When κa is small, we may employ approximate values of the functions J_0 and χ .

Thus
$$\frac{J_0(i\kappa a)}{J'_0(i\kappa a)} = -\frac{2}{i\kappa a} \dots\dots\dots (64).$$

For χ we have ("Stokes on Pendulums," *Camb. Trans.*, Vol. ix.)

$$\chi = \left(\log \left(\frac{1}{2} \kappa r \right) - \pi^{-1} \Gamma' \left(\frac{1}{2} \right) \right) \left(1 + \frac{\kappa^2 r^2}{2^2} + \frac{\kappa^4 r^4}{2^2 \cdot 4^2} + \dots \right) - \left(\frac{\kappa^2 r^2}{2^2} S_1 + \frac{\kappa^4 r^4}{2^2 \cdot 4^2} S_2 + \dots \right) \dots\dots (65),$$

where
$$S_n = 1^{-1} + 2^{-1} + 3^{-1} + \dots + n^{-1} \dots\dots\dots (66).$$

When κa is small,

$$\begin{aligned} \chi(a) &= \log \left(\frac{1}{2} \kappa a \right) - \pi^{-1} \Gamma' \left(\frac{1}{2} \right), \\ \chi'(a) &= a^{-1}, \end{aligned}$$

so that
$$\frac{\chi(a)}{\chi'(a)} = a \left\{ \log \left(\frac{1}{2} \kappa a \right) - \pi^{-1} \Gamma' \left(\frac{1}{2} \right) \right\} \dots\dots\dots (67).$$

If we suppose that $\sigma' = \sigma$, $V' = 0$, (63) becomes, for small values of κa ,

$$(n + \kappa V)^2 + \kappa^2 a^2 \left\{ \log \frac{8}{\kappa a} + \pi^{-1} \Gamma' \left(\frac{1}{2} \right) \right\} n^2 = 0 \dots\dots\dots (68).$$

Writing μ^2 for
$$\kappa^2 a^2 \left\{ \log \frac{8}{\kappa a} + \pi^{-1} \Gamma' \left(\frac{1}{2} \right) \right\},$$

we get
$$n = \kappa V \frac{-1 \pm i\mu}{1 + \mu^2} \dots\dots\dots (69),$$

whence approximately

$$h = H e^{\pm \mu \kappa V t} \cos \kappa (Vt - x) \dots\dots\dots (70).$$

On Self-Strained Frames of Six Joints. By Prof. CROFTON, F.R.S.

[Read Nov. 14th, 1878.]

In a frame of n joints, it is known that there must be $2n-3$ bars uniting these joints, in order to render the frame rigid, or *stiffen* it. If such a frame be subjected to the action of n given equilibrating forces, acting at the joints, and we consider the problem of determining the stresses in the $2n-3$ bars, we find that the equilibrium of the forces at each joint requires two conditions, or equations: giving in all $2n$ equations to be satisfied by the given forces and their directions, and the $2n-3$ unknown stresses. If these $2n-3$ unknowns were

eliminated from the $2n$ equations, we should find 3 equations to be satisfied by the given forces and their directions, which must be equivalent to the three conditions necessary in order that these forces should hold the frame in equilibrium, and therefore are already given by the conditions of the problem. There are, therefore, only $2n-3$ independent equations to determine the $2n-3$ stresses. It would appear, then, that we always have exactly the number of equations required, and that the problem is always a determinate one. In general it is so, and always so in the case of frames of 3, 4, or 5 joints, as is easy to verify (excepting a case of 5 joints, 3 of which are *in directum*). But when we come to frames of 6 joints (or any higher number), just stiffened by the proper number of bars, there are cases when the frame may be *self-strained*, that is, in a state of internal stress (of various degrees of intensity) independently of any external forces. This shows us that in these cases the problem above cannot be a determinate one, as we had supposed; because, whatever be the internal stresses in the frame when acted on by a given system of external forces, we can always superpose upon them a new system of stresses, due to any self-strained state of the frame, and we shall thus have a new state of internal equilibrium, under the action of the same external forces. We see thus that the $2n-3$ equations above referred to cannot in these cases be all independent. I do not propose to consider the analytical question, but only to determine the conditions that a frame of 6 joints, just stiffened by means of 9 bars, shall be capable of internal stress, without the action of external forces.

I. That a hexagonal frame $ABCDEF$, stiffened by means of diagonal bars AD , BE , CF , shall be capable of being self-strained, it is necessary and sufficient that the 6 joints lie on a conic.

For, suppose the frame in a state of stress, the 6 sides in thrust, and the three diagonals in tension; then any side, as AF , is in equilibrium under the four stresses in the bars 1, 5, 7, 9, as marked in the figure. Hence, producing 1, 5 to meet in V , we see that the resultant of the stresses in 7 and 9 passes through V , that is, it acts in OV . In the

same way, the resultant of the stresses in 7 and 9, as acting on the bar OD , must act in OU . But these two directions are identical; consequently the three points V, O, U lie on a straight line; and this proves that $ABCDEF$ lie on a conic—being one of the forms of Pascal's theorem.

This condition is also *sufficient*. For, assuming any tension for the bar 7, T_7 suppose: we can find a value for T_9 such as to make the resultant of the two act in OV ; we can then determine T_1T_8 so as to make their resultant equal and opposite to that of T_7T_9 ; consequently the bar 6 would be in equilibrium. We can also determine T_2T_4 , as acting on the bar 3, so as to have a resultant equal and opposite to that of T_7T_9 : so that OD would be in equilibrium. Also, considering the equilibrium of the bar BE , the resultant of the stresses in 1 and 5 is equal and opposite to that of the stresses in 2 and 4; so that BE is in equilibrium. Consequently, there is equilibrium at every joint. The stresses in all the other bars follow from the original value assumed for T_7 , and all vary proportionally to that value.

The same proof applies if the hexagon be re-entrant.

II. There is only one other case when a frame of 6 joints and 9 bars can be self-strained.

When the hexagon is stiffened by the bars AE, BD, FC . Here the rigid triangle AFE is in equilibrium under the three

stresses in BA, DE, FC as marked in the figure; consequently, these three directions produced must meet in a point V . Any values for these three stresses which satisfy the conditions for equilibrium at V , will then keep the triangle AFE in equilibrium; and also the triangle BCD ; hence the frame will be in equilibrium. The only condition required being that the lines BA, DE, CF shall meet in one point.

It is easy to show that every possible case of a self-strained frame of 6 joints is reducible to one or other of the above cases—omitting cases where 3 or more joints lie in a straight line—it being obviously necessary that it be stiffened in such a manner that 3 bars meet at every joint.*

* Thus, suppose the joint A joined with B, C, D . There are two cases:—(1.) If two of the latter, as B, C , are joined, then we shall easily see that DEF must also be joined; so that we have the seven bars $AB, AC, BC, DE, DF, EF, AD$; the two remaining ones must be BE, CF , or else BF, CE . This gives us Case II. above, of two rigid triangles with their vertices joined. (2.) When B, C, D are not joined—Here we must have AB, AC, AD , and BE, BF, CE, CF, DE, DF ; so that we have the closed hexagon $ABEDFC$, with the three diagonals AD, BE, EC ; that is, Case I. above.

I have not yet attempted to extend the inquiry to frames of 7 or more joints. There are even cases where a frame of *less* than the number $2n-3$ of bars required for stiffness may be self-strained. Thus, a frame of the form of a regular octagon, with four diagonal bars crossing at the centre, would evidently be in equilibrium, if the diagonals be all in tension, of the same amount, and the sides in compression of the amount requisite to give equilibrium at each joint: yet the frame is not even rigid, and would require a thirteenth bar to render it so.*

The Calculus of Equivalent Statements (Third Paper).

By HUGH MCCOLL, B.A.

[Read November 14th, 1878.]

The following formulæ are all either self-evident or easily verified, and some of them will be found useful in abbreviating the operations of the calculus:—

- (1) $1' = 0, \quad 0' = 1;$
- (2) $1 = 1 + a = 1 + a + b = 1 + a + b + c, \text{ \&c.};$
- (3) $(ab + a'b')' = a'b + ab',$
 $(a'b + ab')' = ab + a'b';$
- (4) $a : a + b : a + b + c, \text{ \&c.};$
- (5) $(a + A)(a + B)(a + C) \dots = a + ABC \dots;$
- (6) $(a : b) : a' + b;$
- (7) $(a = b) = (a : b)(b : a);$
- (8) $(a = b) : ab + a'b';$
- (9) $(A : a)(B : b)(C : c) \dots : (ABC \dots : abc \dots);$
- (10) $(A : a)(B : b)(C : c) \dots : (A + B + C + \dots : a + b + c + \dots);$
- (11) $(A : x)(B : x)(C : x) \dots = (A + B + C + \dots : x);$
- (12) $(x : A)(x : B)(x : C) \dots = (x : ABC \dots);$
- (13) $(A : x) + (B : x) + (C : x) + \dots : (ABC \dots : x);$
- (14) $(x : A) + (x : B) + (x : C) + \dots : (x : A + B + C + \dots).$

RULE 19.—To test the equivalence of any two statements; say

$$f(a, b, c, \dots) \text{ and } \phi(a, b, c, \dots).$$

* It is easy to show that the only condition necessary for any octagonal frame, with 4 diagonal bars joining opposite vertices, to be capable of internal stress, is that the 4 intersections of each pair of opposite sides shall be in *directum*.