(u) \left[\frac{A_2}{2!} \frac{du}{dx} + \frac{d}{dx} \frac{A_3}{3!} \right] + \dots + \phi^{n-1}(u) \left[\frac{A_{n-2}}{(n-2)!} \frac{du}{dx} + \frac{d}{dx} \frac{A_{n-1}}{(n-1)!} \right] \\ &+ \phi^n(u) \left[\frac{A_{n-1}}{(n-1)!} \frac{du}{dx} + \frac{d}{dx} \frac{A_n}{n!} \right] + \phi^{n+1}(u) \frac{A_n}{n!} \frac{du}{dx} \\ &= \frac{B_1}{1!} \phi'(u) + \frac{B_2}{2!} \phi''(u) + \frac{B_3}{3!} \phi'''(u) + \dots + \frac{B_n}{n!} \phi^n(u) + \frac{B_{n+1}}{(n+1)!} \phi^{n+1}(u), \end{aligned}$$

which is of the same form as (1).

We shall now determine the coefficients represented by the A 's in (1). Since the A 's are independent of y , they will have the same value whatever $\phi(u)$ might be.

Letting therefore:

$$y=u, \text{ then } \frac{d^n u}{dx^n} = A_1;$$

$$y=u^2, \text{ ,, } \frac{d^n u^2}{dx^n} = 2uA_1 + A_2;$$

$$y=u^3, \text{ ,, } \frac{d^n u^3}{dx^n} = 3u^2A_1 + 3uA_2 + A_3;$$

.

$$y=u^\kappa, \text{ ,, } \frac{d^n u^\kappa}{dx^n} = \binom{\kappa}{1} u^{\kappa-1} A_1 + \binom{\kappa}{2} u^{\kappa-2} A_2 + \dots + \binom{\kappa}{\kappa-1} u A_{\kappa-1} + A_\kappa.$$

Solving these equations for the A 's, we obtain

$$A_1 = \frac{d^n u}{dx^n},$$

$$A_2 = \frac{d^n u^2}{dx^n} - \binom{2}{1} u \frac{d^n u}{dx^n},$$

$$A_3 = \frac{d^n u^3}{dx^n} - \binom{3}{1} u \frac{d^n u^2}{dx^n} + \binom{3}{2} u^2 \frac{d^n u}{dx^n}.$$

Let us now assume

$$A_{\kappa} = \sum_{\lambda=0}^{\kappa} (-1)^{\lambda} \binom{\kappa}{\lambda} u^{\lambda} \frac{d^{\kappa}}{dx^{\kappa}} u^{\kappa-\lambda}. \quad (2)$$

Now if $y = u^{\kappa+1}$, then from (1) follows

$$\frac{d^n u^{\kappa+1}}{dx^n} = \binom{\kappa+1}{1} u^{\kappa} A_1 + \binom{\kappa+1}{2} u^{\kappa-1} A_2 + \dots + \binom{\kappa+1}{\kappa} u A_{\kappa} + A_{\kappa+1}$$

and

$$A_{\kappa+1} = \frac{d^n u^{\kappa+1}}{dx^n} - \sum_{\lambda=1}^{\kappa} \binom{\kappa+1}{\lambda} u^{\kappa-\lambda+1} A_{\lambda}.$$

But

$$A_{\lambda} = \sum_{\gamma=0}^{\lambda} (-1)^{\gamma} \binom{\lambda}{\gamma} u^{\gamma} \frac{d^{\lambda}}{dx^{\lambda}} u^{\lambda-\gamma},$$

for values of λ from 1 to κ inclusive.
Therefore

$$A_{\kappa+1} = \frac{d^n u^{\kappa+1}}{dx^n} - \sum_{\lambda=1}^{\kappa} \sum_{\gamma=0}^{\lambda} (-1)^{\gamma} \binom{\kappa+1}{\lambda} \binom{\lambda}{\gamma} u^{\kappa+1-\lambda+\gamma} \frac{d^{\lambda}}{dx^{\lambda}} u^{\lambda-\gamma}.$$

Denoting the double summation by S and letting $\lambda = \kappa + 1 - \alpha$, and in turn $\alpha + \gamma = \beta$, we have

$$\begin{aligned} S &= \sum_{\alpha=1}^{\kappa} \sum_{\beta=\alpha}^{\kappa+1} (-1)^{\beta-\alpha} \binom{\kappa+1}{\alpha} \binom{\kappa+1-\alpha}{\beta-\alpha} u^{\beta} \frac{d^{\kappa+1-\beta}}{dx^{\kappa+1-\beta}} \\ &= \sum_{\alpha=1}^{\kappa} \sum_{\beta=\alpha}^{\kappa} (-1)^{\beta-\alpha} \binom{\kappa+1}{\alpha} \binom{\kappa+1-\alpha}{\beta-\alpha} u^{\beta} \frac{d^{\kappa+1-\beta}}{dx^{\kappa+1-\beta}}, \quad (3) \end{aligned}$$

since for $\beta = \kappa + 1$, $\frac{d^{\kappa+1-\beta}}{dx^{\kappa+1-\beta}} = 0$.

Now
$$\sum_{\alpha=1}^{\kappa} \sum_{\beta=\alpha}^{\kappa} M_{\alpha, \beta} = \sum_{\beta=1}^{\kappa} \sum_{\alpha=1}^{\beta} M_{\alpha, \beta}.$$

This can be shown in the following way:

$$\begin{aligned} \sum_{\alpha=1}^{\kappa} \sum_{\beta=\alpha}^{\kappa} M_{\alpha, \beta} &= M_{1,1} + M_{1,2} + M_{1,3} + \dots + M_{1,\kappa} \\ &\quad + M_{2,2} + M_{2,3} + \dots + M_{2,\kappa} \\ &\quad + M_{3,3} + \dots + M_{3,\kappa} \\ &\quad + \dots \dots \dots \\ &\quad + M_{\kappa,\kappa}. \end{aligned}$$

Adding by columns we have

$$\sum_{a=1}^{\kappa} \sum_{\beta=a}^{\kappa} M_{a,\beta} = M_{1,1} + (M_{1,2} + M_{2,2}) + \dots + (M_{1,\kappa} + M_{2,\kappa} + \dots + M_{\kappa,\kappa})$$

$$= \sum_{\beta=1}^{\kappa} \sum_{a=1}^{\beta} M_{a,\beta}.$$

By means of (4) the form (3) changes to

$$S = \sum_{\beta=1}^{\kappa} (-1)^{\beta} \sum_{a=1}^{\beta} (-1)^a \binom{\kappa+1}{a} \binom{\kappa+1-\alpha}{\beta-a} u^{\beta} \frac{d^n u^{\kappa+1-\beta}}{dx^n}$$

$$= \sum_{\beta=1}^{\kappa} (-1)^{\beta} u^{\beta} \frac{d^n u^{\kappa+1-\beta}}{dx^n} \sum_{a=1}^{\beta} (-1)^a \binom{\kappa+1}{a} \binom{\kappa+1-\alpha}{\beta-a}.$$

Now

$$\binom{\kappa+1}{a} \binom{\kappa+1-\alpha}{\beta-a} = \frac{(\kappa+1)!}{(\kappa+1-\alpha)! a!} \cdot \frac{(\kappa+1-\alpha)!}{(\kappa+1-\beta)! (\beta-a)!} \cdot \frac{\beta!}{\beta!}$$

$$= \frac{(\kappa+1)!}{(\kappa+1-\beta)! \beta!} \cdot \frac{\beta!}{(\beta-a)! a!} = \binom{\kappa+1}{\beta} \binom{\beta}{a}.$$

Therefore

$$S = \sum_{\beta=1}^{\kappa} (-1)^{\beta} \binom{\kappa+1}{\beta} u^{\beta} \frac{d^n u^{\kappa+1-\beta}}{dx^n} \sum_{a=1}^{\beta} (-1)^a \binom{\beta}{a}.$$

But $\sum_{a=1}^{\beta} (-1)^a \binom{\beta}{a} = \sum_{a=0}^{\beta} (-1)^a \binom{\beta}{a} - 1 = (1-1)^{\beta} - 1 = -1,$

and (5) becomes

$$S = \sum_{\beta=1}^{\kappa} (-1)^{\beta} \binom{\kappa+1}{\beta} u^{\beta} \frac{d^n u^{\kappa+1-\beta}}{dx^n}.$$

Hence

$$A_{\kappa+1} = \frac{d^n u^{\kappa+1}}{dx^n} + \sum_{\beta=1}^{\kappa} (-1)^{\beta} \binom{\kappa+1}{\beta} u^{\beta} \frac{d^n u^{\kappa+1-\beta}}{dx^n}$$

$$= \sum_{\beta=0}^{\kappa} (-1)^{\beta} \binom{\kappa+1}{\beta} u^{\beta} \frac{d^n u^{\kappa+1-\beta}}{dx^n}, \text{ since } \frac{d^n u^{\kappa+1-\kappa-1}}{dx^n} = 0,$$

and $A_{\kappa+1}$ is of the same form as the one assumed for A_{κ} in (2).

II. The following is another method for proving the given theorem.

Let κ successive operations $u \frac{d}{du}$ each on $\phi(u)$, that is

$$\left[\left(u \frac{d}{du} \right) \left(u \frac{d}{du} \right) \left(u \frac{d}{du} \right) \dots \kappa \text{ operations} \right] \phi(u)$$

wherein u and $\frac{d}{du}$ are not permutable, be designated by

$$\left(u \frac{d}{du} \right)^\kappa \phi(u),$$

then

$$\left(u \frac{d}{du} \right)^\kappa \phi(u) = \sum_{\kappa=1}^{\infty} \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} (\kappa - \alpha)^n u^\kappa \frac{d^\kappa \phi(u)}{du^\kappa}.$$

Let $u = e^x$, then $\left(u \frac{d}{du} \right)^n = \left(u \frac{\frac{dx}{du}}{\frac{du}{dx}} \right)^n = \left(\frac{d}{dx} \right)^n$,

therefore

$$\frac{d^n \phi(u)}{dx^n} = \sum_{\kappa=1}^{\infty} \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} (\kappa - \alpha)^n e^{\kappa x} \phi^{(\kappa)}(e^x).$$

But

$$\begin{aligned} (\kappa - \alpha)^n e^{\kappa x} &= e^{a x} (\kappa - \alpha)^n e^{(\kappa - a)x} \\ &= e^{a \kappa} \frac{d^n e^{(\kappa - a)x}}{dx^n} \\ &= u \frac{d^n u^{\kappa - a}}{dx^n}. \end{aligned}$$

Hence

$$\frac{d^n \phi(u)}{dx^n} = \sum_{\kappa=1}^{\infty} \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} u^a \frac{d^n}{dx^n} u^{\kappa - a} \phi^{(\kappa)}(u).$$

III. A third way of proving the theorem is as follows:—

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

$$\frac{d^2 y}{dx^2} = \frac{d^2 u}{dx^2} \cdot \frac{dy}{du} + \left(\frac{du}{dx} \right)^2 \frac{d^2 y}{du^2},$$

but

$$\left(\frac{du}{dx} \right)^2 = \frac{1}{2!} \left[\frac{d^2 u^2}{dx^2} - \left(\frac{2}{1} \right) u \frac{d^2 u}{dx^2} \right],$$

therefore

$$\frac{d^2 y}{dx^2} = \sum_{\kappa=1}^{\infty} \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} u^a \frac{d^2 u^{\kappa - a}}{dx^2} \frac{d^\kappa y}{du^\kappa}. \quad (1)$$

Again
$$\frac{d^3y}{dx^3} = \frac{d^3u}{dx^3} \frac{dy}{du} + 3 \frac{d^2u}{dx^2} \cdot \frac{du}{dx} \frac{d^2y}{du^2} + \left(\frac{du}{dx}\right)^3 \frac{d^3y}{du^3}.$$

Since
$$\frac{d^3u^2}{dx^3} = 6 \frac{d^2u}{dx^2} \frac{du}{dx} + 2u \frac{d^3u}{dx^3},$$

and
$$\frac{d^3u^3}{dx^3} = 3u^2 \frac{d^3u}{dx^3} + 18u \frac{du}{dx} \frac{d^2u}{dx^2} + 6 \left(\frac{du}{dx}\right)^3,$$

therefore
$$3 \frac{d^2u}{dx^2} \cdot \frac{du}{dx} = \frac{1}{2!} \left[\frac{d^3u^2}{dx^3} - \binom{2}{1} u \frac{d^3u}{dx^3} \right],$$

and
$$\left(\frac{du}{dx}\right)^3 = \frac{1}{3!} \left[\frac{d^3u^3}{dx^3} - \binom{3}{1} u \frac{d^3u^2}{dx^3} + \binom{3}{2} u^2 \frac{d^3u}{dx^3} \right],$$

and hence

$$\begin{aligned} \frac{d^3y}{dx^3} &= \frac{d^3u}{dx^3} \frac{dy}{du} + \frac{1}{2!} \left[\frac{d^3u^2}{dx^3} - \binom{2}{1} u \frac{d^3u}{dx^3} \right] \frac{d^2y}{du^2} \\ &\quad + \frac{1}{3!} \left[\frac{d^3u^3}{dx^3} - \binom{3}{1} u \frac{d^3u^2}{dx^3} + \binom{3}{2} u^2 \frac{d^3u}{dx^3} \right] \frac{d^3y}{du^3}, \end{aligned}$$

or written symbolically

$$\frac{d^3y}{dx^3} = \sum_{\kappa=1}^3 \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} u^\alpha \frac{d^3u^{\kappa-\alpha}}{dx^3} \frac{d^{\kappa}y}{du^{\kappa}}. \quad (2)$$

Let us assume that this form holds for all values of n from 1 to n inclusive, that is

$$\frac{d^ny}{dx^n} = \sum_{\kappa=1}^n \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} u^\alpha \frac{d^nu^{\kappa-\alpha}}{dx^n} \frac{d^{\kappa}y}{du^{\kappa}}. \quad (3)$$

We shall then prove that this form holds also for $\frac{d^{n+1}y}{dx^{n+1}}$.

Differentiating (3) gives

$$\begin{aligned} \frac{d^{n+1}y}{dx^{n+1}} &= \sum_{\kappa=1}^n \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} u^\alpha \frac{d^nu^{\kappa-\alpha}}{dx^n} \frac{d^{\kappa+1}y}{du^{\kappa+1}} \frac{du}{dx} \\ &\quad + \sum_{\kappa=1}^n \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} \frac{d}{dx} \left\{ u^\alpha \frac{d^nu^{\kappa-\alpha}}{dx^n} \right\}. \end{aligned}$$

Now
$$\begin{aligned} &\sum_{\kappa=1}^n \left[\sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} \frac{d}{dx} \left\{ u^\alpha \frac{d^nu^{\kappa-\alpha}}{dx^n} \right\} \right] \\ &= \sum_{\kappa=2}^n \sum_{\alpha=1}^{\kappa-1} (-1)^\alpha \alpha u^{\alpha-1} \frac{du}{dx} \frac{d^nu^{\kappa-\alpha}}{dx^n} + \sum_{\kappa=1}^n \sum_{\alpha=0}^{\kappa-1} (-1)^\alpha \binom{\kappa}{\alpha} u^\alpha \frac{d^{n+1}u^{\kappa-\alpha}}{dx^{n+1}}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d^{n+1}y}{dx^{n+1}} &= \sum_{\kappa=2}^{n+1} \left[\sum_{a=0}^{\kappa-1} (-1)^a \binom{\kappa-1}{a} u^a \frac{d^n u^{\kappa-a-1}}{dx^n} \right] \frac{1}{(\kappa-1)!} \frac{du}{dx} \frac{d^\kappa y}{du^\kappa} \\ &\quad - \sum_{\kappa=2}^n \left[\sum_{a=0}^{\kappa-2} (-1)^a \binom{\kappa}{a+1} (\alpha+1) u^a \frac{d^n u^{\kappa-a-1}}{dx^n} \right] \frac{1}{\kappa!} \frac{du}{dx} \frac{d^\kappa y}{dx^\kappa} \\ &\quad + \sum_{\kappa=1}^n \left[\sum_{a=0}^{\kappa-1} (-1)^a \binom{\kappa}{a} u^a \frac{d^{n+1} u^{\kappa-a}}{dx^{n+1}} \right] \frac{1}{\kappa!} \frac{d^\kappa y}{dx^\kappa} \cdot \cdot \cdot \quad (4) \end{aligned}$$

But

$$\left(\frac{\kappa}{\alpha+1} \right) \frac{\alpha+1}{\kappa!} = \binom{\kappa-1}{\alpha} \frac{1}{(\kappa-1)!},$$

therefore

$$\begin{aligned} \frac{d^{n+1}y}{dx^{n+1}} &\left[\sum_{a=0}^{n-1} (-1)^a \binom{n}{a} u^a \frac{d^n u^{n-a}}{dx^n} \right] \frac{1}{n!} \frac{du}{dx} \frac{d^{n+1}y}{du^{n+1}} \\ &+ \sum_{\kappa=1}^n \frac{1}{\kappa!} \sum_{a=0}^{\kappa-1} (-1)^a \binom{\kappa}{a} u^a \frac{d^{n+1} u^{\kappa-a}}{dx^{n+1}} \frac{d^\kappa y}{du^\kappa} \cdot \cdot \cdot \quad (5) \end{aligned}$$

We must now show that

$$\begin{aligned} \frac{1}{n!} \frac{du}{dx} \frac{d^{n+1}y}{du^{n+1}} \left[\sum_{a=0}^{n-1} (-1)^a \binom{n}{a} u^a \frac{d^n u^{n-a}}{dx^n} \right] \\ = \frac{1}{n+1} \frac{d^{n+1}}{dx^{n+1}} \left[\sum_{a=0}^n (-1)^a \binom{n+1}{a} u^a \frac{d^{n+1} u^{n+1-a}}{dx^{n+1}} \right]. \end{aligned}$$

i. e.

$$(n+1) \sum_{a=0}^{n+1} (-1)^a \binom{n}{a} u^a \frac{d^n u^{n-a}}{dx^n} \frac{du}{dx} = \sum_{a=0}^n (-1)^a \binom{n+1}{a} u^a \frac{d^{n+1} u^{n+1-a}}{dx^{n+1}},$$

or if we let $n-\alpha=\beta$, we must show that

$$(n+1) \sum_{\beta=1}^n (-1)^\beta \binom{n}{\beta} u^{n-\beta} \frac{du}{dx} \frac{d^n u^\beta}{dx^n} = \sum_{\beta=0}^n (-1)^\beta \binom{n+1}{\beta+1} u^{n-\beta} \frac{d^{n+1} u^{\beta+1}}{dx^{n+1}} \cdot \cdot \cdot \quad (6)$$

Now

$$\frac{d^{n+1} u^{\beta+1}}{dx^{n+1}} = (\beta+1) \frac{d^n}{dx^n} \left(u \frac{du}{dx} \right),$$

and (6) reduces to

$$\sum_{\beta=1}^n (-1)^\beta \binom{n}{\beta} u^{n-\beta} \frac{du}{dx} \frac{d^n u^\beta}{dx^n} = \sum_{\beta=0}^n (-1)^\beta \binom{n}{\beta} u^{n-\beta} \frac{d^n \left(u^\beta \frac{du}{dx} \right)}{dx^n}.$$

By Leibnitz's theorem for the derivative of a product we have

$$\sum_{\beta=0}^n (-1)^{\beta} \binom{n}{\beta} u^{n-\beta} \sum_{\gamma=1}^n \binom{n}{\gamma} \frac{d^{n-\gamma}}{dx^{n-\gamma}} u^{\beta} \frac{d^{\gamma+1} u}{dx^{\gamma+1}} = 0. \quad (7)$$

In order to prove (7), it is only necessary to show that it holds for $u=x^p$, when it becomes

$$x^{pn-n+p-1} \left[\sum_{\beta=0}^n \sum_{\gamma=1}^n (-1)^{\beta} \frac{(p\beta)!}{\beta! \gamma! (n-\beta)! (n-\gamma)! (p\beta-n+\gamma)! (p-\gamma+1)!} \right] = 0,$$

which is evidently independent of x . But when x is a constant, n is also a constant, and therefore (7) holds for all values of u . So that:

$$(n+1) \sum_{\alpha=0}^{n-1} (-1)^{\alpha} \binom{n}{\alpha} u^{\alpha} \frac{du}{dx} \frac{d^n u^{n-\alpha}}{dx^n} = \sum_{\alpha=0}^n (-1)^{\alpha} \binom{n+1}{\alpha} u^{\alpha} \frac{d^{n+1}}{dx^{n+1}} u^{n+1-\alpha},$$

and therefore

$$\frac{d^{n+1} y}{dx^{n+1}} = \sum_{\kappa=1}^{n+1} \frac{1}{\kappa!} \sum_{\alpha=0}^{\kappa-1} (-1)^{\alpha} \binom{\kappa}{\alpha} u^{\alpha} \frac{d^{n+1} u^{\kappa-\alpha}}{dx^{n+1}} \frac{d^{\kappa} y}{du^{\kappa}}.$$

The following examples will illustrate the theorem.

To calculate

$$(i.) \quad \frac{d^{\kappa}}{dx^{\kappa}} \frac{x^m}{(1-x^n)^p}.$$

By Leibnitz's theorem for the derivative of a product of two functions,

$$\begin{aligned} \frac{d^{\kappa}}{dx^{\kappa}} \frac{x^m}{(1-x^n)^p} &= \sum_{\alpha=0}^{\kappa} \binom{\kappa}{\alpha} \frac{d^{\kappa-\alpha}}{dx^{\kappa-\alpha}} x^m \frac{d^{\alpha}}{dx^{\alpha}} (1-x^n)^{-p} \\ &= \kappa! \sum_{\alpha=0}^{\kappa} \binom{m}{\kappa-\alpha} \frac{x^{m-\kappa+\alpha}}{\alpha!} \frac{d^{\alpha}}{dx^{\alpha}} (1-x^n)^{-p} \\ &= \kappa! \left[\binom{m}{\kappa} \frac{x^{m-\kappa}}{(1-x^n)^p} + \sum_{\alpha=1}^{\kappa} \binom{m}{\kappa-\alpha} \frac{x^{m-\kappa+\alpha}}{\alpha!} \frac{d^{\alpha}}{dx^{\alpha}} (1-x^n)^{-p} \right]. \end{aligned}$$

Let $x^n=u$, and $(1-u)^{-p}=y$, then

$$\begin{aligned} \frac{d^{\alpha} y}{dx^{\alpha}} &= \sum_{\beta=1}^{\alpha} \frac{1}{\beta!} \sum_{\gamma=0}^{\beta-1} (-1)^{\gamma} \binom{\beta}{\gamma} x^{n\gamma} \frac{d^{\alpha}}{dx^{\alpha}} x^{n(\beta-\gamma)} \frac{p(p+1)(p+2) \dots (p+\beta-1)}{(1-x^n)^{p+\beta}} \\ &= \sum_{\beta=1}^{\alpha} \frac{1}{\beta!} \sum_{\gamma=0}^{\beta-1} (-1)^{\gamma} \binom{\beta}{\gamma} \frac{(n\beta-\gamma)!}{(n\beta-\gamma-\alpha)!} \frac{(p+\beta-1)!}{(p-1)!} \frac{x^{n\beta-\alpha}}{(1-x^n)^{p+\beta}}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d^\kappa}{dx^\kappa} \frac{x^m}{(1-x^n)^p} &= \kappa! \left[\binom{m}{\kappa} \frac{x^{m-\kappa}}{(1-x^n)^p} + \sum_{a=1}^{\kappa} \binom{m}{\kappa-a} x^{m-\kappa+a} \right. \\ &\quad \times \sum_{\beta=1}^a \sum_{\gamma=0}^{\beta-1} (-1)^\gamma \binom{\beta}{\gamma} \binom{n\beta-\gamma}{\alpha} \binom{p+\beta-1}{\beta} \frac{x^{n\beta-\alpha}}{(1-x^n)^{p+\beta}} \Big] \\ &= \kappa! \frac{x^{m-\kappa}}{(1-x^n)^p} \left[\binom{m}{\kappa} + \sum_{a=1}^{\kappa} \binom{m}{\kappa-a} \right. \\ &\quad \times \sum_{\beta=1}^a \binom{p+\beta-1}{\beta} \frac{x^{m\beta}}{(1-x^n)^\beta} \sum_{\gamma=0}^{\beta-1} (-1)^\gamma \binom{\beta}{\gamma} \binom{n\beta-\gamma}{\alpha} \Big]. \end{aligned}$$

(ii.) $\frac{d^n}{dx^n} \frac{1}{(1-x^3+x^7)^p}$. Let $x^3-x^7=u$ and $(1-u)^{-p}=y$,
then

$$\frac{d^n y}{dx^n} = \sum_{\kappa=1}^n \frac{1}{\kappa!} \sum_{a=0}^{\kappa-1} (-1)^a \binom{\kappa}{a} u^a \frac{d^n}{dx^n} u^{\kappa-a} \frac{d^\kappa y}{du^\kappa}.$$

Now

$$\frac{d^\kappa y}{du^\kappa} = \frac{d^\kappa}{du^\kappa} (1-u)^{-p} = \kappa! \binom{p+\kappa-1}{\kappa} (1-u)^{-(p+\kappa)},$$

and

$$\begin{aligned} \frac{d^n}{dx^n} u^{\kappa-a} &= \frac{d^n}{dx^n} (x^3-x^7)^{\kappa-a} = \frac{d^n}{dx^n} [x^3(1-x^4)]^{\kappa-a} \\ &= \frac{d^n}{dx^n} x^{3(\kappa-a)} \sum_{\beta=0}^{\kappa-a} (-1)^\beta \binom{\kappa-a}{\beta} x^{4\beta} \\ &= n! \sum_{\beta=0}^{\kappa-a} (-1)^\beta \binom{\kappa-a}{\beta} \binom{3\kappa-3a+4\beta}{n} x^{3\kappa-3a+4\beta-n}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d^n y}{dx^n} &= n! \sum_{\kappa=1}^n \frac{1}{\kappa!} \sum_{a=0}^{\kappa-1} (-1)^a \binom{\kappa}{a} (x^3-x^7)^a \\ &\quad \times \sum_{\beta=0}^{\kappa-a} (-1)^\beta \binom{\kappa-a}{\beta} \binom{3\kappa-3a+4\beta}{n} x^{3\kappa-3a+4\beta-n} \kappa! \binom{p+\kappa-1}{\kappa} (1-x^3+x^7)^{p-\kappa} \\ &= \frac{n!}{x^n(1-x^3+x^7)^p} \sum_{\kappa=1}^n \sum_{a=0}^{\kappa-1} \binom{\kappa}{a} \sum_{\beta=0}^{\kappa-a} (-1)^{a+\beta} \binom{\kappa-a}{\beta} \\ &\quad \times \binom{3\kappa-3a+4\beta}{n} \binom{p+\kappa-1}{\kappa} \frac{(x^3-x^7)^{a\beta} x^{3\kappa-3a+4\beta}}{(1-x^3+x^7)^\kappa} \\ &= \frac{n!}{x^n(1-x^3+x^7)^p} \sum_{\kappa=1}^n \sum_{a=0}^{\kappa-1} \sum_{\beta=0}^{\kappa-a} (-1)^{a+\beta} \binom{\kappa}{a} \binom{\kappa-a}{\beta} \\ &\quad \times \binom{3\kappa-3a+4\beta}{n} \binom{p+\kappa-1}{\kappa} \frac{(1-x^4)^a x^{3\kappa+4\beta}}{(1-x^3+x^7)^\kappa}. \end{aligned}$$

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