If we put

$$\Psi_1 = \mu' \psi_1 + R' \frac{d\psi_1}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

we obtain the values of μ' and R' by writing in (23) and (24) throughout $c_{11} - p^2 a_{11}$, $c_{22} - p^2 a_{22}$, $c_{12} - p^2 a_{12}$, . . . for c_{11} , c_{22} , c_{12} , . . . respectively.

A simpler case, which may be worth special mention, arises when all the coefficients b_{12} , b_{13} , . . . vanish. We have then

$$\mu' = c_{11} - p^2 a_{11} - \Sigma \frac{(c_{12} - p^2 a_{12})^2 (c_{22} - p^2 a_{22})}{(c_{22} - p^2 a_{22})^2 + p^2 b_{22}^2}, \quad . \quad . \quad (27)$$

$$R' = b_{11} + \Sigma \frac{(c_{12} - p^2 a_{12})^2 b_{22}}{(c_{22} - p^2 a_{22})^2 + p^2 b_{22}^2}. \quad . \quad . \quad . \quad . \quad (28)$$

This case, when there are two degrees of freedom, is considered in my book on the 'Theory of Sound,' § 117.

If all the frictional coefficients b_{22} , b_{33} , . . . disappear, we have

$$R' = b_{11}, \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and

$$\mu' = c_{11} - p^2 a_{11} - \Sigma \frac{(c_{12} - p^2 a_{12})^2}{c_{22} - p^2 a_{22}}. \quad . \quad . \quad . \quad . \quad (30)$$

Whenever, during its increase, p approaches and passes through one of the values proper to the free vibrations of the system supposed to be vibrating under the condition that ψ_1 is constrained by a suitable force Ψ_1 to remain zero, μ' rises to $-\infty$ and passes through to $+\infty$.

LII. *On the Self-induction and Resistance of Straight Conductors.* By Lord RAYLEIGH, *Sec. R.S., D.C.L.**

IN connection with the experimental results of Professor Hughes†, I have recently been led to examine more minutely the chapter in Maxwell's 'Electricity and Magnetism' (vol. ii. ch. xiii.) in which the author calculates the self-induction of cylindrical conductors of finite section. The problems being virtually in two dimensions, the results give the ratio $L : l$, where L is the coefficient of self-induction, and l the length considered. And since both these quantities are linear, the ratio is purely numerical. In some details the formulæ, as given by Maxwell, require correction, and in some directions the method used by him may usefully be

* Communicated by the Author.

† Inaugural Address to the Society of Telegraph Engineers, January 1886.

pushed further. The present paper may thus be regarded partly as a review, and partly as a development of Maxwell's chapter.

The problems divide themselves into two classes. In the first class the distribution of the currents is supposed to be the same as it would be if determined solely by resistance, undisturbed by induction; in particular the density of current in a cylindrical conductor is assumed to be uniform over the section. The self-induction calculated on this basis can be applied to alternating currents, only under the restriction that the period of the alternation be not too small in relation to the other circumstances of the case. If this condition be not satisfied, the investigation must be modified so as to include a determination of the distribution of current. A problem of this class considered by Maxwell (§ 689) relates to the "Electromotive Force required to produce a Current of Varying Intensity along a Cylindrical Conductor."* In connection therewith another problem of the same class will here be treated, in which the mathematical conditions are simpler, and the results more readily apprehended.

In § 685 Maxwell takes the problem of two cylindrical conductors, the first of which conveys the outgoing and the second the (numerically equal) return current. The external radii are a_1, a_1' ; the internal radii a_2, a_2' ; b the distance between the centres. A possible difference in the magnetic quality is contemplated, the permeabilities for the material composing the cylinders being denoted by μ, μ' , and that of the intervening space by μ_0 .

The first correction I have to note relates merely to a slip of the pen. The result (22) should run

$$\frac{L}{l} = 2\mu_0 \log \frac{b^2}{a_1 a_1'} + \frac{1}{2} \mu \left[\frac{a_1^2 - 3a_2^2}{a_1^2 - a_2^2} + \frac{4a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} \right] \\ + \frac{1}{2} \mu' \left[\frac{a_1'^2 - 3a_2'^2}{a_1'^2 - a_2'^2} + \frac{4a_2'^4}{(a_1'^2 - a_2'^2)^2} \log \frac{a_1'}{a_2'} \right].$$

As printed in Maxwell's book the square brackets are omitted. This error does not affect the following formula (23), in which the cylinders are supposed to be solid. By putting a_2, a_2' equal to zero, we get

$$\frac{L}{l} = 2\mu_0 \log \frac{b^2}{a_1 a_1'} + \frac{1}{2} (\mu + \mu').$$

* That some of the results arrived at experimentally by Hughes might be attributed to unequal distribution of current in the conductors was pointed out by Prof. Forbes in the course of a discussion which followed the delivery of Prof. Hughes's address.

It must, however, be remarked that in the derivation of (22) Maxwell appears to have overlooked the effect of the matter composing one conductor in disturbing the lines of induction due to the current in the other. On this account the formula is correct only when the permeabilities μ are all equal, and the results cannot be applied to iron wires without reservation. It would seem, however, that the error is of small importance when the wires are distant. The application to wires in contact, contemplated in § 688, will hold good only for the non-magnetic metals.

If we write c^2 for $(a_1^2 - a_2^2)$, so that c is the radius of the solid cylinder of equal sectional area, we have in (22),

$$\begin{aligned} \frac{a_1^2 - 3a_2^2}{a_1^2 - a_2^2} + \frac{4a_2^4}{(a_1^2 - a_2^2)^2} \log \frac{a_1}{a_2} \\ = \frac{3c^2 - 2a_1^2}{c^2} - \frac{2(a_1^2 - c^2)^2}{c^4} \log \left(1 - \frac{c^2}{a_1^2} \right) \\ = \frac{2}{3} \frac{c^2}{a_1^2} + \text{terms in } \frac{c^4}{a_1^4}. \end{aligned}$$

Hence, when the thickness of the cylinders is relatively small,

$$\frac{L}{l} = 2\mu_0 \log \frac{b^2}{a_1 a_1'} + \frac{\mu}{3} \frac{c^2}{a_1^2} + \frac{\mu'}{3} \frac{c'^2}{a_1'^2}.$$

If b, c, c' be given, the self-induction diminishes with increase of a_1, a_1' , especially when μ, μ' are much greater than μ_0 .

When μ is constant throughout, the "geometric mean distance" (§§ 691, 692) may conveniently be introduced. If A_1, A_2 be the areas occupied by the outgoing and return currents, we have

$$\frac{L}{l} = 4\mu_0 \left[\log R_{A_1 A_2} - \frac{1}{2} \log R_{A_1} - \frac{1}{2} \log R_{A_2} \right],$$

where R_{A_1}, R_{A_2} are to be understood as in (5), (6), (9) § (692).

For two circular areas,

$\log R_{A_1 A_2} = \log b, \quad \log R_{A_1} = \log a_1 - \frac{1}{4}, \quad \log R_{A_2} = \log a_1' - \frac{1}{4},$
if a_1, a_1' be the radii and b the distance between the centres; so that

$$\frac{L}{l} = 2\mu_0 \log \frac{b^2}{a_1 a_1'} + \mu_0,$$

as before.

In § 692 the value of R_A is given for rectangles and circular

rings. For an ellipse of semi-axes a and b Prof. J. J. Thomson gives*

$$\log R = \log \frac{a+b}{2} - \frac{1}{4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which of course the case of the circle is included.

It is evident that for a given area R is least when the figure is circular. In that case we have

$$R = .4393 \sqrt{A}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the case of the square,

$$R = .44705 \sqrt{A}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For the ellipse, Prof. Thomson's result leads to

$$R = .4393 \sqrt{A} (1 + \frac{1}{32} e^4 + \dots), \quad . \quad . \quad . \quad (4)$$

showing the small effect of moderate eccentricity when the area is given.

As examples of very elongated forms we may take the ellipse or the rectangle. In the latter case the value approximates to that applicable to a line given by Maxwell (5) § 692. If the length be a ,

$$R = ae^{-\frac{3}{2}} = \frac{Ae^{-\frac{3}{2}}}{b}, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

increasing without limit for a given area as b decreases.

It has been pointed out that Maxwell's result (22) § 685 is not rigorous, unless μ be constant. In order to put a case in which the lines of induction follow a simple law in spite of the presence of iron, we may suppose that the conductors are co-axial cylindrical shells. The outgoing current of total strength C travels in the interior cylinder of radii a_2, a_1 ; the return current of strength $-C$ in the outer cylinder of radii a_2', a_1' .

In Maxwell's notation we have the equations

$$\frac{dH}{dr} = -\mu\beta, \quad \beta r = 4\pi \int_0^r w r dr,$$

so that

$$\begin{aligned} \iint H w dx dy &= 2\pi \int_0^\infty H w r dr = \frac{1}{2} \int H d(\beta r) \\ &= \frac{1}{2} [H \beta r] + \frac{1}{2} \int_0^\infty \mu \beta^2 r dr. \end{aligned}$$

Now βr vanishes both at zero and infinity, so that we may

* In a private letter.

take

$$\iint H w \, dx \, dy = \frac{1}{2} \int_0^{\infty} \mu \beta^2 r \, dr, \quad \quad (6)$$

in which β represents the magnetic force, everywhere perpendicular to r .

In the integration from 0 to ∞ there are five regions to be considered. In the first, from 0 to a_2 , there is no magnetic force. In the second, from a_2 to a_1 , the magnetic force depends upon the total current travelling through the strata which are internal with respect to the point in question. In terms of the total current C we have

$$\beta = \frac{2C}{a_1^2 - a_2^2} \left(r - \frac{a_2^2}{r} \right). \quad \quad (7)$$

The permeability is here supposed to be μ .

In the third region, between the cylinders, the permeability is μ_0 , and the magnetic force is given by

$$\beta = \frac{2C}{r}. \quad \quad (8)$$

Within the second cylinder the permeability is μ' , and

$$\beta = \frac{2C}{r} - \frac{2C}{a_1'^2 - a_2'^2} \left(r - \frac{a_2'^2}{r} \right). \quad \quad (9)$$

In the fifth region, from a_1' to ∞ , $\beta = 0$.

Effecting the integrations, as indicated in (6), we obtain the value of

$$\iint H w \, dx \, dy,$$

which gives $2T/l$; and (if L be the coefficient of self-induction) $T = \frac{1}{2} LC^2$. The result is

$$\begin{aligned} \frac{L}{l} = & 2\mu_0 \log \frac{a_2'}{a_1} \\ & + \frac{2\mu}{a_1^2 - a_2^2} \left\{ \frac{a_1^2 - 3a_2^2}{4} + \frac{a_2^4}{a_1^2 - a_2^2} \log \frac{a_1}{a_2} \right\} \\ & + \frac{2\mu'}{a_1'^2 - a_2'^2} \left\{ \frac{a_2'^2 - 3a_1'^2}{4} + \frac{a_1'^4}{a_1'^2 - a_2'^2} \log \frac{a_1'}{a_2'} \right\}. \quad . \quad (10) \end{aligned}$$

Perhaps the most interesting application of the general result is to trace the diminution of self-induction as the two currents are brought into closer and closer proximity. Let us suppose that the intervening space is reduced without limit, so that $a_2' = a_1$. Suppose further that $\mu' = \mu$, and that both conductors have the same sectional area πc^2 , so that

$$a_1^2 - a_2^2 = a_1'^2 - a_2'^2 = c^2.$$

Under these circumstances we have

$$\frac{L}{l} = \frac{2\mu}{c^4} \left\{ -r^2 c^2 + (r^2 - c^2)^2 \log \frac{r}{\sqrt{(r^2 - c^2)}} + (r^2 + c^2)^2 \log \frac{\sqrt{(r^2 + c^2)}}{r} \right\}, \quad (11)$$

in which r is written for the radius of the common surface.

If c is small in comparison with r , (11) becomes

$$\frac{L}{l} = \frac{2\mu c^2}{3r^2} \left\{ 1 + \frac{1}{10} \frac{c^2}{r^2} + \dots \right\}, \quad (12)$$

showing that when the sectional areas are given, the self-induction diminishes without limit as the radius (r) increases.

If b denote the thickness of the walls, we have ultimately

$$c^2 = 2br,$$

and

$$\frac{L}{l} = \frac{4\mu b}{3r}. \quad (13)$$

If the material composing the conductors be soft iron, the self-induction will be several hundred times greater than in the case of copper or other non-magnetic metal.

I now pass on to § 689, in which Maxwell solves a problem of the second class, relative to the self-induction of a cylindrical conductor, regard being had to the disturbance from uniformity in the distribution of the current over the section, due to induction. I will introduce the permeability μ , which in this question Maxwell treats as unity. His equations (14), (15), thus become

$$\begin{aligned} \mu C = & - \left(\alpha \mu \frac{dT}{dt} + \frac{2\alpha^2 \mu^2}{1^2 \cdot 2^2} \frac{d^2 T}{dt^2} + \dots \right. \\ & \left. + \frac{n\alpha^n \mu^n}{1^2 \cdot 2^2 \dots n^2} \frac{d^n T}{dt^n} + \dots \right), \\ AC - S = & T + \alpha \mu \frac{dT}{dt} + \frac{\alpha^2 \mu^2}{1^2 \cdot 2^2} \frac{d^2 T}{dt^2} \\ & + \dots + \frac{\alpha^n \mu^n}{1^2 \cdot 2^2 \dots n^2} \frac{d^n T}{dt^n} + \dots, \end{aligned}$$

where α , equal to l/R , represents the conductivity (for steady currents) of unit of length of the wire.

If $\phi(x)$ denotes the function

$$1 + x + \frac{x^2}{1^2 \cdot 2^2} + \dots + \frac{x^n}{1^2 \cdot 2^2 \dots n^2} + \dots, \quad (14)$$

these equations may be written

$$\frac{dS}{dt} = AC - \phi \left(\alpha \mu \frac{d}{dt} \right) \cdot \frac{dT}{dt},$$

$$C = -\alpha \phi' \left(\alpha \mu \frac{d}{dt} \right) \cdot \frac{dT}{dt}.$$

Moreover, if E denote "the electromotive force due to other causes than the induction of the current upon itself,"

$$\frac{E}{l} = \frac{dS}{dt}.$$

To apply this to periodic currents following the harmonic law, where all the functions are proportional to e^{ipt} , we may replace d/dt by ip . Hence, eliminating $\frac{dT}{dt}$, we get

$$\frac{\alpha E}{lC} = \frac{E}{RC} = ip\alpha \cdot A + \frac{\phi(ip\alpha\mu)}{\phi'(ip\alpha\mu)}. \quad \dots \quad (15)$$

To interpret (15), we have to separate the real and imaginary parts of ϕ/ϕ' . Now if x be small, we get in ascending powers of x by ordinary division,

$$\frac{\phi(x)}{\phi'(x)} = 1 + \frac{1}{2}x - \frac{1}{12}x^2 + \frac{1}{48}x^3 - \frac{1}{180}x^4 + \frac{13}{8640}x^5 - \dots; \quad (16)$$

so that

$$\frac{\phi(ip\alpha\mu)}{\phi'(ip\alpha\mu)} = 1 + \frac{1}{12}p^2\alpha^2\mu^2 - \frac{1}{180}p^4\alpha^4\mu^4 + \dots$$

$$+ i \left\{ \frac{1}{2}p\alpha\mu - \frac{1}{48}p^3\alpha^3\mu^3 + \frac{13}{8640}p^5\alpha^5\mu^5 - \dots \right\}. \quad (17)$$

Thus, if we write

$$E = R'C + ipL'C, \quad \dots \quad (18)$$

we find

$$R' = R \left\{ 1 + \frac{1}{12}p^2\alpha^2\mu^2 - \frac{1}{180}p^4\alpha^4\mu^4 + \dots \right\}$$

$$= R \left\{ 1 + \frac{1}{12} \frac{p^2l^2\mu^2}{R^2} - \frac{1}{180} \frac{p^4l^4\mu^4}{R^4} + \dots \right\}, \quad \dots \quad (19)$$

$$L' = l \left\{ A + \mu \left(\frac{1}{2} - \frac{1}{48}p^2\alpha^2\mu^2 + \frac{13}{8640}p^4\alpha^4\mu^4 - \dots \right) \right\}$$

$$= l \left\{ A + \mu \left(\frac{1}{2} - \frac{1}{48} \frac{p^2l^2\mu^2}{R^2} + \frac{13}{8640} \frac{p^4l^4\mu^4}{R^4} - \dots \right) \right\}, \quad (20)^*$$

* For the case $\mu=1$, (19) and (20) follow readily from Maxwell's equation (18) § 690.

in which R' represents the effective resistance and L' the effective self-induction.

If the rate of alternation be very slow, so that p is small, these equations give (as was to be expected), $R' = R$, and

$$L' = l\{A + \frac{1}{2}\mu\}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

representing the self-induction for steady or slowly alternating currents. If we include the next terms, we see that as the frequency increases, the self-induction begins to diminish. At the same time the resistance begins to increase.

These results are merely very special cases of a general law*, from which we may learn that as the frequency of alternation gradually increases from zero to infinity, there is a steady rise of resistance and accompanying fall of self-induction. The application of the general idea to the present case is very simple. At slow rates of alternation the distribution of current, being such as to make the resistance a minimum, is uniform over the section; and this distribution, since it involves magnetization of the outer parts of the cylinder, leads to considerable self-induction, especially in iron. On the other hand, when the rate of alternation is very rapid, the endeavour is to make the self-induction a minimum irrespective of resistance. This object is attained by concentration of the current into the outer layers. The magnetization of the conductor is thus more and more avoided, but of course at the expense of increased resistance. We may gather from the general argument, what (19) and (20) in their actual forms do not tell us, that as p increases without limit, R' also becomes infinite, while the part of L' depending upon the magnetization of the conductor tends to zero.

The increase of resistance proper (not merely of the "throttling" due to the combined effect of resistance and self-induction) in iron wires of moderate diameter subjected to varying currents, is one of the most striking of Prof. Hughes's results. So far as I am aware, neither Maxwell nor any other theorist had anticipated that the alteration of resistance would be important under such circumstances†.

In order to see under what conditions the alteration of resistance (and of self-induction) would become sensible, we have to examine the value of $\frac{1}{12} \frac{p^2 l^2 \mu^2}{R^2}$. We will take first

* See preceding article.

† In the paper referred to I have quoted Maxwell's calculation of increased resistance and diminished self-induction due to the operation of currents in a *secondary* circuit.

the case of an iron wire of .4 centim. diameter. The specific resistance of iron in C.G.S. measure is about 10^4 ; so that

$$\frac{R}{l} = \frac{10^4}{\pi \times .04}.$$

Thus, if $p=2\pi/\tau$, τ being the complete period,

$$\frac{1}{12} \frac{p^2 l^2 \mu^2}{R^2} = \frac{5.2 \mu^2}{10^{10} \tau^2} \cdot \cdot \cdot \cdot \cdot \quad (22)$$

The value of μ is more difficult to assign. It must be remembered that it is only in a very rough sense that μ can be treated as a constant at all. For small degrees of magnetization and for solid iron we may perhaps take $\mu=300$. In the case of the hissing sounds of a microphone-clock, working by a scraping contact, τ must be less than $\frac{1}{10000}$ second. Taking it, however, at this value, we get

$$\frac{1}{12} \frac{p^2 l^2 \mu^2}{R^2} = 47 \text{ nearly,}$$

which shows that under these circumstances the resistance and self-induction are entirely different from what they would be for slow rates of alternation.

We will now consider the case of copper, where $\mu=1$. The specific resistance may be taken to be 1640. If a be the radius in centimetres,

$$\frac{1}{12} \frac{p^2 l^2 \mu^2}{R^2} = \frac{\pi^4}{3} \frac{a^4}{1640^2 \tau^2} = \frac{1.2 a^4}{10^6 \tau^2} \cdot \cdot \cdot \cdot \cdot \quad (23)$$

From an alternate-current machine we may have currents of period .01 second (100 positive and 100 negative pulses per second). In such a case our fraction becomes .12 a^4 . This shows that for diameters of 1 centim. and over, the augmentation of resistance in the mains of an alternate-current system will rapidly become of commercial importance. A remedy may be found in the use of a more elongated section, or in subdivision of the main into a number of detached parts.

In physical experiments, such as those in which absolute resistance is determined by the method of the revolving coil*, (23) may be neglected, as we may see by supposing $a=\frac{1}{10}$, $\tau=\frac{1}{4}$.

The ultimate form of (15), when p is very great, may be

* See for example Phil. Trans. 1882. Experiments to Determine the Value of the B.A. Unit of Resistance in Absolute Measure.

We might commence with the investigation of steady currents; but it will be sufficient to regard them as alternating (e^{ipt}). The results applicable to steady currents can always be deduced by simply putting p equal to zero.

Assuming, then, that the component currents u, v vanish, as well as the components of magnetic force γ, α , we have, in Maxwell's notation, the equations

$$\rho w = -\frac{d\Psi}{dz} - \frac{dH}{dt} = -\frac{d\Psi}{dz} - ipH, \quad \dots \quad (28)$$

$$\frac{d\beta}{dx} = 4\pi w, \quad \mu\beta = -\frac{dH}{dx}; \quad \dots \quad (29)$$

so that

$$w = \frac{1}{4\pi} \frac{d\beta}{dx} = -\frac{1}{4\pi\mu} \frac{d^2H}{dx^2}, \quad \dots \quad (30)$$

and

$$\frac{d^2H}{dx^2} = \frac{4\pi\mu}{\rho} \frac{d\Psi}{dz} + i \cdot \frac{4\pi\mu p}{\rho} \cdot H \dots \quad (31)$$

We will now apply (31) to that conducting strip which lies on the positive side of the origin. Since, by hypothesis, $\frac{d\Psi}{dz}$, the rate at which the potential varies, is independent of x , the solution for regular periodic motion may be written

$$H = \frac{i}{p} \frac{d\Psi}{dz} + Ae^{m(x-a)} + Be^{-m(x-a)}, \dots \quad (32)$$

$$\mu\beta = -mAe^{m(x-a)} + mBe^{-m(x-a)}, \dots \quad (33)$$

in which A and B are constants, so far arbitrary, and

$$m^2 = \frac{4\pi\mu pi}{\rho} \dots \quad (34)$$

One relation between A and B is supplied by the condition that the magnetic force β must vanish at the external surface, where $x=a+b$. Hence

$$Ae^{mb} = Be^{-mb} \dots \quad (35)$$

If C be the total current corresponding to width y , we have

$$\frac{C}{y} = \int_a^{a+b} w \, dx = \frac{1}{4\pi} (\beta_{a+b} - \beta_a) = -\frac{1}{4\pi} \beta_a = \frac{m}{4\pi\mu} (A - B),$$

by (33); so that

$$A - B = \frac{4\pi\mu C}{my} \dots \quad (36)$$

These equations determine A and B.

Another condition is afforded by the consideration that, on account of the symmetry, H must vanish when $x=0$, or $H_0=0$. Within the insulator, whose permeability we take to be unity, $w=0$, so that by (29) $\beta=\beta_a$, and

$$H_a - H_0 = -\int \beta \, dx = -\beta_a a.$$

Hence from (32), by equating the values of H_a ,

$$\frac{4\pi aC}{y} = \frac{i}{p} \frac{d\Psi}{dz} + A + B. \quad \dots \quad (37)$$

Now by (35), (36),

$$A + B = -\frac{4\pi\mu C}{my} \frac{e^{mb} + e^{-mb}}{e^{mb} - e^{-mb}},$$

so that

$$-\frac{d\Psi}{dz} = ipC \frac{4\pi a}{y} + \frac{4\pi\mu ipC}{my} \frac{e^{mb} + e^{-mb}}{e^{mb} - e^{-mb}}. \quad \dots \quad (38)$$

In (38) the first term represents a part of the effective self-induction, and contributes nothing to the effective resistance. This self-induction, per unit length, is simply $4\pi a/y$, and is independent of p . The second term, being neither wholly real nor wholly imaginary, contributes both to self-induction and to resistance. If, separating the real and imaginary parts of the right-hand member of (38), we write

$$-l \frac{d\Psi}{dz} = R'C + L'.ipC; \quad \dots \quad (39)$$

then R' represents the resistance and L' the self-induction of length l of the conductor measured parallel to z .

From (34)

$$m = \sqrt{\left(\frac{2\pi\mu p}{\rho}\right)} \cdot (1+i) = q(1+i), \quad \dots \quad (40)$$

if we write for brevity

$$q = \sqrt{\left(\frac{2\pi\mu p}{\rho}\right)}. \quad \dots \quad (41)$$

The general expressions for R' and L' are somewhat complicated. If the rate of alternation be slow, p , and with it m and q , are small. In this case (38) may be written approximately

$$-\frac{d\Psi}{dz} = ipC \frac{4\pi a}{y} + \frac{\rho C}{by} \left(1 + i \frac{4\pi\mu p b^2}{3\rho}\right);$$

so that

$$R' = \frac{\rho l C}{by}, \quad \dots \quad (42)$$

in accordance with Ohm's law, and

$$L' = \frac{4\pi al}{y} + \frac{4\pi}{3} \frac{\mu bl}{y} \dots \dots \dots (43)$$

If $a=0$, the first term in (43) disappears, and we get a simplified result which should agree with one found previously (13). We may compare them by replacing y with $2\pi r$. The apparent discrepancy that the self-induction by (13) is twice as great as by (43) depends merely upon a slight difference in the way of reckoning. The result in (13) refers to a *double* length l , one part from the outgoing and the other from the return conductor.

At the other extreme, when p is very great, a simple result again applies. In that case $(e^{mb} + e^{-mb}) / (e^{mb} - e^{-mb})$ may be replaced by unity, and (38) becomes

$$-\frac{d\Psi}{dz} = ipC \frac{4\pi a}{y} + (1+i) \frac{C \sqrt{2\pi\mu\rho\rho}}{y}.$$

Thus

$$R' = \frac{l \sqrt{2\pi\mu\rho\rho}}{y}, \dots \dots \dots (44)$$

$$L' = \frac{4\pi al}{y} + \frac{l \sqrt{2\pi\mu\rho}}{y \sqrt{p}} \dots \dots \dots (45)$$

These formulæ show that the resistance increases without limit with p , being proportional to $\sqrt{p}$, and that the self-induction diminishes towards the limit $4\pi al/y$. If a be zero, that is if the insulating layer be infinitely thin, the self-induction diminishes without limit as p increases, being proportional to $p^{-\frac{1}{2}}$. Another important point is that when p is great enough, the values of R' and L' are independent of b , the thickness of the strips. The meaning of this is, of course, that, under such circumstances, the currents concentrate themselves more and more towards the inner parts, in the endeavour to diminish the effective self-induction.

The distribution of current in the extreme case, where it is not limited by the thinness of the strips, is readily expressed. We have in general

$$w = \frac{mC}{y} \frac{e^{m(x-a-b)} + e^{-m(x-a-b)}}{e^{mb} - e^{-mb}}, \dots \dots \dots (46)$$

becoming, when $b = \infty$,

$$w = \frac{mC}{y} e^{-m(x-a)}; \dots \dots \dots (47)$$

or, on introducing the value of m from (40),

$$w = \frac{\sqrt{2} \cdot qC}{y} e^{-q(x-a)} \left\{ \cos \left(\frac{1}{4}\pi - qx + qa \right) + i \sin \left(\frac{1}{4}\pi - qx + qa \right) \right\}. \quad (48)$$

The thickness through which the current is important is found from

$$q(x-a) = 1,$$

or

$$x-a = \sqrt{\left(\frac{\rho}{2\pi\mu p} \right)}, \quad . \quad . \quad . \quad . \quad (49)$$

diminishing as p increases.

It should be remarked in conclusion that when a is very small and l very great, there may be a sensible accumulation of electricity upon the inner surfaces of the strips acting as plates of a condenser. In such a case u will no longer vanish, w will become a function of z , and our results will require modification.

April 3, 1886.

LIII. *Observations on Invisible Heat-Spectra and the Recognition of hitherto Unmeasured Wave-lengths, made at the Allegheny Observatory.* By Prof. S. P. LANGLEY, Allegheny, Pa.*

[Plate IV.]

IT is known to all that the surface-temperature of this planet depends upon the properties of radiant heat and the relation to them of the action of its atmosphere. It has been usual to compare this action to that of the glass cover of a hot-bed; for glass, it is also well known, grows opaque to dark heat, and continuously so as its wave-length increases, thus letting the solar light-heat pass freely through it to the soil, while it is comparatively impervious to the dark heat returned from the latter. But this analogy must not be interpreted too literally. Whether the atmosphere is pervious to the soil's heat we do not here discuss, but it has of late † been shown that the air does not behave otherwise like glass, as it was supposed to do, but, except for the absorption-bands, grows—not more opaque—but more transmissible, to *solar* heat, up to its greatest observed wave-length, and that hence our views of the nature of the yet uncomprehended heat-storing action which maintains organic life on the earth must be modified. The little that the spectroscope tells us about the atmospheres of other

* Communicated by the Author, having been read before the American Association for the Advancement of Science, at Ann Arbor, 1885.

† See Amer. Journ. of Science, March 1883; Professional Papers of U.S. Signal Service, No. 15, Expedition to Mount Whitney.