THE EFFECT OF A UNIFORM MAGNETIC FIELD ON THE MOTION OF ELECTRONS BETWEEN COAXIAL CYLINDERS.

By Albert W. Hull.

Synopsis.

Equations of motion for electrons are developed starting from a cylindrical cathode and moving toward a co-axial cylindrical anode, in a uniform magnetic field parallel to the common axis. The electrons will reach the anode if the ratio of potential difference to magnetic field is greater than a critical value, and will fail to reach it if the ratio is less than this value.

In the case of a *small cathode* in the axis of an anode of radius R at potential V, the critical magnetic field is

$$H = \sqrt{\frac{8m}{e}} \frac{V^{1/2}}{R}.$$

For a *small anode* of radius r_0 at potential V, in the axis of a cathode of radius R_0, the critical field is

$$H = \frac{r_0}{R_0} \sqrt{\frac{8m}{e}} \frac{V^{1/2}}{R_0} + \frac{2mv_0}{eR_0}.$$

In this case the initial velocity v_0 of the electrons cannot be neglected. This equation also applies, with appropriate R_0 and e/m, to *positive ions* produced by electrons from an *internal cathode*.

If the radii of *both cylinders are large* the solution reduces to the familiar one of plane parallel plates.

The *equation of the path* of the electrons is deduced, on the assumption that the space charge distribution is the same as without magnetic field. The path is given by $r = R(\sin \frac{2}{3}\theta)^{3/2}$. This is a close approximation to the true path, as calculated from the space charge distribution recently worked out by Langmuir. Experimental curves showing current at constant potential as a function of magnetic field, for different anode diameters, voltages, filament temperatures, degrees of symmetry, etc., are in agreement with theory within the limit of experimental error.

The *internal cathode* tube offers a means of measuring e/m, and the *internal anode* a means of measuring the distribution of initial velocities.

Introduction.

THE motion of electrons in uniform electric and magnetic fields has been studied by many investigators.[1] A few special cases of non-uniform fields have also been investigated; for example, the magnetic field due to current in a long straight wire,[2] and the electric field between a straight filament and concentric cylinder. In the latter case

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[1] See, for example, J. J. Thomson, Conduction of Electricity through Gases, second Ed., pp. 104–116.

[2] J. J. Thomson, l.c., p. 108; O. W. Richardson, Roy. Soc. Proc., 90, 174, 1914.

the effect of the space charge of the moving electrons has been taken account of.[3]

A case of special interest, which appears not to have been investigated, is that in which the electric field is radial and symmetrical about a straight line, and the magnetic field is uniform and parallel to this line. This applies to an evacuated tube containing a straight filament and a concentric circular cylinder, in the axis of a long straight solenoid. The theoretical conditions can be accurately realized, and the apparatus is easy to construct and operate.
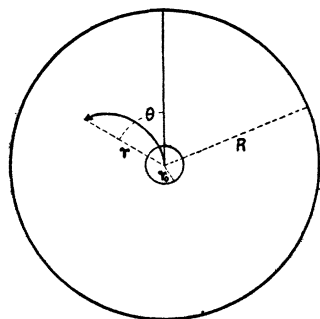

Fig. 1.

Small cathode in axis of cylindrical anode.

Relation between Magnetic Field, Voltage, and "Range" of the Electrons.—The equations of motion of an electron starting from a point at distance r_0 from an axis (Fig. 1), with initial velocity components

$$u_0 = \left(\frac{dr}{dt}\right)_0; \quad v_0 = r_0\left(\frac{d\theta}{dt}\right)_0; \quad w_0 = \left(\frac{dz}{dt}\right)_0;$$

and moving under the influence of a constant magnetic field of intensity H parallel to the axis, and a symmetrical radial electric field of intensity F (function of r only) are:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = F\frac{e}{m} - H\frac{e}{m}r\frac{d\theta}{dt}, \tag{1}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = H\frac{e}{m}\frac{dr}{dt}, \tag{2}$$

$$\frac{d^2z}{dt^2} = 0. \tag{3}$$

Integrating (2) with respect to time gives:

$$\frac{d\theta}{dt} = \frac{r_0^2}{r^2}\left(\frac{v_0}{r_0} - \frac{He}{2m}\right) + \frac{He}{2m}. \tag{4}$$

The maximum radial distance from the axis that an electron can travel may be found either by equating the total work eV done by the electric force to the kinetic energy of tangential motion

$$\frac{1}{2}m\left(r\frac{d\theta}{dt}\right)^2,$$

given by Eq. 4; or by equating the radial velocity dr/dt to zero. The

[3] Langmuir, PHYS. REV., 2, 450, 1913; Schottky, Ann. d. Phys., 44, 1011, 1914.

latter is found from integrating Eq. 1 with respect to r, which gives:

$$\left(\frac{dr}{dt}\right)^2 = 2\frac{e}{m}V_r - H^2\left(\frac{e}{2m}\right)^2 r^2\left(1 - \frac{r_0^2}{r^2}\right)^2 - H\frac{e}{m}r_0v_0\left(1 - \frac{r_0^2}{r^2}\right)$$

$$+ v_0^2\left(1 - \frac{r_0^2}{r^2}\right) + u_0^2, \quad (5)$$

where

$$V_r = \int_{r_0}^{r} F dr$$

is the potential difference between the starting point r_0 and the point r.

Putting $dr/dt = 0$ gives the maximum distance r_m from the axis that the electron can reach, in terms of the potential V_r at this distance and the magnetic field H:

$$V_r = H^2\frac{e}{8m}r_m^2\left(1 - \frac{r_0^2}{r_m^2}\right)^2 + \left(\frac{Hr_0v_0}{2} - \frac{v_0^2}{2e/m}\right)\left(1 - \frac{r_0^2}{r_m^2}\right) - \frac{u_0^2}{2e/m}. \quad (6)$$

If a conducting cylinder of radius r_m is inserted in the position of the surface r_m, all the electrons will reach it if its potential is greater than V_r, and none will reach it if its potential is less than V_r. *Hence if one observes the current to the cylinder as a function of V with constant H, or as a function of H with constant V, this current should fall abruptly to zero at the critical values of V and H.* This is found to be true experimentally (Figs. 7–10).

The above equations refer to the motion of both electrons and ions, in either direction, starting at any point with any initial velocity. They become much simpler under practical conditions. Four special cases will be considered.

CASE 1. INTERNAL CATHODE.—*Electrons start from a straight filament of radius r_0 and travel to a concentric cylindrical anode of radius R, large compared with the filament (Fig. 1).*

Equation 6 then becomes

$$V = H^2\frac{e}{8m}R^2 + H\frac{r_0v_0}{2} - \frac{u_0^2 + v_0^2}{2e/m}. \quad (7)$$

The last term on the right-hand side represents the initial energy of emission of the electrons, expressed in the same units as V.

If the initial energies are those of thermal emission, between zero and 2 volts, then the last two terms are negligible for the great majority of electrons, whose initial energies are less than 1/10 volt; and they are negligible even for electrons with the highest initial energies when V and H are large.

For example, in the case of a 4 mil filament in the axis of a 1 inch cylinder,

$$r_0 = .005 \text{ cm.}; \qquad R = 1.25 \text{ cm.};$$

$$\sqrt{u_0{}^2 + v_0{}^2} = 2 \times 10^7 \text{ (0.1 volt initial energy).}$$

If $H = 10$,

$$\left.\begin{array}{l} V = 3.5 \times 10^8 \pm 5 \times 10^5 - 10^7 \text{ E.M.U.} \\ \quad = 3.5 \pm .005 - 0.1 \text{ volts} \\ \text{If } H = 100 \\ V = 350 + .05 - 0.1 \text{ volts,} \end{array}\right\} \text{ initial energy } \frac{1}{10} \text{ volt,}$$

while for the very fastest electrons, with initial energies of 2 volts $(\sqrt{u_0{}^2 + v_0{}^2} = 8 \times 10^7)$.

$$\left.\begin{array}{llr} \text{If } H = & 10, V = & 3.5 \pm .02 - 2 \text{ volts} \\ \text{If } H = & 100, V = & 350 \pm 0.2 - 2 \quad `` \\ \text{If } H = & 1000, V = & 35000 \pm 2 - 2 \quad `` \end{array}\right\} \text{ initial energy 2 volts.}$$

It is seen that the term involving the first power of H is negligible under all conditions, so that Eq. 7 becomes

$$V = H^2 \frac{e}{8m} R^2 - \frac{1}{2} \frac{m}{e} (u_0{}^2 + v_0{}^2). \tag{7'}$$

The only correction due to initial velocities is the addition of the initial energy of emission, $\frac{1}{2}m(u_0{}^2 + v_0{}^2)$ to the work eV done by the electric field, as in all thermionic applications. This correction is negligible at reasonable voltages. Hence in all practical cases, for electrons moving from a straight filament to a concentric cylinder,

$$V = H^2 \frac{e}{8m} R^2 \tag{8}$$

or

$$H = \sqrt{\frac{8m}{e}} \frac{V^{1/2}}{R}. \tag{9}$$

V = potential difference between filament and cylinder.

R = radius of cylindrical anode.

H = magnetic field just sufficient to prevent electrons reaching anode.

The value of H corresponding to the middle of the steep portion of the $I - H$ curve (Figs. 7–10) can be accurately measured, so that Eq. 9 may be used for the experimental determination of e/m.

If e/m is taken as 1.77×10^7 E.M.U. and V is expressed in volts, H and R in Gauss and cm. resp., Eqs. (8) and (9) become

$$V = .0221 \, H^2 R^2, \tag{8'}$$

$$H = 6.72 \frac{V^{1/2}}{R}. \tag{9'}$$

CASE 2. *Internal Anode.—Electrons start from an outer cylinder of radius R_0 and move toward a concentric inner cylinder or wire of radius r_0, small compared with R_0 (Fig. 2).*

Eq. 6 becomes in this case:

$$V = \beta^2 \frac{e}{8m}\left(HR_0 - 2\frac{m}{e}v_0\right)^2 \quad (10)$$

or

$$H = \frac{1}{\beta R_0}\sqrt{\frac{8\,m}{e}}\,V^{1/2} + \frac{2\,mv_0}{eR_0}. \quad (11)$$

V = potential difference between outer and inner cylinders, reckoned positive when the inner cylinder is positive.

R_0 = radius of outer cylinder.

H = magnetic field just sufficient to prevent electrons reaching inner cylinder.

β = ratio of radii of outer and inner cylinders

v_0 = tangential initial velocity of electrons.

Fig. 2.

Small anode in axis of cylindrical cathode.

For electrons with very small tangential initial velocities this becomes:

$$V = \beta^2 \frac{e}{8m} H^2 R_0^2. \quad (12)$$

This is the same as Eq. 8 for electrons starting from the filament, except for the factor β^2. The voltage against which the electrons can travel in the same magnetic field is β^2 times greater than in case 1; or, conversely, with the same potential difference between cathode and anode, the magnetic field required to prevent electrons reaching the anode is only $1/\beta$ as great with internal anode as with internal cathode. This is found to be true experimentally (see Figs. 15–20).

The effect of initial velocities is much greater in this case, however, than in the case of internal cathode, so that in practical cases only a very small fraction of the electrons have a sufficiently small initial velocity to obey Eq. 12. For the rest the critical magnetic field will depend on the initial velocity, and the curves representing current as a function of magnetic field will not be steep, as in the case of electrons starting from the filament, but will fall off slowly (see Figs. 15–20).

The magnitude of the correction for initial velocities can best be illustrated by an example. If the diameters of inner and outer cylinders are in the ratio 1 : 25, viz., 12 mils (r_0 = .0152 cm.) and 3/10 inches
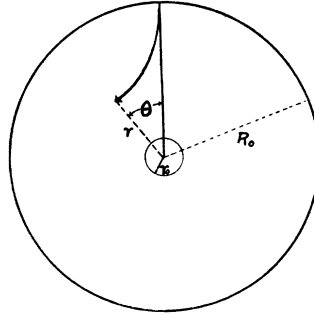
(R = 0.38 cm.) respectively, and the potential difference 50 volts, Eqs.
7 and 11 give, for the cases of external anode and internal anode respectively,

$$H_1 = \frac{\sqrt{8m/e}}{R} \sqrt{V + \frac{1}{2}\frac{m}{e}(u_0^2 + v_0^2)}$$

$$= 17.7\sqrt{50 + 2.82 \times 10^{-16}(u_0^2 + v_0^2)} \quad \text{(external anode)},$$

$$H_2 = \frac{\sqrt{8m/e}}{\beta R} V^{1/2} + \frac{2m}{eR}v_0 = 5 + 3 \times 10^{-7} v_0 \quad \text{(internal anode.)}$$

The following table gives the values of H_1 and H_2 for different values
of v_0 (taking $u_0 = 0$).

v_0		H_1 (External Anode).	H_2 (Internal Anode).
Cm./sec.	Volts.	Gauss.	Gauss.
0 × 10⁷..........	0.	125	5.0
1 "	.028	125 + .035	5.0 ± 3.0
2 "	.112	125 + .141	5.0 ± 6
3 "	.25	125 + .317	5.0 ± 9
4 "	.45	125 + .563	5.0 ± 12
5 "	.70	125 + .880	5.0 ± 15
6 "	1.01	125 + 1.27	5.0 + 18
7 "	1.37	125 + 1.72	5.0 ± 21
8 "	1.80	125 + 2.25	5.0 ± 24

$r_0 = .0152$ cm.; $R = 0.38$ cm.; $\beta = 25$, $V = 50$ volts.

C<small>ASE</small> 3.   *Parallel Planes.—The radii of both cylinders are large compared with their distance apart.*
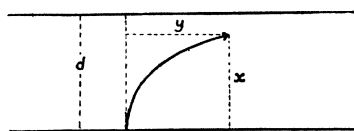

Fig. 3.

Plane parallel plates.

This applies to electrons moving between plane parallel plates, and includes the case of uniform electric and magnetic fields at right angles to each other. Putting

$$r - r_0 = x; \qquad rd\theta = dy; \qquad \frac{r}{r_0} = 1;$$

Eq. 6 becomes

$$V_x = H^2 \frac{e}{2m} x^2 + Hv_0 x - \frac{u_0^2}{2e/m}, \tag{13}$$

where x is the maximum distance that an electron starting from the
plane $x = 0$, with initial velocity u_0 and v_0 in the x and y directions
respectively, can travel from this plane in a magnetic field H parallel to
Z; and V is the potential at the point x.

With large values of H and x the effect of initial velocities is small, and (13) becomes

$$V = H^2 \frac{e}{2m} d^2,$$ (14)

where d is the distance between plates, V the potential difference between them, and H the critical magnetic field that is just sufficient to prevent the electrons reaching the anode plate.

This value of V is 4 times greater than that against which electrons starting from a filament are able to travel the same distance in the same magnetic field (Eq. 8); or, for electrodes the same distance apart and at the same potential difference, only half as strong a magnetic field is required to turn electrons back in the case of parallel plates as is the case of filament and concentric cylindrical anode.

The effect of initial velocities is much greater in the case of the parallel plates, however, than for the filament. For example, for plates 1 cm. apart and electrons with 2 volts initial energy ($v_0 = 8 \times 10^7$), Eq. (13) gives

$$\text{for } H = 10, \; V = 8.8 \pm 8 - 2 \atop \text{`` } H = 100, \; V = 880 \pm 80 - 2 \Big\} \; 2 \text{ volt electrons}$$

while even for electrons of only $1/10$ volt initial energy ($v_0 = 2 \times 10^7$)

$$\text{for } H = 10, \; V = 8.8 \pm 2 - 0.1 \text{ volts} \atop \text{`` } H = 100, \; V = 880 \pm 20 - 0.1 \phantom{00} \text{``} \Big\} \frac{1}{10} \text{ volt electrons.}$$

Hence, the curve representing current as a function of magnetic field will not be as steep in the case of plane parallel plates as in the case of filament and cylindrical anode.

CASE 4. *Positive ions are produced at a point r cm. from a straight filament where the potential is V with respect to the filament, with initial velocities u_0 and v_0, and move toward the filament in a magnetic field H.*

Equations (10) and (11) apply, with

$$\frac{e}{m} = \frac{10^4}{M},$$

where M is the molecular weight of the ion. The small value of e/m, as compared with that for electrons, more than offsets the factor β^2, so that a magnetic field which is capable of turning back electrons moving from a filament to concentric cylindrical anode will not be able to prevent the positive ions produced by these electrons from reaching the filament, except in the case of small diameter filament (large β) and ions of low atomic weight. The relative effect, upon the value of critical magnetic field, of initial velocities and voltage respectively is the same as in the

case of electrons, for the same initial energy per unit charge. Hence the cutoff by a magnetic field of positive ions moving toward a filament will not be sharp except in the case of very strong magnetic and electric fields, and will be practically ineffective in the case of small diameter filaments and heavy ions.

Paths of the Electrons.

The phenomena discussed thus far are independent of the potential distribution in the tube. The complete determination of the paths of the electrons requires a knowledge of this potential distribution, which depends on the space charge of the electrons. Certain properties of the motion can be obtained, however, without this knowledge, and an approximate solution can be obtained on the assumption that the space charge distribution is the same as that which exists without the magnetic field. There is experimental evidence that this assumption is not far from the truth.

The angular velocity of the electron is given by Eq. 4 above;

$$\frac{d\theta}{dt} = \frac{r_0^2}{r^2}\left(\frac{v_0}{r_0} - \frac{He}{2m}\right) + \frac{He}{2m}. \tag{4}$$

In the case of an electron starting from a filament of radius r_0 with initial velocity

$$v_0 \leq r_0 \frac{He}{2m},$$

the first term on the right hand side becomes negligible compared with the second as soon as the electron has travelled a small distance r equal to about 10 times the radius of the filament. From this point on its angular velocity is constant and equal to $He/2m$. In the usual case this distance is of the order of $1/2$ mm. For example, in the case of a 1 mil fil. ($r_0 = .0012$ cm.) and electrons with initial velocities of 2×10^6 cm./sec., if $H = 200$ gauss,

$$\frac{v_0}{r_0} = 1.7 \times 10^9; \qquad \frac{He}{2m} = 1.8 \times 10^9; \qquad 10r_0 = .012;$$

the electron acquires 99 per cent. of its maximum angular velocity within $1/10$ mm. of the filament. Hence

$$\frac{d\theta}{dt} = \frac{He}{2m}\begin{cases} r > .01 \text{ cm.,} \\ H > \dfrac{2mv_0}{er_0}, \end{cases} \tag{15}$$

that is, for electrons starting with small initial velocity from a 1 mil filament, *except in the very small region within $1/10$ mm. of the filament, that angular velocity of the electrons is constant and equal to $He/2m$.*

If an electron starts with initial angular velocity v_0/r_0 large compared with $He/2m$, this angular velocity will fall off rapidly at first, at a rate proportional to $1/r^3$, then less rapidly as it approaches the limiting value $He/2m$, and will not reach its limiting value in as short a distance as that given by Eq. (15). For example, in the case of the 1 mil filament and magnetic field of 200 gauss considered above, electrons with $1/10$ volt initial energy ($v_0 = 2 \times 10^7$ cm./sec.) must travel 1 mm. before their angular velocity falls to within 1 per cent. of its constant value, and electrons with 2 volts initial energy ($v_0 = 8 \times 10^7$) must travel 4 mm. before their angular velocity is constant within 1 per cent.

From Eqs. 4 and 5 one obtains for the angle ϕ between the radius vector and the tangent to the path, neglecting initial velocities:

$$\tan \varphi = \frac{r d\theta}{dr} = \pm \frac{H \frac{e}{2m} (r^2 - r_0^2)}{\sqrt{2 \frac{e}{m} V r^2 - \left(H \frac{e}{2m} \right)^2 (r^2 - r_0^2)}} . \tag{16}$$

Since V is a function of r only, this has the same numerical value for an electron traveling in toward the axis as for one traveling out, hence *the electrons return in paths of exactly the same shape as those in which they go out.* This shows that the electrons, if they fail to reach the anode, do not continue circling around the cathode in approximately circular orbits forever (or until they strike gas molecules), but return periodically to the cathode. The total angle they describe before returning depends on the distribution of electric intensity as a function of r. If the space charge distribution were the same as without magnetic field this angle would be 270 degrees. The actual space charge, which has recently been calculated by Langmuir,[1] is slightly greater than without magnetic field and gives a total angle of 288.5 degrees.

There are two special cases of intensity distribution that lead to very simple solutions of the path. The first represents the initial condition, before space charge has had time to build up. The second represents very nearly the stationary condition.

CASE I. *Space charge is negligible.*

The electric intensity is that due to the potential difference V between the cylinders, viz.,

$$F_r = \frac{V}{r \log \frac{R}{r_0}} ,$$

where R and r_0 are the radii of cylinder and filament respectively. If r_0

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[1] These calculations will be published soon by Dr. Langmuir.

is very small, the intensity is negligible except very close to the filament. The electron will therefore move with constant velocity, and its path will be a circle of radius

$$\frac{\sqrt{2\dfrac{m}{e}V}}{H}.$$

This solution represents the initial condition, before the electrons which fail to reach the cylinder have had time to pile up in the space in appreciable quantity. The steady state, however small the emission, must be that of full space charge.

CASE 2. *Space Charge the Same as Without Magnetic Field.*—Consider the case of electrons starting with zero initial velocities from a filament of very small diameter. The potential at any distance r from the axis is given by the equation[1]

$$i_0 = \frac{2\sqrt{2}}{9}\sqrt{\frac{e}{m}}\frac{V^{3/2}}{r}, \tag{17}$$

where i_0 is the maximum electron current per cm. length from the filament to a concentric cylinder of radius r at potential V. Equations (4) and (5) give

$$\left.\begin{array}{l}\left(\dfrac{d\theta}{dt}\right)^2 = \dfrac{\alpha^2}{r^4}(r^2 - r_0{}^2),\\[3mm] \left(\dfrac{dr}{dt}\right)^2 = 2\dfrac{e}{m}V - \dfrac{\alpha^2}{r^2}(r^2 - r_0{}^2)\end{array}\right\} ; \alpha = H\frac{e}{2m},$$

from which, if $r_0 = 0$,[2]

$$\frac{d\theta}{dr} = \frac{\alpha}{\sqrt{2e/m\ V - \alpha^2 r^2}}$$

and substituting the value of V from (17)

$$\frac{d\theta}{dr} = \frac{\alpha}{\sqrt{(9i_0 e/m)^{2/3}r^{2/3} - \alpha^2 r^2}}. \tag{18}$$

Eq. (18) may be written,

$$d\theta = \frac{3/2dx}{\sqrt{\beta^2/\alpha^2 - x^2}}, \text{ where } \begin{cases} x = r^{2/3}, \\[2mm] \beta^2 = \left(9i_0\dfrac{e}{m}\right)^{2/3}, \\[2mm] \alpha = \dfrac{He}{2m}. \end{cases}$$

[1] Langmuir, PHYS. REV., **2**, 455, 1913.

[2] The assumption that $r_0 = 0$ leads to no mathematical discontinuities. The solution thus obtained (Eq. 19) will therefore represent the actual solution, viz., the solution for finite r_0, to any desired degree of accuracy, if r_0 is taken sufficiently small.

The integral of this is

$$\theta = \frac{3}{2}\sin^{-1} x \,\frac{\alpha}{\beta} + \theta_0 \,.$$

Substituting the values of α, β and x, and determining the constant θ_0 from the condition that θ shall be zero when r is zero,[1]

$$\theta = \frac{3}{2}\sin^{-1}\left(\sqrt[3]{\frac{e^2}{72 i_0 m^2}\,H r^{2/3}} \right)$$

or

$$r = 6 \,\sqrt{2 i_0}\,\frac{m}{e}\,H^{-3/2}\left(\sin \frac{2}{3}\theta \right)^{3/2}, \tag{19}$$

substituting for i_0 its value from Eq. 17 in terms of the diameter R and potential V_0 respectively of the outer cylinder,

$$r = 4 \,\sqrt[4]{2}\left(\frac{m}{e} \right)^{3/4} \frac{V^{3/4}}{R^{1/2} H^{3/2}}\left(\sin \frac{2}{3}\theta \right)^{3/2}. \tag{20}$$

If r_{max} represents the maximum value of r in Eq. 20, that is the greatest radial distance the electron can travel in magnetic field H when the potential on the anode, of radius R, is V_0, Eq. (20) may be written

$$r = r_{max}\left(\sin \frac{2}{3}\theta \right)^{3/2}, \tag{21}$$

where

$$r_{max} = 4\sqrt[4]{2}\left(\frac{m}{e} \right)^{3/4} \frac{V^{3/2}}{R^{1/2} H^{3/2}}.$$

When $r_{max} = R$, i.e., when the electrons can just reach the outer cylinder, this reduces to

$$R = \sqrt{\frac{8m}{e}}\,\frac{V^{1/2}}{H},$$

which is identical with Eq. (9).
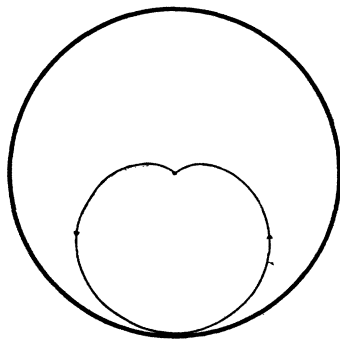
Fig. 4.

Plot of approximate equation of path, $r = R[\sin^{2/3}\theta^{3/2}]$.
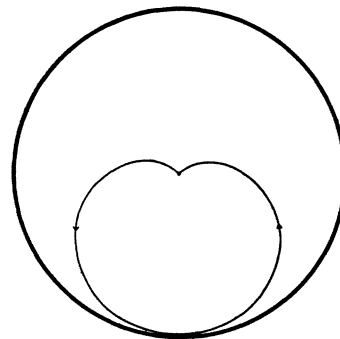
Fig. 5.

Exact path of electron (Langmuir's solution).

[1] See footnote 2, previous page.
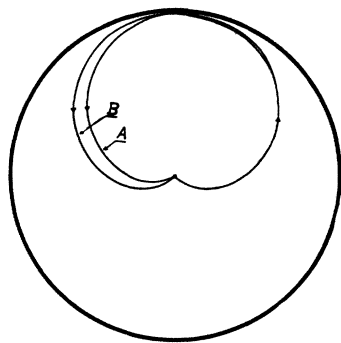
Fig. 6.

Path of electron. Degree of approximation represented by Eq. 21.
A, approximate path, Eq. 21; B, exact path (Langmuir).

The path given by Eq. 20 is shown in Fig. 4. The exact path, as calculated by Langmuir, is shown in Fig. 5. For the purpose of comparison the two are plotted together in Fig. 6. The difference is inappreciable up to $r = 9/10$ $r_{max'}$, i.e., Eq. 21 represents the true path very closely except in the region near the maximum range. In this region the radial velocity is less than that given by Eq. 21, making the total angle traversed per cycle greater, viz., 288.5 degrees instead of 270 degrees.

EXPERIMENTS.

The experimental work was done in coöperation with Mr. E. F. Hennelly. More than 1,000 tubes have been studied, of various shapes and sizes, all conforming as closely as possible to the symmetry conditions postulated above. The following examples are typical:

A. Internal Cathode. Straight Filament in Axis of Cylinder. General Characteristics.—The diameter of the filament is always small compared with that of the anode, so that r_0^2/R^2 is negligible compared with unity. Hence, the relation between critical magnetic field, voltage, and anode radius is that given by equation (9):

$$H = \sqrt{\frac{8m}{e}} \frac{V^{1/2}}{R} . \qquad (9)$$

H is the value of the magnetic field that is just sufficient to prevent electrons from reaching an anode of radius R at potential V with respect to filament. If V is kept constant and H gradually increased, the current should remain constant until H reaches the critical value given by equation (9), and should then fall abruptly to zero.

Figure 7, for which I am indebted to Dr. K. H. Kingdon, shows a typical example. The tube contained a straight tungsten filament .012 cm. in diameter, about 1 mm. from the axis of an anode 3.86 cm. in diameter, 12 cm. long. A rotating commutator applied voltage and heating current alternately, so that measurements were made while the filament was an equipotential surface. The potential difference between cathode and anode was 50 volts, and the filament temperature high enough so that the current was limited only by space charge.

It is seen that the current maintains nearly its full space charge value

until the critical magnetic field is reached, and then falls very abruptly. The steepness is as great as to be expected from the degree of symmetry, *i.e.*, with filament 1 mm. out of center.   The effect of high initial velocities is observable near the bottom of the curve.
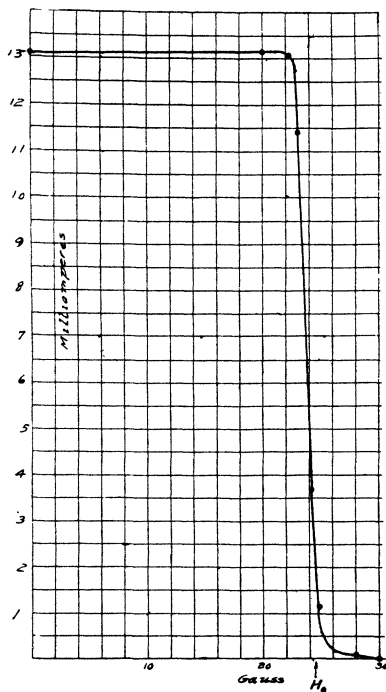

Fig. 7.

Typical Gauss-Ampere characteristic at 50 volts.

The theoretical "cut off" field is shown by the dotted curve, as calculated from Eq. 11, viz.,

$$H = 6.72 \sqrt{\frac{50}{1.93}} = 24.56 \text{ Gauss.}$$

Effect of Heating Current through Filament.—The characteristic of Fig. 7 was taken with a rotating commutator, adjusted to cut off the heating current during the brief intervals when measurements were being made. If measurements are made while the heating current is flowing, the curves are less steep, due to the voltage drop along the filament.   This effect is more marked at low than at high anode voltages.   Most of the data which follows was taken with filament current flowing.

No effect due to the magnetic field of the heating current has been observed.

Effect of Anode Voltage.—Fig. 8 shows the characteristic of a tube with anode 3/4 in. diameter × 1½ in. long and cathode 4 mils diameter, at three different anode potentials, viz., 10.2, 21.7, and 57 volts. These measurements were taken by Dr. K. H. Kingdon with rotating commutators.
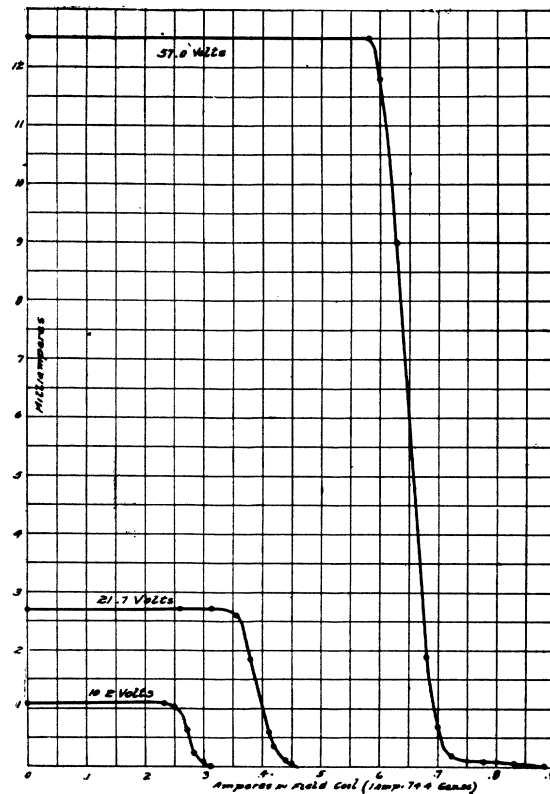
Fig. 8.

Characteristic of ¾ inch dia. anode, at different potentials.

Fig. 9 shows the $I - H$ characteristic of a tube with 1 in. × 4 in. anode and 10 mil cathode at four different anode voltages, 60, 125, 200, and 250 volts. The potential drop in the filament was 10 volts, and anode potentials were measured from the negative end of the filament. Hence in the 60-volt test, one end of the filament was at 50 volts with respect to the anode, and the other at 60.

It will be observed that the maximum currents in Figs. 8 and 9 are in the ratio of the 3/2 power of the voltage, and the critical magnetic fields are in the ratio of the 1/2 power of the voltage.

Figs. 10 and 11 show the lower portions of curves taken at higher poten-
tials.   The anode of Fig. 10 was 2 in. diam. × 2 in. long, that of Fig. 11
was 1 in. diam. × 4 in. long.   The upper portions of these curves could
not be obtained because of the excessive heating of the anode.   The full
current at 9,000 volts would have been 100 amps.   Its reduction to 35
milliamperes was due solely to the magnetic field.
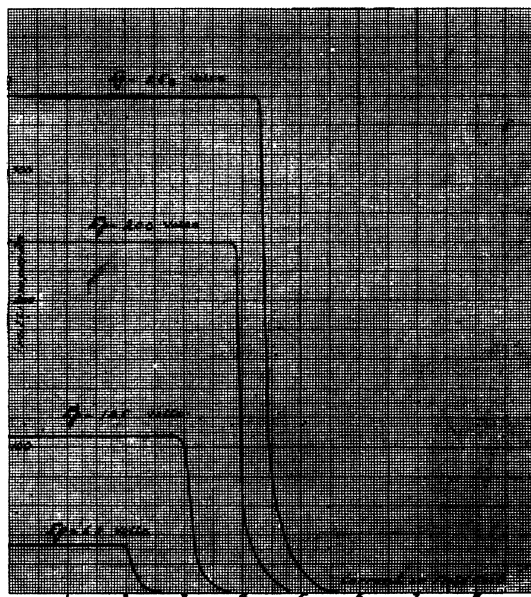
Fig. 9.

Characteristic of 1 inch dia. anode, at different potentials.

Effect of Filament Temperature.—Fig. 12 shows the characteristics of a
tube of the same construction as that of Fig. 9, at three different filament
temperatures.   The potential difference was 250 volts.   The steep
portions of the curves are identical.   There is a slight transition region
between the horizontal and vertical portion of each curve.   In the upper
curve the current is limited by space charge, in the other two by filament
temperature.

Effect of Diameter of Anode.—Fig. 13 shows the characteristics, at 250
volts, of 4 tubes of different diameters, from 2 inches to 3/8 inch.   In
order to bring them within the limits of the same plot, the currents in
the smaller tubes have been limited by filament temperature to the same
maximum value as for the largest diameter.   The critical magnetic fields
are proportional to the diameters within the limits of accuracy with
which these diameters were known.

Effect of Symmetry.—Symmetry, both of electric and magnetic field, is essential for sharp cut-off. A numerical estimate of the effect of asymmetry is difficult, but an example will illustrate it. Fig. 14 shows two characteristics taken with the same tube. In curve A the filament was parallel to the magnetic field; in curve B the magnetic field was inclined 20 degrees to the filament.
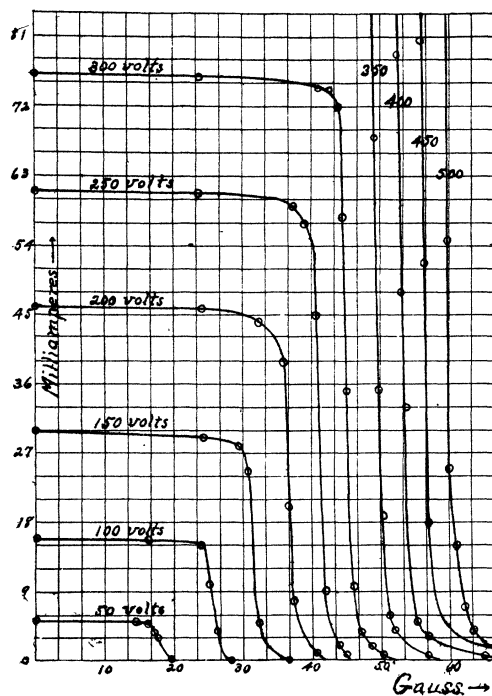

Fig. 10.

Characteristic of 2 inch dia. anode, at different potentials.

The effect of small pieces of iron in the magnetic field is very striking. With the magnetic field adjusted to the critical value, the point of a pen brought near the tube is sufficient to reduce the current to half value.

Effect of Gas.—The equations for cut-off are independent of potential distribution in the tube. The cut-off should not be affected, therefore, by the presence of positive ions, even in sufficient numbers to completely neutralize the space charge of the electrons, provided the mean free path is large compared with the dimensions of the tube. Hence with low gas pressure one should observe currents which are not limited by space charge, but are completely controllable by a magnetic field. At higher gas pressures the magnetic field becomes less and less effective, due to

the fact that every electron which collides with a molecule will suffer a change, in direction at least, of its velocity, and a large fraction of these velocity changes will be in such direction as to require a stronger field to keep the electrons from the anode.   It is probable, also, that the impact of positive ions on the cathode gives rise to the emission of electrons of relatively high initial velocity, some of which, according to Eq. 7, require a stronger field to turn them back.
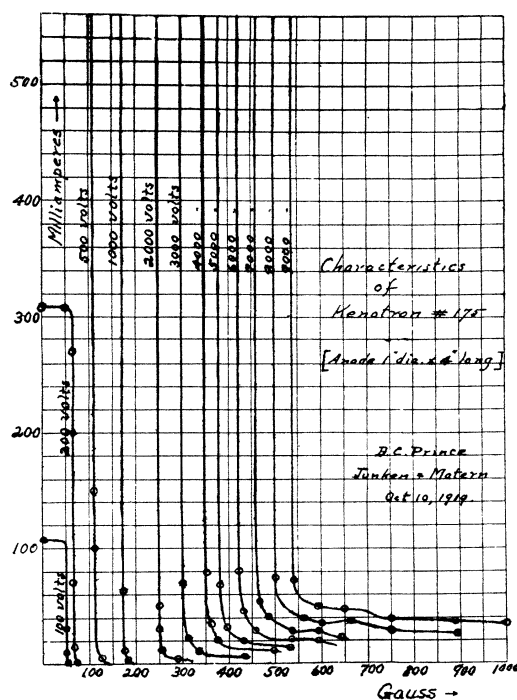

Fig. 11.

Characteristic of 1 inch dia. anode, at high potentials.

Figure 15 shows the typical behavior of tubes containing small amounts of gas.   Curve A is the Gauss-ampere characteristic in good vacuum. The maximum current is limited by space charge, which increases slightly, with corresponding decrease in current, as the critical magnetic field is approached.   The steepness of cut-off is limited mainly by symmetry and potential drop along the filament.   Curve B shows the effect of a few hundredths of a bar of gas.   The space charge of electrons is partially neutralized by positive ions so that the maximum current is slightly larger, especially in the neighborhood of critical magnetic field, where the paths of the electrons are longer and the ionization greater.   The cut-off
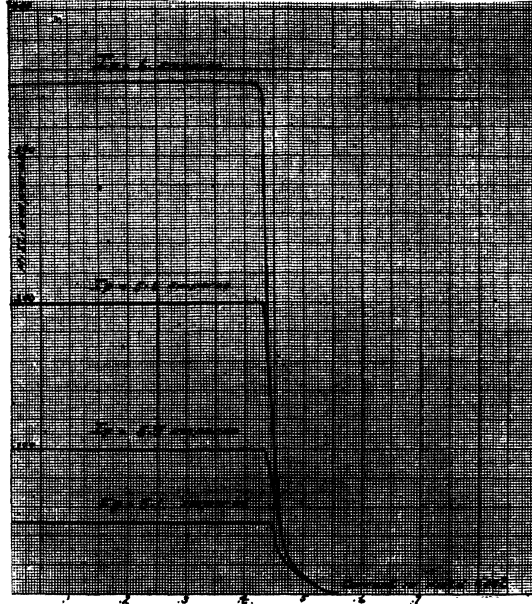
Fig. 12.

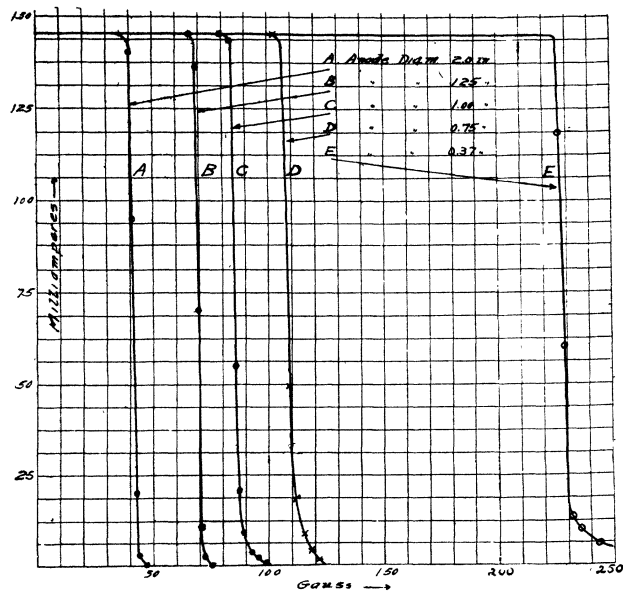Characteristic of 1 inch dia. anode, at 250 volts, at different filament temperatures.

Fig. 13.

Characteristics of different anodes, at 250 volts.

is not appreciably affected.   Curve C shows the effect of a larger amount of gas, of the order of 1 bar.   The maximum current is very large, but the critical magnetic field is only slightly greater than for good vacuum. The curve has a long tail, however, representing the electrons which have collided or have abnormal initial velocities, and hence require a field much stronger than the critical value to keep them from the anode.   At higher pressures, of the order of 10–20 bars, the magnetic cut-off is much less steep (Curve D) and a considerable fraction of the electrons are entirely uncontrollable by the magnetic field.   At still higher pressures magnetic fields of this order of magnitude are practically ineffective.
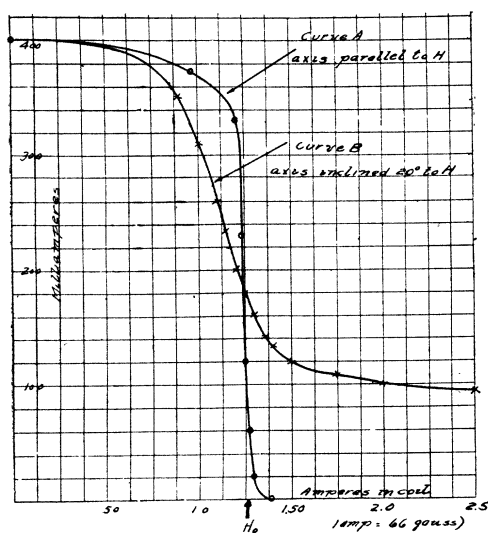
Fig. 14.

Effect of lack of symmetry.   Magnetic field not parallel to axis.

B. Internal Anode.   Straight Rod in Axis of Helical Filament.   General Characteristics.—Equations (10) and (11) apply.   The effect of initial velocities of the electrons is large, except in strong magnetic fields. Hence the curves representing current to the central anode as a function of magnetic field, at constant anode potential, will not be as steep as in the first case.   On the other hand, the magnetic field required to produce a marked change in current is very much smaller than in the first case, so that the earth's magnetic field produces an easily measurable effect, and the field of the current which heats the filament practically prevents any electrons reaching the anode at potentials less than about 100 volts.

A typical example is shown in Fig. 16.[1]   The cathode was a helical

[1] For this and the following curves I am indebred to Mr. C. G. Found.

coil 0.63 cm. in diameter, 1.7 cm. long, consisting of 10 turns of 10 mil tungsten wire. The anode was a 10 mil wire in the axis of the helix. A rotating commutator applied heating current and voltage alternately, so that the measurements were taken when no current was flowing through the filament. The measurements were taken with the tube vertical, and the earth's magnetic field was not compensated. The abscissas are the values of the impressed magnetic field. It will be noted that the curve is symmetrical, not about the axis of ordinates, but about the ordinate $H = 0.6$ approximately, the vertical component of the earth's magnetic field.
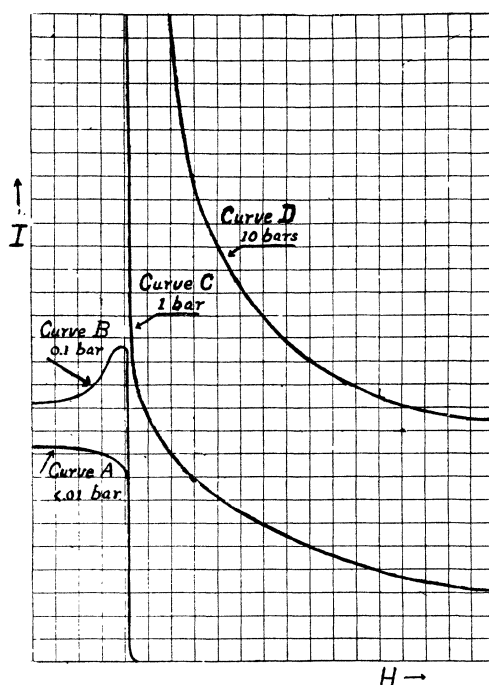
Fig. 15.

Effect of gas pressure upon magnetic cutoff.

The value of magnetic field just sufficient to prevent electrons of zero tangential initial velocity from reaching the anode, according to Eq. 12, is shown at H_0. It is seen that about half of the electrons, viz., those whose initial velocities are in the same direction as the magnetic deflection, are kept from the anode by a field less than H_0. The remainder, whose initial velocities are in the opposite direction to the deflection produced by the magnetic field, require a field greater than H_0.

It is possible to calculate the exact current to be expected at each value of magnetic field, assuming the tangential initial velocities to be distributed according to Maxwell's law.[1]   The magnetic field just sufficient to keep back electrons with initial velocities less than v_0' or greater than v_0'' is given by Equation (11), viz.,

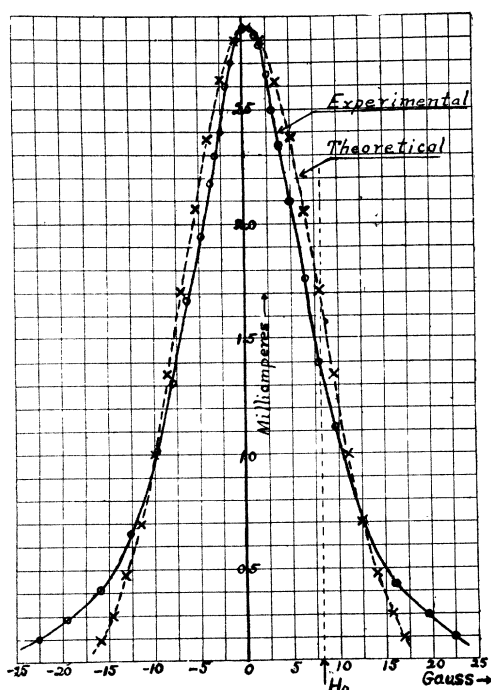$$H = \pm \frac{1}{\beta R_0} \sqrt{\frac{8m}{e} V + 2 \frac{mv_0}{eR_0}} \qquad (11)$$

Fig. 16.

Typical internal anode characteristic at 110 volts. O, experimental points (solid curve).
X, theoretical points (broken curve).

v_0' and v_0'' correspond to the plus and minus sign of the radical respectively.   In the present case, with $\beta = 25$, $R_0 = .38$ cm. and $V = 110$ volts, this becomes

$$H = \pm 7.5 + 3v_0.$$

The values of H required for different values of v_0 are given in Table II. All electrons with initial velocities *between* any pair of values v_0' and v_0''

[1] Richardson and Cooke, Phil. Mag., 16, 353, 1908.

will *not* be deflected from the anode by the corresponding H, *i.e.*, will reach the anode. This number, which represents the relative current to the anode for each value of H, is given by Maxwell's distribution law as

$$i_H = \int_{v_0'}^{v_0''} f(v_0)dv_0 = \int_{v_0'}^{v_0''} \left(\frac{m}{2\pi R_1 T}\right)^{1/2} e^{-\frac{mv_0^2}{2R_1 T}} dv_0 , \qquad (22)$$

where m = mass of electron = 9.01×10^{-28},

     R_1 = gas constant for 1 molecule = 1.372×10^{-16} ergs/degree,

     T = absolute temperature = 2,200 degrees,

     v_0 = initial (tangential) velocity of electron in cm./sec.

The values of i_H given by equation (22) are tabulated in column 4 of Table II. These values are percentages of the total number of electrons emitted by the filament. It is seen that only 83 per cent. of the electrons emitted reach the anode *even without magnetic field*. For comparison with the observed currents the values in column 4, multiplied by 3.44 to reduce them to the same scale, are tabulated in column 5, and plotted as crosses in Fig. 16. The agreement is remarkably good considering the degree of symmetry of the apparatus. With more perfectly constructed apparatus it should be possible to test accurately the law of distribution of initial velocities of the emitted electrons.

TABLE II.

nitial Velocity of Electrons .		Critical Magnetic Field (from Eq. 11).	Fraction of Emitted Electrons with Velocities between v_0' and v_0'' .	Number of Electrons Reaching Anode.	
v_0' cm./sec. $\times 10^7$.	v_0'' cm./sec. $\times 10^7$.	H Gauss.	$i_H = \int_{v_0'}^{v_0''} f(v_0)dv_0$.	Milliamperes Calculated $3.44\, i_H$)	Observed Milliamperes.
− 2.5	+ 2.5	0	0.828	2.85	2.85
− 2.0	+ 3.0	1.5	.812	2.80	2.78
− 1.5	+ 3.5	3.0	.765	2.63	2.45
− 1.0	+ 4.0	4.5	.692	2.38	2.10
− 0.5	+ 4.5	6.0	.600	2.07	1.80
0	+ 5.0	7.5	.497	1.71	1.45
+ 0.5	+ 5.5	9.0	.391	1.35	1.15
+ 1.0	+ 6.0	10.5	.292	1.01	0.97
+ 1.5	+ 6.5	12.0	.207	.71	.72
+ 2.0	+ 7.0	13.5	.138	.47	.56
+ 2.5	+ 7.5	15.0	.086	.30	.47
+ 3.0	+ 8.0	16.5	.050	.17	.39
+ 4.0	+ 9.0	19.5	.015	.05	.28

Magnetic Field of Heating Current.—Fig. 17 shows the effect of the

magnetic field of the heating current in the filament.   The measurements
were taken with a tube having the same cathode as that of Fig. 16, viz.
a helix of .63 cm. diam. but with a 40 mil (1 mm.) anode.   Curve A is
taken with rotating commutator, that is the measurements were made
while no heating current was flowing.   Curve B was taken with D.C.
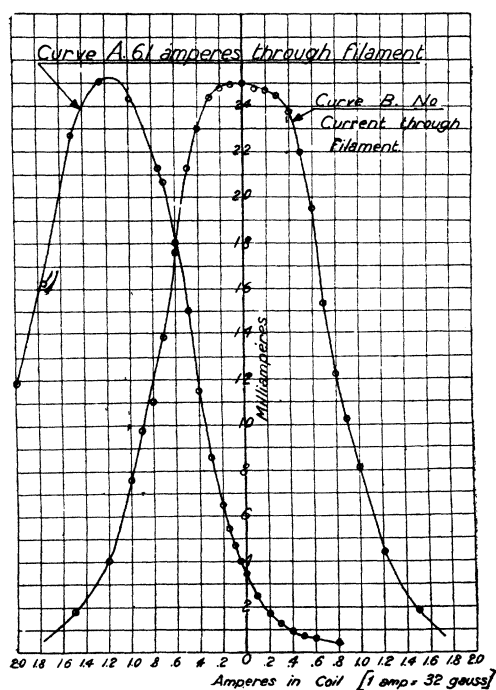heating current of 6.1 amperes.

Fig. 17.

Internal anode.   Effect of magnetic field of heating current in filament.

The magnetic field of the heating current reduced the electron current
to 13 per cent. of its value with no magnetic field.   The two curves are
similar except that one is displaced to the left by 48 gauss, which is the
approximate value of the field produced by the heating current of 6.1
amperes through a coil of 6 turns per cm. (10 turns in 1.7 cm.).

Effect of Voltage.—Figs. 18 and 19 show the variation of current with
magnetic field for the same tube at different anode potentials.   The
tube is the same as of Fig. 17, viz., cathode .63 cm. diameter, 1.7 cm.
long, anode 1 mm. diameter.

In Fig. 18 the currents are all plotted to the same scale, showing the
relative magnitudes, and the relative variations in milliamperes with
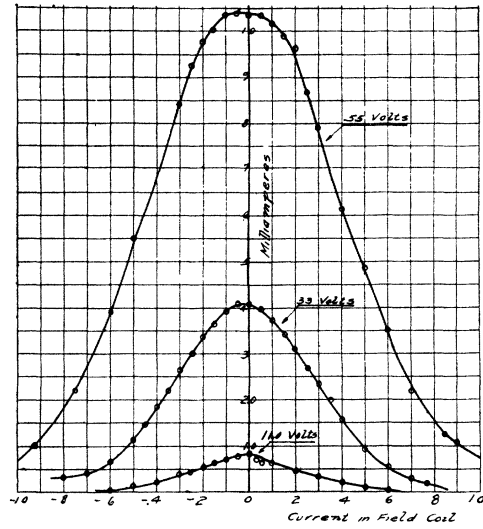
Fig. 18.

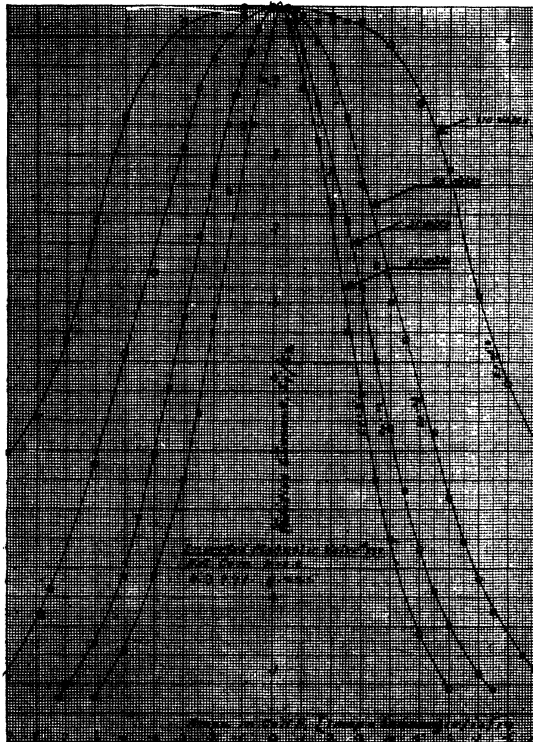Characteristic of 1 mm. internal anode at different potentials.

Fig. 19.

Characteristic of 1 mm. internal anode at different potentials. Currents expressed as fractions of maximum at each voltage.

magnetic field. The maximum currents at the different voltages are proportional to the 3/2 power of the voltages.
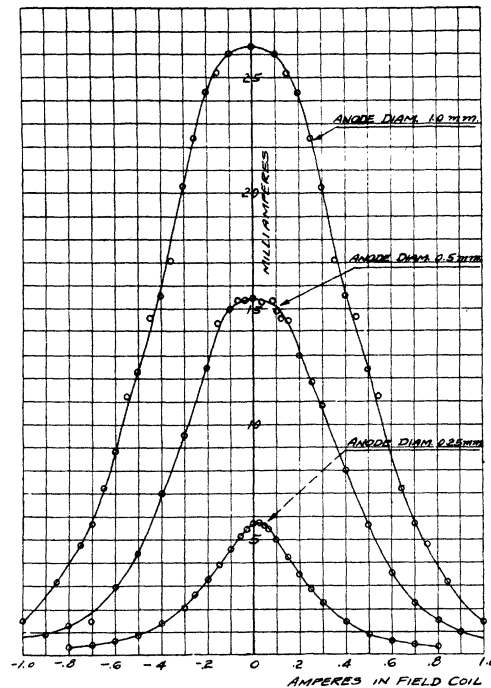

Fig. 20.

Different internal anodes at same potential (110 volts).

In Fig. 19 the maximum at each voltage is taken as unity, in order to show the relative *percentage* variation with magnetic field. It is seen that the field required to produce the same percentage decrease is larger the higher the voltage. At lower voltages, the variation is less rapid than the square root of the voltage, since the initial velocity term in equation 11 predominates. At the highest voltage the curve has a fairly flat top, showing smaller relative effect of initial velocities, and approaching the steep cutoff of internal cathode tubes.

Effect of Anode Diameter.—Figures 20, 21, and 22 show the characteristics of three similar tubes with different anode diameters. The cathodes of the three tubes were the same, viz., 10 turns of 10 mil tungsten wire wound in a helix 0.63 cm. in diam. 1.7 cm. long. The anodes were 0.25 mm., 0.50 mm., and 1.0 mm. diam. respectively.

Figure 20 shows the actual currents in the three tubes at the same anode voltage, plotted to the same scale. The difference in the maxima

is due partly to space charge, but mostly to initial velocities, since by Eq. (10) only part of the electrons emitted can reach the anode even in the absence of magnetic field, viz., those whose initial energy of tangential motion, expressed in equivalent volts, is less than V/β. (V is the potential difference between anode and cathode, and β the ratio of cathode diameter to anode diameter.) Thus, the fraction that can reach the anode falls off very rapidly with increasing β, that is, with smaller anode diameter.
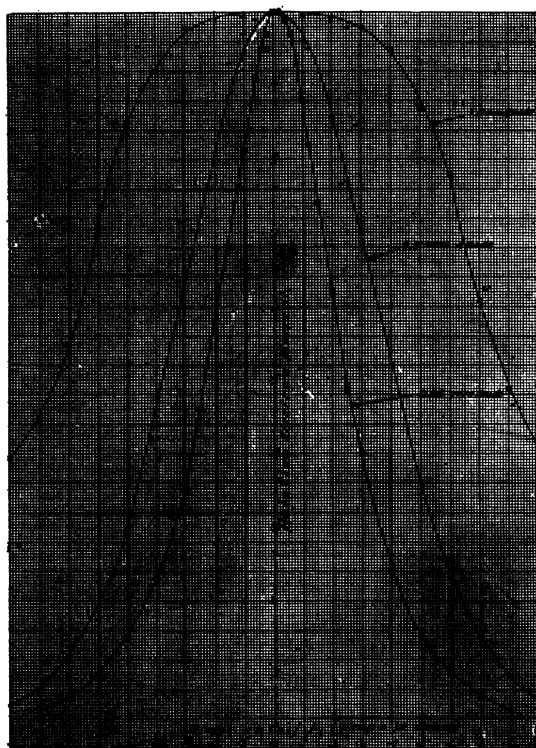
Fig. 21.

Different internal anodes at same potential (110 volts). Currents expressed as fractions of maximum for each anode.

In Fig. 21, the same curves are reproduced on different scales, the maximum of each being taken as unity. It will be noted that the smaller anode diameter gives the greater percentage decrease of current with magnetic field, and that weaker fields are required for smaller anode diameter, as predicted by Eq. 11.

Fig. 22 shows the characteristic of the same three tubes at different voltages, so chosen that the maximum space-charge limited current was
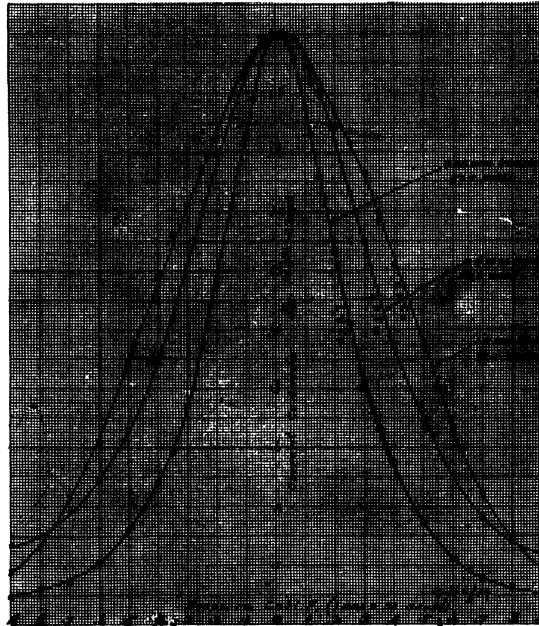
Fig. 22.

Comparison of different internal anodes at different potentials, so chosen as to give same
maximum current for each.

the same for each.   It is seen that even at the higher voltage required
for equal current, the smaller anode is the more susceptible to magnetic
field.

CONCLUSION.

The agreement between theory and experiment is well within the limits
prescribed by the degree of symmetry of the apparatus.   Experiments
with more perfect apparatus are in progress by Dr. K. H. Kingdon, which,
it is hoped, will show the limitations of the theory, and bring to light any
new facts not hitherto taken account of.

Several useful applications have been investigated, which will be
discussed elsewhere.

The above solution assumes constant e/m, and therefore applies only to
potential differences of the order of 50,000 volts or less.   Dr. Leigh
Page has very kindly worked out the solution for the case of variable e/m,
which is presented in the following paper.

RESEARCH LABORATORY, GENERAL ELECTRIC CO.,
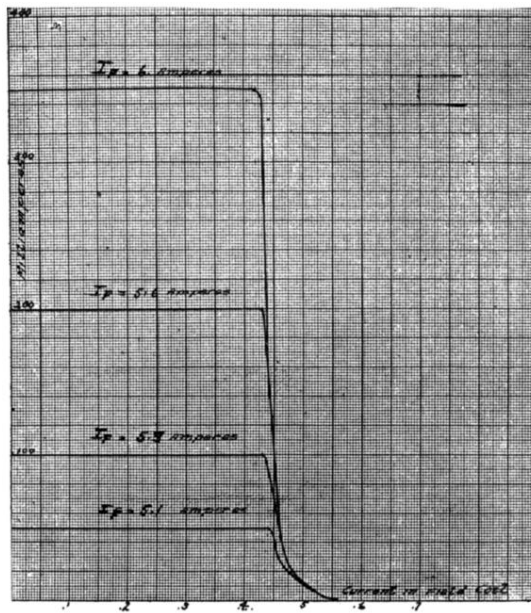SCHENECTADY, N. Y.,
March 3, 1921.

Fig. 12.

Characteristic of 1 inch dia. anode, at 250 volts, at different filament temperatures.
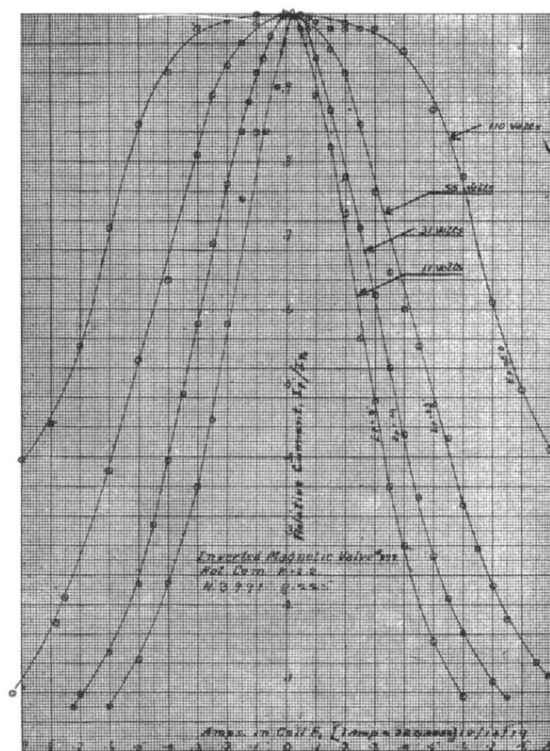
Fig. 19.

Characteristic of 1 mm. internal anode at different potentials. Currents expressed as fractions of maximum at each voltage.
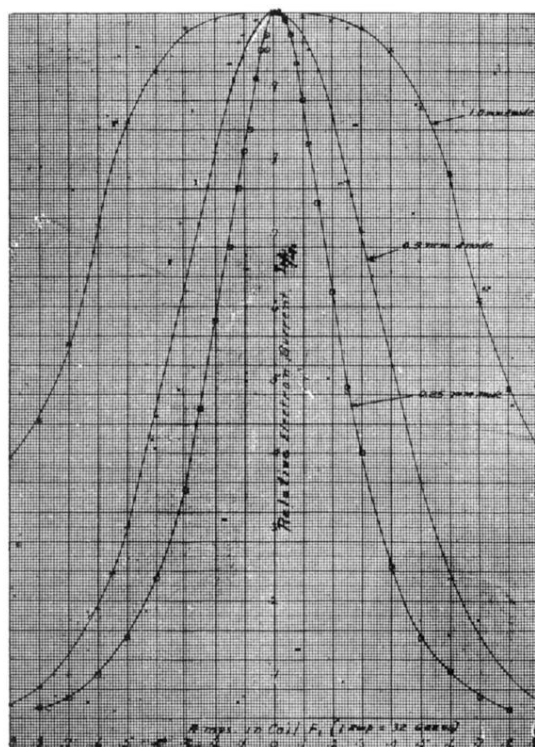
Fig. 21.

Different internal anodes at same potential (110 volts).   Currents expressed as fractions of
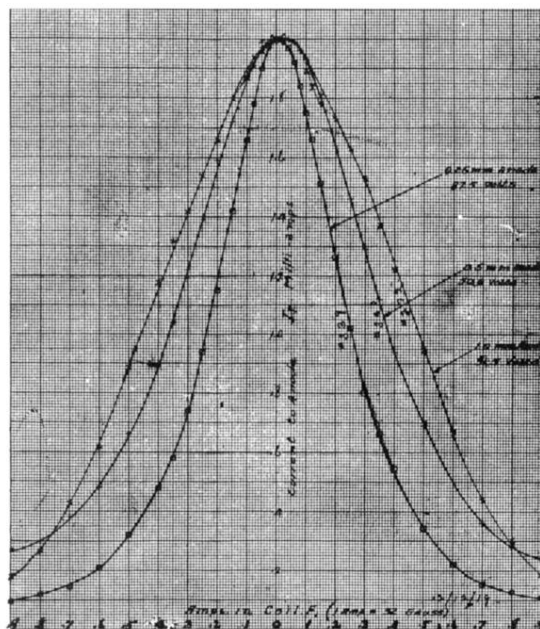maximum for each anode.

Fig. 22.

Comparison of different internal anodes at different potentials, so chosen as to give same
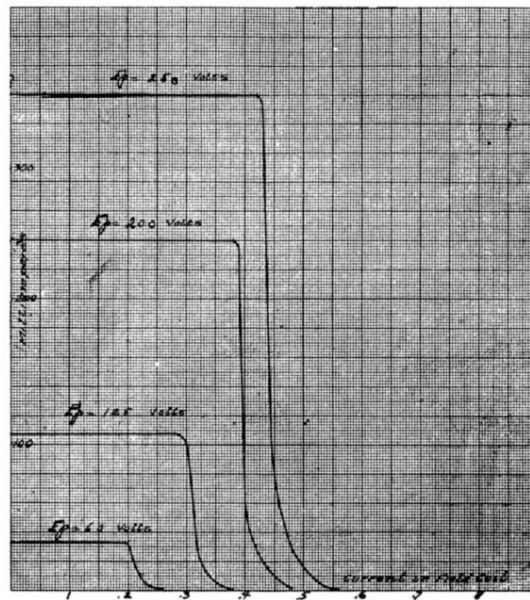maximum current for each.

Fig. 9.

Characteristic of 1 inch dia. anode, at different potentials.