IX.—On the Conditions for the Reversibility of the Order of Partial Differentiation. By W. H. Young, Sc.D., F.R.S.
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§ 1. IN the case of a function of two variables $f(x, y)$ there is, in general, no connection between the results of proceeding to the various limits, upper, lower, and intermediate, first with respect to x and then with respect to y , and first with respect to y and then with respect to x . We cannot even assert that the repeated upper limit obtained in one way is not less than the repeated lower limit obtained in the other way.

The so-called necessary and sufficient conditions for the equality of the two repeated limits do little more than express in ϵ -language the fact of the equality. There is, however, one simple case in which we can assert with confidence the existence of the repeated limit, this is, when a unique double limit exists.

It is no accident, therefore, that the simplest and best-known set of conditions for the reversibility of the order of partial differentiation—that associated with the names of Dini and Schwarz—is based on this property.

The main object of the first of the three parts of the present paper is to obtain these Schwarz-Dini conditions in a more precise form than has yet been given to them. It is found that the axial cross through the point (a, b) considered may be omitted in obtaining the limit of the values of $\frac{d}{dy} \cdot \frac{df}{dx}$ in the neighbourhood of the point (a, b) , and indeed that the existence of $\frac{d}{dy} \cdot \frac{df}{dx}$ on the axial cross is nowhere required for the purposes of the proof, except at the point (a, b) itself.

It is hoped that the mode of proof adopted, in which the ϵ -method is avoided, will be found somewhat simpler and more concise than the usual mode of presentation.

In the second part of the paper I obtain certain properties of the repeated partial differential coefficients, or, more generally, of the derivatives with respect to one variable of the differential coefficient, or of the derivatives, with respect to the other variable. I am thus led to state

and prove certain extensions of the Schwarz-Dini conditions considerably less narrow than those previously given.

In the third part of the paper I obtain another set of sufficient conditions which I believe to be new both in the form given to them and in the mode of proof. These are as follows:—

If both $\frac{d^2f}{dx^2}$ and $\frac{d^2f}{dy^2}$ are continuous functions of the ensemble (x, y) at the point (a, b) and both $\frac{d}{dy} \cdot \frac{df}{dx}$ and $\frac{d}{dx} \cdot \frac{df}{dy}$ exist at the point (a, b) , then these two latter repeated partial differential coefficients are equal at (a, b) .

Stolz in his *Grundzüge der Differenzial- und Integralrechnung*, vol. i. pp. 141-147, has given a set of conditions, also due to Schwarz, included in mine as a very special case. The additional demands made by Schwarz on the function consist of (1) the existence of $\frac{d}{dy} \cdot \frac{df}{dx}$ and $\frac{d}{dx} \cdot \frac{df}{dy}$ in a closed neighbourhood of the point (a, b) , and (2) the continuity of $\frac{d}{dy} \cdot \frac{df}{dx}$ with respect to y and of $\frac{d}{dx} \cdot \frac{df}{dy}$ with respect to x at the point (a, b) .

The interesting question then arises: What is the relation of these conditions to those for the equality of two repeated limits? The method of proof adopted shows that in this case also the equality virtually depends on the existence of a unique double limit.

It may be noted also that in their relation to the repeated partial differential coefficients $\frac{d}{dy} \cdot \frac{df}{dx}$ and $\frac{d}{dx} \cdot \frac{df}{dy}$, or to the repeated derivatives, the new conditions are less narrow than the Schwarz-Dini conditions. They do not demand even the continuity of the derivatives of derivatives at the point, still less the existence and continuity of $\frac{d}{dy} \cdot \frac{df}{dx}$ in the neighbourhood of the point considered.

PART I.

§ 2. With the object of rendering what follows as complete in itself as possible, we begin by formulating a few properties in the theory of limits, on the proper grasp of which the understanding of the subject hinges. We have first to explain exactly what we mean by a double and a repeated limit.

If $x_1, x_2, \dots, x_n, \dots$ be a series of values of a variable x , forming

a sequence with a as limiting value, the value a being expressly excluded from being a possible value for any x^n , and further, the series

$$f(x_1), f(x_2), \dots f(x_n), \dots$$

has only one limiting value u (which may be $+\infty$ or $-\infty$), then we say that u is one of the single limits of $f(x)$ at a , and write

$$u = \lim_{x \rightarrow a} f(x) = \text{one of the } \lim_{x \rightarrow a} f(x).$$

The function $f(x)$ may or may not be defined for the value $x=a$; but, if defined, the value $f(a)$ must be disregarded in considering the limits at a .

If $f(x, y)$ is a function of two independent variables x and y , it becomes a function of x alone when we keep y constant, and has a corresponding set of simple limits,

$$\lim_{x \rightarrow a} f(x, y).$$

If there is only one such limit for each value of y ,* this limit defines a function of y , and has, as such, a set of limits for $y=b$; these are called *the repeated limits of $f(x, y)$ first with respect to x and then with respect to y* , and written

$$\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y).$$

Similarly, if there is only one limit when y is kept constant,

$$\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y)$$

denotes *the repeated limits of $f(x, y)$ first with respect to y and then with respect to x* .

The idea of *double limits* of $f(x, y)$ is different.

Let $x_1, x_2, \dots$ be a sequence of values of x having a as limit, but not including a , and $y_1, y_2, \dots$ a sequence of values of y having b as limit, but not including b , then $f(x, y)$ has at the points $(x_1, y_1), (x_2, y_2), \dots$

* The same is true if a law is given by means of which one of the limits is defined for each value of y ; e.g. the maximum limit. In this case a number of quantities can be identified as limits of limits of $f(x, y)$, or repeated limits; e.g. the upper upper limit, the upper lower limit, etc. The whole set of such limits, which may theoretically be denoted by

$$\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y),$$

can only be regarded as *perfectly* defined when all possible laws which can be used are in some manner specified.

a series of values and at (a, b) a corresponding set of limits.* If one of these is A (which may be $+\infty$ or $-\infty$), then A is said to be a *double limit* of $f(x, y)$ for the ensemble of values (a, b) .

If the variables x and y are continuous, we may regard the ensemble (x, y) as a point of the plane. The points $(x_1, y_1), (x_2, y_2), \dots$ then form a sequence with the point (a, b) as limiting point. They do not, however, form the most general such sequence, since the individual points are explicitly excluded from lying on the axial cross

$$\begin{aligned}x &= a, \\y &= b,\end{aligned}$$

through the limiting point.†

Lemma 1.—*Any repeated limit is a double limit.*

For convenience we shall use $F(m, n)$ for $f(x_m, y_n)$.

By the definition of a repeated limit there is then a sequence of values of $F(m, n)$, keeping n constant, having a unique limit $v(n)$, when m is indefinitely increased; and the quantities $v(n)$, as n is indefinitely increased, have the repeated limit, say u , as one of their limits; that is

$$\begin{aligned}v(n) &= \lim_{m \rightarrow \infty} F(m, n) \\u &= \text{one of the } \lim_{n \rightarrow \infty} v(n).\end{aligned}$$

First consider the quantities $v(n)$ to be all finite.

Represent the values of the various functions on a straight line,

$$\begin{aligned}f(m, n) &\text{ by the point } P_{m, n}, \\v(n) &\text{ by the point } Q_n, \\u &\text{ by the point } Q.\end{aligned}$$

Then, under the given conditions, Q is the limiting point of the sequence $Q_1, Q_2, \dots$ and Q_n is, for each value of n , the limiting point of the sequence $P_{1, n}, P_{2, n}, \dots$

Since all these points, except possibly Q , are finite points, any interval d containing Q as internal or end-point determines a point Q_n where n is the first integer greater than some chosen number such that Q_n lies inside d . The interval d then determines a point $P_{m, n}$ such that n is the first integer greater than a chosen number, and such that $P_{m, n}$ lies inside d .

Taking a sequence of such intervals d_1, d_2 having Q as sole common

* These may be regarded as falling under the heading of simple limits, since they are $\lim_{n \rightarrow \infty} F(n)$, where $F(n) = f(x_n, y_n)$.

† It is unnecessary here to enter into the modifications necessary when the point (a, b) is at infinity.

internal or end-point, we get in this manner a constantly increasing set of integers $m_1, m_2, \dots$ and another $n_1, n_2, \dots$ such that Q is the sole limiting point of the points P_{m_i, n_i} . The corresponding series $F(m_i, n_i)$ has then u for limit, which shows that u is a double limit of $F(m, n)$.
Q.E.D.

If a finite number of the quantities $v(n)$ are infinite, we can omit them from consideration. If, however, this is not the case, u is $+\infty$ or $-\infty$ and is equal to all but a finite number of the $v(n)$'s. In this trivial case it is clear that, corresponding to any $v(n)=u$, we can find an integer m such that $F(m, n)$ is numerically greater than any chosen quantity, and in this way we can construct a series $F(m_i, n_i)$ having u as limit.

Thus in any case the theorem is demonstrated.

Lemma 2.—If $f(x, y)$ has only one double limit at a point (a, b) , and the simple limit

$$\text{Lt}_{h=0} f(a+h, b+k)$$

is unique for all values of k in a certain neighbourhood of the zero point, then the repeated limit

$$\text{Lt}_{k=0} \text{Lt}_{h=0} f(a+h, b+k)$$

exists.

This is an immediate consequence of Lemma 1.

N.B.—If the simple limit is not unique, it follows from Lemma 1 that, however we define the limiting function as a function of y , all its limits are double limits of $f(x, y)$. Hence if, as in Lemma 2, $f(a+h, b+k)$ has only one double limit, all the limits of all possible limiting functions coincide. In particular the upper upper and lower lower limits are equal.

NOTE 1.—It may evidently happen that even when $\text{Lt}_{x=a} f(x, y)$ does not exist as a unique limit, the upper and lower limits, and therefore all intermediate limits, have one and the same definite limit as y approaches b . If we agree with some writers to call this the repeated limit, it is evident that Lemma 2 still holds, omitting the assumption as to the uniqueness of the simple limit.

NOTE 2.—The existence of a unique limit of $f(x, y)$ for each fixed value of x does not, of course, involve the existence of a unique limit when y varies with x , which would be concomitant to the existence of a unique double limit.

§ 3. A rectangle whose axes are parallel to the axes of coordinates, and whose lower left-hand and upper right-hand corners are respec-

tively the points (a, b) , $(a+h, b+k)$, will be referred to as the rectangle $(a, b; a+h, b+k)$, and its double incrementary ratio defined as follows:—

$$m(a, b; a+h, b+k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}. \quad (1)$$

Here h and k may have any positive or negative values.

Using $m_x(b, b+k)$ for the single incrementary ratio of $f(x, y)$ when x is constant,

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k}. \quad (2)$$

we clearly have the following identity:—

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h}. \quad (3)$$

so that $m(a, b; a+h, b+k)$ is the incrementary ratio with respect to x of the incrementary ratio with respect to y of $f(x, y)$, and similarly it is the incrementary ratio with respect to y of the incrementary ratio with respect to x .

If in (2), keeping x constant, we let k approach zero in any manner, and there is only one limit, this is, by definition, the partial differential coefficient with respect to y , $f_y(x, b)$, or $\frac{df}{dy}$. Forming the single incrementary ratio of $f_y(x, b)$ for the pair of values $x=a$ and $x=a+h$, if this has a unique limit as h approaches zero, this is, by definition, the repeated differential coefficient with respect first to y and then to x , and will be denoted by f_{xy} or $\frac{d}{dx} \cdot \frac{df}{dy}$ or $\frac{d^2f}{dxdy}$.

If f_y exists, but its single incrementary ratio has more than one limit, the various limits are the various derivatives at the point (a, b) of f_y , and may be called the repeated derivatives of $f(x, y)$ first with respect to y and then to x .

If f_y does not exist, the upper and lower left- and right-hand derivatives which appear instead form functions whose derivatives with respect to x may be treated as the repeated derivatives of $f(x, y)$ first with respect to y and then to x .

The following theorem will for convenience be referred to as the Repeated Theorem of the Mean.

The Repeated Theorem of the Mean.

If $f(x, y)$ is a finite continuous function of x at every point of the closed rectangle $(a, b; a+h, b+k)$, and has, except possibly on the bounding ordinates,

$$x=a \text{ and } x=a+h,$$

a differential coefficient f_x with respect to x , which is itself a finite continuous function of y with, at every internal point of the rectangle, a differential coefficient

$$f_{yx}(x, y) = \frac{d}{dy} \cdot \frac{df}{dx} = \frac{d^2f}{dydx},$$

then there is an internal point of the rectangle at which the repeated differential coefficient $f_{yx}(x', y')$ is equal to the double incrementary ratio of the rectangle, viz.

$$m(a, b; a+h, b+k) = f_{yx}(x', y').$$

For, applying the Theorem of the Mean to the identity (3), we get

$$m(a, b; a+h, b+k) = \frac{d}{dx} m_x(b, b+k) \text{ at some point } x' \text{ between } a \text{ and } (a+h) \text{ not inclusive}$$

$$= \frac{f_x(x', b+k) - f_x(x', b)}{k},$$

whence the result follows by a second application of the Theorem of the Mean, since f_x is continuous with respect to x on the sides $y=b$ and $y=b+k$ of the rectangle.

NOTE.—Assuming the simple result that, when $f(x, y)$ is a continuous function of the ensemble (x, y) the double incrementary ratio assumes at points internal to a rectangle $(a, b; a+h, b+k)$ every value between its upper and lower bounds,* this shows that if f_{yx} then exists and is finite at every internal point of this rectangle it also assumes every value between its upper and lower bounds.

§ 4. Theorem 1 (on the existence of f_{yx} at the point (a, b)).—If f_x exists in a closed neighbourhood of a point (a, b) , while in the completely open neighbourhood, excluding the axial cross,† it has a differential coefficient f_{yx} with respect to y , then, if f_{yx} has only one double limit as we approach the point (a, b) in any manner by means of points not on the axial cross, f_{yx} exists also at the point (a, b) itself.

For, by the Repeated Theorem of the Mean,

$$m(a, b; a+h, b+k) = f_{yx}(x', y'),$$

where the point (x', y') does not lie on the axial cross, and has (a, b) as limiting point when h and k each approach zero in any manner without assuming the value zero.

Since $f_{yx}(x, y)$, and therefore $f_{yx}(x', y')$, has only one double limit at the point (a, b) , the same is true of $m(a, b; a+h, b+k)$ when h and k have

* A proof of this is given in § 7 below.

† That is, not on $x=a$ nor on $y=b$.

zero as limit. Hence, a repeated limit being a double limit (Lemma 1), all the repeated limits of $m(a, b; a+h, b+k)$ are equal.

But, by hypothesis, f_x is defined on the ordinate $x=a$, so that

$$\frac{f_x(a, b+k) - f_x(a, b)}{k} = \lim_{h=0} m(a, b; a+h, b+k),$$

and has, therefore, by what has been shown, only one limit when k has zero as limit; that is to say, f_x has a differential coefficient f_{yx} with respect to y at the point (a, b) .

NOTE.—It should be noticed that it does not follow that f_{yx} is continuous at (a, b) with respect to either variable, still less with respect to the ensemble (x, y) . In fact, f_{yx} need not exist on the axial cross, except at the point (a, b) itself. It will be proved below (§ 8, Theorem 7, Cor. 2), however, that when it does exist on the axial cross, it is continuous at (a, b) .

Theorem 2 (the Dini-Schwarz Theorem).—*If in addition to the requirements of the preceding theorem, f_y exists along the line $y=b$ at and in the neighbourhood of the point (a, b) , then f_{xy} also exists at the point (a, b) and has the same value as f_{yx} .*

For, in this case,

$$\frac{f_y(a+h, b) - f_y(a, b)}{h} = \lim_{k=0} m(a, b; a+h, b+k),$$

and has, therefore, as was shown in the preceding proof, only one limit when h has zero as limit; that is to say, f_y has a differential coefficient f_{xy} with respect to x at the point (a, b) .

Since the value of $f_{xy}(a, b)$, like that of $f_{yx}(a, b)$, is thus the unique double limit of $m(a, b; a+h, b+k)$,

$$f_{xy}(a, b) = f_{yx}(a, b). \quad \text{Q.E.D.}$$

NOTE 1.—It has nowhere been assumed that the unique limit postulated is finite; it may be $+\infty$ or $-\infty$.

NOTE 2.—The arguments used in proving the Repeated Theorem of the Mean, as well as Theorems 1 and 2, which depend on it, being based on the Theorem of the Mean in one dimension, do not require the full assumption that f_{yx} exists at every point internal to the rectangle. *It suffices, in fact, if there is no distinction of right and left* with respect to the derivatives of f_x regarded as a function of y , as follows*

* Thinking of the representation in two dimensions, right and left with respect to y is, of course, "up and down."

from the more general statement of the Theorem of the Mean given in the paper "On Derivates and the Theorem of the Mean," by W. H. and G. Chisholm Young, *Quart. Journal of Math.*, Oct. 1908.

NOTE 3.—It should be noticed that, without making any properly two-dimensional hypothesis, *we can prove that f_{yx} exists at the point (a, b) if we postulate that f_{yx} exists and is finite along the ordinate $x=a$ in some open neighbourhood of the point (a, b) but not at the point itself, and that it has a unique limit as we approach the point (a, b) along that ordinate.*

This is an immediate consequence of the Theorem of the Mean for a single variable applied to f_x regarded as a function of y at points of the ordinate $x=a$.

For this reason, as well as from the fact that the axial cross can, as we have seen, be omitted, the usual statements of the Schwarz-Dini conditions seem to leave something to be desired.*

It may further be remarked that in Note 2 it is sufficient if the derivates of f_x with respect to y on the ordinate $x=a$ present no distinction of right and left.† All these sets of conditions are, of course, sufficient but not necessary. It is obvious that sufficient conditions of a less restricted character can be formulated; some of these will be found below.

NOTE 4.—In the proof of the existence of a unique double limit for $m(a, b; a+h, b+k)$, the assumption (1) that f_x exists in the open neighbourhood of the point (a, b) , excluding the ordinate $x=a$, was rendered necessary in order to apply the Repeated Theorem of the Mean.

This being postulated, the further assumption (2) that f_x also exists on the ordinate $x=a$, at and in the neighbourhood of the point (a, b) , is needed in order to ascribe a meaning to the expression

$$f_x(a, b+k) - f_x(a, b)$$

and so to prove the existence of f_{yx} at the point (a, b) .

Similarly, without postulating (2), the existence of f_{xy} requires the assumption (3) that f_y exists on the line $y=b$, at and in the neighbourhood of the point (a, b) .

* Stolz, *Grundzüge der Differenzial- und Integralrechnung*, 1893, p. 147. Hobson, "Partial Differential Coefficients and Repeated Limits," *Proc. L.M.S.*, 1906, series ii., vol. v. p. 234. See also *Functions of a Real Variable*, p. 318 and the errata. It should be noticed that the account in the book is really earlier than that in the paper.

† See footnote on preceding page.

The assumptions (2) and (3) are independent, and precisely of the same importance in the Schwarz-Dini conditions. Both could be omitted if we suitably enlarged our definition of a repeated differential coefficient, for without them we could still prove that the double incrementary ratio had a unique limit.*

PART II.

§ 5. We now proceed to entirely drop the assumption as to the existence of a differential coefficient with respect to y of $\frac{df}{dx}$, and begin by proving certain properties of the derivatives of $\frac{df}{dx}$ with respect to y .

The following lemma is an immediate consequence of the definitions and Lemma 1.

Lemma 3.—*If f_x exists at a point (a, b) and on its ordinate in the neighbourhood, any derivate of f_x with respect to y is a double limit of $m(a, b; a+h, b+k)$.*

For, putting

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k}$$

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h}.$$

Therefore, by the definition of a differential coefficient,

$$\lim_{h=0} \lim_{k=0} m(a, b; a+h, b+k) = \frac{d}{dx} m_x(b, b+k) \text{ at } x=a,$$

$$= \frac{f_x(a, b+k) - f_x(a, b)}{k}.$$

Hence, by the definition of the derivatives,

$\lim_{k=0} \lim_{h=0} m(a, b; a+h, b+k)$ = the derivatives of f_x with respect to y at the point (a, b) .

Thus these derivatives are repeated limits, and therefore, by Lemma 1, double limits of $m(a, b; a+h, b+k)$. Q.E.D.

Lemma 4.—*If f_x exists at every point of an open rectangle $(a, b; a+h, b+k)$, excluding the bounding ordinates, and is a finite continuous function of y at every internal point, then, if $f(x, y)$ still exists and is a continuous function of y on the boundary,*

$$L \leq m(a, b; a+h, b+k) \leq U,$$

* It should be noticed that the difference between Hobson's conditions, *loc. cit.*, and those given by Schwarz consists in the omission of assumption (3), and that this is only possible in the light of the extended definitions suggested by Hobson. If such a definition be adopted, there is, it would appear, no reason for making the assumption (2) either.

where L and U are the lower and upper bounds of any derivate of f_x with respect to y at points internal to the rectangle.

For, since

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h},$$

and the conditions of the Theorem of the Mean are satisfied,

$$m(a, b; a+h, b+k) = \frac{f_x(x', b+k) - f_x(x', b)}{k} \quad \text{where } a < x' < a+h,$$

= incrementary ratio of function $f_x(x', y)$ at the pair of
values $y = b, y = b+k$.

Hence

$$L_x \leq m(a, b; a+h, b+k) \leq U_x$$

where L_x and U_x are the bounds of this incrementary ratio at pairs of values of y internal to the interval $(b, b+k)$. Since these bounds are the same as those of any derivate of $f_x(x', y)$ with respect to y at points internal to the same interval (that is, at certain points internal to the rectangle), we have, *a fortiori*,

$$L \leq m(a, b; a+h, b+k) \leq U. \quad \text{Q.E.D.}$$

Theorem 3.—If f_x exist in a closed neighbourhood of the point (a, b) , and any derivate of f_x with respect to y has only one double limit as we approach the point (a, b) by means of points not on the axial cross, then f_x has a differential coefficient f_{yx} with respect to y at the point (a, b) .

For, by Lemma 4,

$$L \leq m(a, b; a+h, b+k) \leq U,$$

where L and U are the lower and upper bounds of the derivate in question in the completely open rectangle $(a, b; a+h, b+k)$.

But, under the given condition, L and U evidently have the same limit when h and k each approach zero in any manner whatever. Hence $m(a, b; a+h, b+k)$ has only one double limit, viz. the common limit of L and U .

Further, f_x being defined on the ordinate $x=a$, the derivatives of f_x with respect to y are, by Lemma 3, double limits of $m(a, b; a+h, b+k)$. They are therefore all equal to the unique double limit of $m(a, b; a+h, b+k)$, so that f_x has a differential coefficient at (a, b) with respect to y .

NOTE.—If f_x exist on the ordinate $x=a$, and any derivate with respect to y of f_x is continuous with respect to y at the point (a, b) , then f_{yx} exists at the point (a, b) .

This follows, of course, from the well-known theorem in one dimension.

§ 6. I now proceed to give the property which may be regarded as constituting the *raison d'être* of the sets of sufficient conditions for the reversibility of the order of partial differentiation with which we are at present occupied, in the case when f_x and f_y exist throughout an area. This property is that *all the derivatives of f_x with respect to y , and all the derivatives of f_y with respect to x , have the same upper and lower bounds in every completely open area, and these bounds are the same as those of the double incrementary ratio in the same area.* This follows at once from the following theorem, which results immediately from Lemmas 3 and 4.

Theorem 4.—*The upper and lower bounds of all the derivatives of f_x at all the internal points of a closed rectangle with sides parallel to the axes, throughout which f_x exists, and at every internal point of which f_x is a continuous function of y , are the same and are equal to those of the double incrementary ratio $m(x, y; x', y')$ at pairs of points internal to that rectangle.*

For, if (a, b) is internal to the rectangle, so is a portion of its ordinate. Hence any derivative of f_x with respect to y at (a, b) , being a repeated limit of $m(a, b; a+h, b+k)$, $\geq$ lower bound of $m(x, y; x', y')$, since we can restrict h and k so that the point $(a+h, b+k)$ is also internal to the rectangle considered. Therefore

$$\text{lower bound of derivate} \geq \text{lower bound of } m(x, y; x', y'). \quad (1).$$

But, by Lemma 4,

$$\text{lower bound of derivate} \leq m(x, y; x', y')$$

when the points (x, y) , and (x', y') and $\therefore$ the rectangle $(x, y; x', y')$, are internal to the region. Therefore

$$\text{lower bound of derivate} \leq \text{lower bound of } m(x, y; x', y'). \quad (2).$$

From (1) and (2) the equality of the lower bounds of the derivate and the double incrementary ratio follows. Similarly, the equality of the upper bounds may be proved.

Cor. 1.—*The bounds of the double incrementary ratio in any rectangle $(a, b; a+h, b+k)$ are unaltered if we include in the rectangle any or all of its boundary points.*

For the argument used in proving (2) is equally valid if the points (x, y) and (x', y') are on the boundary. Hence

$$\begin{aligned} \text{lower bound of derivate in the open rectangle} &\leq \text{lower bound of } m(x, y; x', y') \\ &\text{in the closed rectangle.} \end{aligned}$$

But we must take the sign of equality, since the lower bound of

$m(x, y; x', y')$ in the closed rectangle is $\geq$ that in the open rectangle, and therefore, by (1), $\geq$ the lower bound of derivate in the open rectangle.

Cor. 2.—*The bounds of the derivatives of f_x with respect to y in any rectangle $(a, b; a+h, b+k)$ are unaltered if we include the left-hand and the right-hand bounding ordinates of the rectangle, and the bounds of the right-hand (left-hand) derivatives are unaltered if we include the lower (upper) bounding lines.*

For in any of these cases the argument used in proving (1) holds. Thus the result (1) still holds. The lower bound of the derivate having certainly not been increased by the introduction of the new points, (2) also holds, whence it follows that the lower bound of the derivate is still equal to that of the double incrementary ratio.

The results of this theorem and its corollaries may be shortly summed up in the following general statement:—

The bounds of any derivate of f_x in any rectangle $(a, b; a+h, b+k)$, open or closed, are the same as those of the double incrementary ratio in the same rectangle, including or not some or all of its boundary points.

Here the expression “derivate of f_x in a rectangle” is to be so understood that only such derivatives are included at boundary points as result from operations in the rectangle; e.g. at the left-hand bottom corner, only the right-hand derivatives with respect to y of f_x .

Cor. 3.—*The bounds of any derivate of f_x and of the double incrementary ratio in any neighbourhood of a point P are unaltered if we omit the axial cross through P .**

Cor. 4.—*All the derivatives of f_x have the same associated plane limiting functions ϕ and ψ , and these are the same whether the values at the point itself be included or not, and are still the same if in calculating ϕ and ψ we omit the axial cross through P .*

In particular, therefore, all the derivatives of f_x lie between their ϕ and ψ , and if one of these derivatives is continuous with respect to the ensemble (x, y) , so are they all, and they are all equal. Moreover, the same is true if it is only known that one of these derivatives has a unique limit as we approach the point by points not lying on the axial cross.

§ 7. The fact that the bounds of the double incrementary ratio in a rectangle $(a, b; a+h, b+k)$ are unaltered, when we include some or all of its boundary points, is at first sight remarkable, considering that we have not made the assumption that $f(x, y)$ is continuous with respect to the

* Or any isolated set of axial crosses; that is, any set of axial crosses such that, taking any point P internal to the region considered, we can find a region containing P as internal point and intersected by at most one of the axial crosses.

The points of the wedge in the last two hyperplanes, which may be called its basal hyperplanes, do not come into consideration, since $m(x, y; z, w)$ is undefined there.

Let k be any value between the upper and lower bounds (not included) of $m(x, y; x', y')$ in the primary closed rectangle. Then we can find two finite values k_1 and k_2 , both of which are *assumed* by the double incrementary ratio, where

$$k_1 < k < k_2.$$

By what precedes there is a point P_1 of the four-dimensional wedge (5) at which

$$m(x, y; z, w) = k_1,$$

and a point at which

$$m(x, y; z, w) = k_2.$$

If this second point is not on one and the same hyperplane boundary as P_1 , let it be P_2 . In the contrary case take as P_2 a neighbouring point inside the wedge at which

$$m(x, y; z, w) = k'_2,$$

where

$$k < k'_2;$$

this is possible by reason of the continuity of $m(x, y; z, w)$.

The stretch P_1P_2 will then be such that every one of its internal points is internal to the wedge, so that $m(x, y; z, w)$ is definite at every internal point of P_1P_2 , and has at the end-points the definite values k_1 and k_2 (or k'_2).

At every point of the closed stretch P_1P_2 , therefore, $m(x, y; z, w)$ is a continuous function of the ensemble (x, y, z, w) , and therefore assumes at some point internal to P_1P_2 , and therefore to the wedge, the value k intermediate between its values at the end-points P_1 and P_2 . This proves the theorem.*

Cor.—*The upper and lower bounds of the double incrementary ratio in the rectangle are unaltered if we omit any or all the boundary points from the possible positions of the points (x, y) and (x', y') .*

§ 8. Returning to § 6, the result of Cor. 4 is sufficiently interesting to merit separate proof.

Theorem 6.—*If f_x exists throughout a closed area and is a continuous function of y , the value of any derivate of f_x with respect to y at any internal point P of the area lies between the upper and lower bounds of the derivate in a non-axial† neighbourhood of P .*

* The extension of this result to the case when $f(x, y)$ is a quasi-finite function, which, in the case of one variable, was given in the paper quoted on p. 144, line 2, is not elaborated here, the attention being concentrated in this paper, in the first instance, on finite functions.

† That is, omitting the axial cross $x=a$ and $y=b$.

Further, the upper and lower bounds in the non-axial neighbourhood are the same as those in the closed neighbourhood of P.

For

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h} = \frac{d}{dx} m_{a+\theta h}(b, b+k) \\ = \frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k}$$

where $0 < \theta < 1$.

Now this latter single incrementary ratio lies between the upper and lower bounds of the derivatives of $f_x(a+\theta h, b+\theta'k)$, where $0 < \theta' < 1$, θh being constant; *a fortiori*, between the bounds of the same when θh is variable; and again, *a fortiori*, between the upper and lower bounds of any derivate of f_x at all points (x', y') of the neighbourhood not on the axial cross.

This being so, it is true of the unique limit of $m(a, b; a+h, b+k)$ with respect to h , that is, of $\frac{f_x(a, b+k) - f_x(a, b)}{k}$, and therefore of all the limits of this last incrementary ratio, that is, the derivatives of f_x at the point (a, b) .

To prove the second part of the theorem, we only have to notice that the value of any such derivate of f_x on the axial cross being a limit of values in the non-axial neighbourhood, lies between the bounds in that neighbourhood.

Cor. 1.—If ϕ and ψ are the associated plane limiting functions of any derivate of f_x , that derivate lies between ϕ and ψ at every point, and ϕ and ψ are the same whether defined in Baire's* manner or that usually adopted by myself† or calculated always with reference to a non-axial neighbourhood.

It should be noticed that the corresponding results for the first derivatives, obtained from one-dimensional theory, are less extended. Any derivate with respect to x of $f(x, y)$, when $f(x, y)$ is continuous with respect to x throughout a closed area, lies between its upper and lower bounds in any neighbourhood omitting the ordinate of the point. Hence the omission of the ordinate does not affect the bounds, and the plane associated functions are the same whether defined as by Baire, or in my usual manner, or calculated from neighbourhoods omitting the ordinate.

Cor. 2.—If any derivate of f_x has only one limit in a non-axial neighbourhood of a point P, it is continuous at P with respect to the ensemble (x, y) , and therefore in particular is continuous at P with respect to y .

* Closed neighbourhood.

† Open neighbourhood, omitting values at the point itself.

This corollary, together with the one-dimensional theorem given in the note on Theorem 3, constitutes a proof of Theorem 3. (Cp. also the note at the end of Theorem 1. It is unnecessary to do more than call attention to the special case in which f_x has a differential coefficient with respect to y throughout the non-axial neighbourhood.)

§ 9. Theorem 7.—If $f(x, y)$ is a finite continuous function of x at every point of a closed rectangle $(a, b; a+h, b+k)$, and has, except possibly on the bounding ordinates $x=a$ and $x=a+h$, a differential coefficient f_x with respect to x , which is a finite continuous function of y : (1) there is a point internal to the rectangle at which one lower derivate of f_x with respect to $y \geq m(a, b; a+h, b+k)$, and the upper derivate on the other side $\leq m(a, b; a+h, b+k)$; (2) each of the derivatives of f_x with respect to y assumes values $\geq m(a, b; a+h, b+k)$ and values $\leq m(a, b; a+h, b+k)$.

For

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h} = \frac{d}{dx} m_{a+\theta h}(b, b+k) \\ = \frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k}$$

= incrementary ratio of $f_x(a+\theta h, y)$ regarded as a function of y .

Hence, by Theorem 4, p. 10, of the paper "On Derivates and the Theorem of the Mean," already quoted, the first result follows, and by Theorem 5 of the same paper we get the result (2).

§ 10. We now return to the question touched on as to the definition of the second partial differential coefficients. So far we have followed the usual definition adopted by Schwarz and others. This postulates that $\frac{d}{dx} \frac{df}{dy}$ should only be considered to exist at a point (a, b) if $\frac{df}{dy}$ exists on the abscissa $y=b$ at and in the neighbourhood of the point (a, b) , and similarly that $\frac{d}{dy} \frac{df}{dx}$ should only be considered to exist at the point (a, b) if $\frac{df}{dx}$ exists on the ordinate $x=a$ at and in the neighbourhood of the point (a, b) . This definition was historically inevitable. To speak of a second differential coefficient implies that we are dealing with a differential coefficient of a differential coefficient. Extension of knowledge has, however, always led to an extension of meaning, and there is no *a priori* reason why we should not, if we find it convenient, extend the definition as follows:—

If $\frac{df}{dx}$ exists at the point (a, b) but not in the neighbourhood of (a, b) on the ordinate $x=a$, and yet it happens that the derivatives with respect to x of $f(x, y)$ (upper and lower, left and right) should all four have a

differential coefficient with respect to y , and, further, all these differential coefficients are equal, this common value shall be called the partial differential coefficient $\frac{d^2f}{dydx}$.

Hobson, who has recently adopted this definition,* is thus enabled to omit from the Schwarz-Dini conditions the condition that $\frac{df}{dy}$ should exist on the abscissa $y=b$, except at the point (a, b) itself.

It is evident from § 4, Note 4, of the present paper that with this definition we must, if we are to be consistent, omit the condition included in that specifically retained by Hobson in a gloss on his condition (1),† that $\frac{df}{dx}$ exists in the neighbourhood of the point (a, b) on the ordinate $x=a$.

In fact, the assumption that $\frac{df}{dx}$ exists at the point (a, b) is equivalent to the statement that, whatever sequence of values with zero as limit we ascribe to h ,

$$\frac{f(a+h, b) - f(a, b)}{h}$$

has the same limit.

At any point $(a, b+k)$ let us take a sequence of values of h , giving the upper right-hand derivate of $f(x, y)$ with respect to x as the unique limit of

$$\frac{f(a+h_{k,n}, b+k) - f(a+h_{k,n}, b)}{h_{k,n}}.$$

At the point (a, b) , however, any sequence of values of h will do, by the assumption made. Hence the incrementary ratio of this derivate may be written

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(a+h_{k,n}, b+k) - f(a+h_{k,n}, b)}{h_{k,n}} &= \lim_{n \rightarrow \infty} \frac{f(a+h_{k,n}, b) - f(a, b)}{h_{k,n}} \\ &= \lim_{n \rightarrow \infty} m(a, b; a+h_{k,n}, b+k). \end{aligned}$$

Thus the derivates with respect to y of the upper right-hand derivate of $f(x, y)$ with respect to x are repeated limits, and therefore double limits, of the double incrementary ratio, so that they all coincide at (a, b) if a unique double limit exists.

As pointed out in Note 4, § 4, the existence of $\frac{df}{dx}$ on the ordinate $x=a$ is immaterial in the proof of the existence of a unique limit for the double incrementary ratio. Thus the assumption in question is superfluous in

* *Loc. cit.*

† Line 13, *loc. cit.*

proving that all the derivatives with respect to y of the upper right-hand derivative with respect to x are equal at the point (a, b) . For definiteness the upper right-hand derivative was taken, but it is clear that the same proof applies for any derivative. Thus, without assuming the existence of f_x on the ordinate $x=a$ except at the point (a, b) , all the derivatives with respect to y of all the derivatives with respect to x of $f(x, y)$ are equal at the point (a, b) , so that, in Hobson's sense, the partial differential coefficient $\frac{d^2f}{dydx}$ exists at the point (a, b) .

Adopting Hobson's definition, the set of sufficient conditions for the reversibility of the order of partial differentiation discussed by Schwarz, Dini, and Hobson, reduce, in the light of what proceeds, to the following simpler form:—

If (1) $\frac{d^2f}{dydx}$ exists at all points in a non-axial neighbourhood of the point (a, b) ;

(2) The values of $\frac{d^2f}{dydx}$ in a non-axial neighbourhood of the point (a, b) have a unique limit at (a, b) , finite or infinite;

(3) $\frac{df}{dx}$ not only, as is implied in (1), exists, and is a finite and continuous function of y in a non-axial neighbourhood of (a, b) , but also exists on the abscissa $y=b$, at and in the neighbourhood of (a, b) ;

(4) $\frac{df}{dy}$ exists at the point (a, b) ;

then, in Hobson's sense, $\frac{d^2f}{dydx}$ and $\frac{d^2f}{dxdy}$ both exist and are equal to the unique limit specified in (2).

§ 11. It should, however, be noticed that the retention of the existence of the first differential coefficient in Dr Hobson's definition of the second differential coefficient must be regarded, from the point of view adopted by him, as somewhat arbitrary. It seems not less unreasonable to assert the existence of $\frac{d^2f}{dydx}$ when all the derivatives with respect to y of all the derivatives with respect to x coincide at the point (a, b) , which may, of course, be the case without $\frac{df}{dx}$ existing at (a, b) .

Adopting this definition, if we know that the double incrementary ratio has a unique limit at the point (a, b) , we can easily show that all the derivatives of derivatives of $f(x, y)$ are equal at (a, b) . For, *although the derivatives of derivatives are not necessarily themselves repeated limits of*

$m(a, b; a+h, b+k)$, they lie between the upper and lower double limits of $m(a, b; a+h, b+k)$ when h and k are indefinitely diminished.

To prove this let $v(x, y)$ denote any derivate of $f(x, y)$ with respect to x , e.g. the right-hand upper derivate; then, for each value of y there is a sequence of values of h which determines the value of $v(x, y)$ as a limit, viz.

$$v(x, y) = \lim_{n \rightarrow \infty} \frac{f(x + h_{y,n}, y) - f(xy)}{h_{y,n}}.$$

Therefore

$$\text{the derivatives of } v(xy) \text{ at } (a, b) = \lim_{k \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{k} \left\{ \frac{f(x + h_{k,n}, b+k) - f(x, b+k)}{h_{k,n}} - \frac{f(a + h_{0,n}, b+k) - f(a, b)}{h_{0,n}} \right\}.$$

Changing $h_{k,n}$ in the first fraction into $h_{0,n}$, we get $m(a, b; a + h_{0,n}, b+k)$.

Remembering that the sequence $h_{k,n}$ gave us the highest possible limit for the fraction, this change may diminish and cannot increase the value of the right-hand side of the preceding identity. Hence

$$\begin{aligned} \text{the derivatives of } v(x, y) &\geq \lim_{k \rightarrow 0} \lim_{n \rightarrow \infty} m(a, b; a + h_{0,n}, b+k) \\ &\geq \text{a double limit of } m(a, b; a + h, b+k) \\ &\geq \text{lowest double limit of } m(a, b; a + h, b+k). \end{aligned} \quad (1)$$

Similarly, changing $h_{0,n}$ to $h_{k,n}$, we get

$$\text{the derivatives of } v(x, y) \leq \text{greatest double limit of } m(a, b; a + h, b+k), \quad (2)$$

which proves the statement made above.

Hence it follows that, if the limits of $m(a, b; a+h, b+k)$ are finite, the first derivatives with respect to one variable are continuous functions of the other variable.

In particular, this will be the case at every internal point of a rectangle throughout which $m(x, y; x', y')$ is bounded.

It must here be pointed out that, in speaking of the derivatives of derivatives, and in deducing the above results, we have tacitly assumed that the first derivatives are finite, or quasi-finite. Unless this is the case, their derivatives are not properly defined, and it would only be by introducing fresh conventions that we could apply the above results to such cases. Thus, for instance, if $f(x, y)$ is the sum of a function of x and a function of y , $m(x, y; x', y')$ is zero always, the same is therefore true of the repeated derivatives, where they are defined. If the function of x has an infinite derivate, or differential coefficient anywhere, in particular if it is a non-differentiable function of x , in which case such points are dense

everywhere* on the x -axis, for each such value of x this derivate, regarded as a function of y , has its derivatives undefined. We might, of course, agree still to define its derivatives as having the value zero, since a change in the value of y makes no change in that of the function. In this case the above result would hold in a more extended sense.

§ 12. From the inequality of the preceding article it follows that if $m(a, b; a+h, b+k)$ has a unique double limit, all the derivatives of its derivatives coincide at the point (a, b) ; so that, in the sense indicated at the beginning of this article, we might choose to say that $\frac{d^2f}{dydx}$ and $\frac{d^2f}{dxdy}$ both exist at (a, b) and are equal.

In this sense the conditions for this to be the case, obtained by omitting superfluous conditions from the Schwarz-Dini conditions, are as follows:—

(1) $\frac{d^2f}{dydx}$ exists at all points in a non-axial neighbourhood of the point (a, b) ;

(2) The values of $\frac{d^2f}{dydx}$ in a non-axial neighbourhood of the point (a, b) have a unique limit at (a, b) , finite or infinite;

(3) $\frac{df}{dx}$ exists and is a finite and continuous function of y in a neighbourhood of the point (a, b) not necessarily including any point on the ordinate $x=a$.

§ 13. Hitherto we have always had to make some assumption as to the existence of $\frac{df}{dx}$, although it has been found unnecessary to assume its existence at the point itself or on its ordinate. If we make no assumption as to the existence of $\frac{df}{dx}$, it is still possible to enunciate a set of sufficient conditions for the equality of all the derivatives of derivatives at the point (a, b) , which is a direct generalisation of the Schwarz-Dini conditions.

To do this we require a theorem of the bounds analogous to Theorem 4, and to prove this we require a theorem analogous to Theorem 8. We begin with some preliminary remarks and inequalities.

We had the identities

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k} \quad . \quad . \quad . \quad (1)$$

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h} \quad . \quad . \quad (2)$$

* See a paper by the author "On Non-differentiable Functions," *Mess. of Math.*, September 1908.

Hence we see that the various derivatives of the function $m_x(b, b+k)$ at the point $x=a$ are the various limits of $m(a, b; a+h, b+k)$ with respect to h, k being constant.

There are two special sequences of values of h on the right which need consideration. The first is that which yields the upper derivate with respect to x of $f(a, b+k)$, say $f^{+x}(a, b+k)$, as the limit of

$$\frac{f(a+h, b+k) - f(a, b+k)}{h}.$$

This will yield a limit, or limits, of

$$\frac{f(a+h, b) - f(a, b)}{h}$$

not greater than the upper derivate $f^{+x}(a, b)$, and therefore for $m(a, b; a+h, b+k)$ a limit, or limits, not less than

$$\frac{f^{+x}(a, b+k) - f^{+x}(a, b)}{k}. \quad (3)$$

Hence, by what was said above, this last expression is $\leq$ the upper right-hand derivate of $m_x(b, b+k)$.

The second sequence is that which yields $f^{+x}(a, b)$. The consideration of this sequence shows in like manner that the expression (3) is $\geq$ the lower right-hand derivate of $m_x(b, b+k)$.

Thus the expression (3) lies between the upper and lower right-hand derivatives of $m_x(b, b+k)$.

Similarly, considering the sequences which yield respectively the lower right-hand derivatives, we find that the incrementary ratio of the lower right-hand derivate, that is,

$$\frac{f_{+x}(a, b+k) - f_{+x}(a, b)}{k},$$

lies between the upper and lower right-hand derivatives of $m_x(b, b+k)$.

Similarly, the incrementary ratios of the left-hand derivatives of $f(x, y)$ lie between those of $m_x(b, b+k)$.

§ 14. Theorem 8.—If $f(x, y)$ is a finite continuous function of x at every point of a closed rectangle $(a, b; a+h, b+k)$, while any one of the four derivatives (upper, lower, left, and right) of $f(x, y)$ with respect to x is a finite continuous function of y , except possibly on the bounding ordinates, then any derivate with respect to y of this derivate with respect to x assumes values both $\leq$ and also $\geq$ the double incrementary ratio $m(a, b; a+h, b+k)$ of the rectangle at points internal to the rectangle.

For $m_x(b, b+k)$ is then also finite and continuous with respect to x in

the closed x -interval $(a, a+h)$. We may therefore apply Theorem 5 of the paper "On Derivates," already quoted. Hence any derivate of $m_x(b, b+k)$ assumes at points internal to this x -interval values both $\leq$ and also $\geq m(a, b; a+h, b+k)$, since this is the incrementary ratio of $m_x(b, b+k)$ for the pair of x -values a and $a+h$. But, by the preceding article, the incrementary ratio of the upper right-hand derivate

$$\frac{f^{++}(x, b+k) - f^{++}(x, b)}{k}$$

lies between the upper and lower right-hand derivates of $m_x(b, b+k)$ at every point, and therefore also assumes values both $\leq$ and also $\geq m(a, b; a+h, b+k)$.

Since, by the preceding article, a similar argument applies to any other derivate of $f(x, y)$ with respect to x , this proves the theorem.

Theorem 9.—*Under the same assumptions as in the preceding theorem, the upper and lower bounds of the derivates with respect to y of the chosen derivate with respect to x in the completely open rectangle $(a, b; a+h, b+k)$ are the same as those of the double incrementary ratio $m(x, y; x', y')$.*

For, if L' and U' denote the lower and upper bounds of $m(x, y; x', y')$, it follows from the preceding theorem that the repeated derivate considered assumes values $\geq$ any quantity $< U'$ and values $\leq$ any quantity $< L'$.

Hence

the lower bound of the repeated derivate $\leq L' \leq U' \leq$ upper bound of repeated derivate . . . (1)

But, by § 10, the repeated derivate in question lies at any point (x, y) internal to the rectangle between the bounds of $m(x, y; x', y')$, when the point (x', y') approaches (x, y) as limit. Therefore the repeated derivate at the point (x, y) lies between quantities which themselves lie between L' and U' , so that $L' \leq$ chosen derivate at $(x, y) \leq U'$, and therefore

$L' \leq$ lower bound of repeated derivate $\leq$ upper bound of same $\leq U'$. . . (2)

from (1) and (2) L' is the lower and U' the upper bound of the repeated derivate. Q.E.D.

Cor. 1.—*The bounds of the double incrementary ratio in any rectangle $(a, b; a+h, b+k)$ are unaltered if we include in the rectangle any or all of its boundary points.*

For the argument used in proving (1) is equally valid if L' and U' denote the bounds of $m(x, y; x', y')$ in the closed rectangle. Also, since this change does not increase L' nor decrease U' , (2) still holds; hence the

bounds of $m(x, y; x', y')$ in the closed rectangle are the same as those of the repeated derivate in the open one, and therefore are the same as the bounds of $m(x, y; x', y')$ in the open rectangle.

Cor. 2.—*The bounds of a repeated derivate in a rectangle $(a, b; a+h, b+k)$ are unaltered by the inclusion of boundary points, in as far as at each such boundary point the repeated derivate in question is defined by means of sequences of points lying in the rectangle.*

Thus, for instance, in dealing with the right-hand derivates with respect to y of the right-hand derivates with respect to x , we may include the left-hand bottom corner; and in dealing with the right-hand derivates with respect to y of the left-hand derivates with respect to x , we may include the right-hand bottom corner.

For in this case the values of h and k used in the reasoning of § 10 do not take us out of the rectangle, so that (2) still holds, the bounds both of the derivate and the double incrementary ratio being taken with respect to the rectangle including the boundary points in question. Since, by the preceding corollary, the bounds of the double incrementary ratio are the same as in the open rectangle, (1) also still holds, since the introduction of new points does not increase the lower bound or decrease the upper bound of the repeated derivate.

Hence the bounds of the repeated derivate are the same as those of the double incrementary ratio, and are therefore unaltered.

Cor. 3.—*The bounds of any repeated derivate of $f(x, y)$ and of the double incrementary ratio are unaltered, if we omit the axial cross through P.*

Cor. 4.—*All the derivates with respect to y of any derivate of $f(x, y)$ with respect to x , which is a continuous function of y in a neighbourhood of a point P, have at P the same associated plane-limiting functions ϕ and ψ , and these are the same whether the values at the point itself be included or not, and are still the same if in calculating ϕ and ψ we omit the axial cross through P. In particular, therefore, all the derivates of such derivates lie between their ϕ and ψ , and if one of these repeated derivates is continuous with respect to the ensemble (x, y) , so are they all, and they are equal. Moreover, the same is true if it is only known that one of these repeated derivates has a unique limit as we approach the point by points not lying on the axial cross.*

§ 15. It should be noticed that the restriction in the preceding theorem and corollaries, implied in the fact that we only consider derivates with respect to one variable of a function or derivate *which is itself a continuous function of that variable*, is an essential one. In fact, the

theorems of the paper "On Derivates and the Theorem of the Mean," used in proving those of the present paper, do not in general hold if we omit the condition of continuity.

Thus, for instance, denoting by $w(x)$ the function graphically represented by the bisector of the angle between the axes of x and y from the origin to $x = \frac{1}{3}$ both inclusive, and by the line perpendicular to this through the point $(1, 0)$ from $x = \frac{1}{3}$ not inclusive to $x = 1$ inclusive (fig. 1), the function $w(x)$ does not obey the theorems of the paper quoted, although it is continuous except only at $x = \frac{1}{3}$. In fact, the left-hand derivate is $+1$ from $x = 0$ to $x = \frac{1}{3}$, and is -1 afterwards. But the right-hand derivate,

FIG. 1.

though it agrees with the left-hand one, except at the discontinuity $x = \frac{1}{3}$, has there the value $+\infty$.

From this function it is easy to construct a function of x and y which violates the result of the preceding theorem, owing to the fact that it is not a continuous function of x , or that its derivates with respect to x are discontinuous functions of y .

Example 1.—Let

$$f(x, y) = yw(x).$$

Here the left-hand derivate with respect to x is always either y or $-y$, so that the left-hand derivates with respect to y of the left-hand derivate with respect to x are always finite, being either 1 or -1 . But the right-hand derivates of the same derivate, though agreeing with its left-hand derivates except on the ordinate $x = \frac{1}{3}$, have there the value $+\infty$. Thus the upper bound is in the case of one repeated derivate finite and in another infinite.

Example 2.—In the preceding example $f(x, y)$ was itself discontinuous with respect to x . If we write

$$f(x, y) = xw(y),$$

we get a function which is continuous with respect to x , and which yet

violates the result of Cor. 4 owing to the fact that the first derivatives with respect to x are not continuous with respect to y .

In both these examples the appearance of infinite values for certain of the first derivatives disturbs the existence of certain of the second derivatives. The following example is, for this reason, perhaps more striking.

Example 3.—

$$\begin{aligned} f(x, y) &= y - x, \text{ if } x \leq y, \\ &= x^2 - y^2, \text{ if } x > y. \end{aligned}$$

The region considered is the square $(0, 0; 1, 1)$.

Here $f(x, y)$ is a continuous function of the ensemble (x, y) ; its derivatives are, however, discontinuous.

$$\begin{aligned} f_{-x} = f^{-x} &= -1, \text{ if } x \leq y, \\ &= 2x \text{ if } x > y. \end{aligned}$$

Therefore the right-hand derivatives with respect to y of the left-hand derivative with respect to x is zero everywhere, but the left-hand derivative with respect to y , though zero except on the line $x = y$, has here the value $+\infty$, since it is the limit of

$$\frac{-1 - 2x}{h}.$$

Again,

$$\begin{aligned} f_{+x} = f^{+x} &= -1, \text{ if } x < y, \\ &= 2x \text{ if } x \geq y. \end{aligned}$$

Therefore the left-hand derivative with respect to y of the right-hand derivative with respect to x is zero everywhere, but the right-hand derivative with respect to y , though zero except on the line $x = y$, has here the value $+\infty$. Hence

$$L = 0,$$

while, with an obvious notation for the upper bounds,

$$U_{+y, -x} = U_{-y, +x} = 0, \quad U_{-y, -x} = U_{+y, +x} = +\infty.$$

§ 16. Of course, if all the derivatives with respect to y of some derivative of $f(x, y)$ with respect to x are finite at any point, that first derivative is bound to be continuous at the point with respect to y . Hence, as a special case of Cor. 4, we have the following:—

If $f(x, y)$ is a continuous function of x and of y , and if the derivatives with respect to one variable of one or more derivatives of $f(x, y)$ with respect to the other variable are all finite in a neighbourhood of a point P , they all have the same plane associated limiting functions ϕ and ψ at P , and lie between them.

Hence if one of them is a continuous function of the ensemble (x, y) , so are they all, and they are equal. The same is true if one of them has a unique limit for all modes of approach to the point by means of points not on the axial cross through the point.

PART III.

§ 17. Lemma.—If f_x is a continuous function of the ensemble (x, y) at the point (a, b) , and f_y exists at the point (a, b) , then

$$\frac{f(a+h, b+k) - f(a+h, b)}{k}$$

has a unique double limit f_y for all modes of approach of the point $(a+h, b+k)$ to the point (a, b) such that $\left|\frac{k}{h}\right|$ has not zero for one of its limits.

For

$$\begin{aligned} \frac{f(a+h, b+k) - f(a, b+k)}{h} + \frac{f(a, b+k) - f(a, b)}{k} &= f(a+h, b+k) - f(a, b) \\ &= \frac{f(a+h, b+k) - f(a+h, b)}{k} + \frac{f(a+h, b) - f(a, b)}{h} \end{aligned}$$

Hence, using the Theorem of the Mean,

$$f_x(a+\theta h, b+k)h + (f_y(a, b) + \eta_k)k = \frac{f(a+h, b+k) - f(a+h, b)}{k}k + (f_x(a, b) + \eta'_h)h,$$

where

$$0 < \theta < 1,$$

θ being an otherwise unknown function of h and k ; η_k is a function of k , but not of h , which vanishes with k ; η'_h is a function of h , but not of k , which vanishes with h .

Therefore

$$\left| \frac{f(a+h, b+k) - f(a+h, b)}{k} - f_y(a, b) \right| \leq \left| f_x(a+\theta h, b+k) - f_x(a, b) \right| \left| \frac{h}{k} \right| + |\eta_k| + |\eta'_h| \left| \frac{h}{k} \right|.$$

Let us approach in such a way that

$$\left| \frac{k}{h} \right| \geq m \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where m is any positive quantity, finite or infinite. Then the right-hand side of the last inequality has the limit zero, therefore the same is true of the left-hand side.

Since this is true for all positive values of m , it is true for all modes of approach such that $\frac{k}{h}$ has not zero for one of its limits, which is equivalent to the statement to be proved.

Theorem 10.—If f_{xx} is a continuous function of the ensemble (x, y) at the point (a, b) , and f_{yx} exists at the point (a, b) , then the double incremental ratio has a unique double limit whose value is $f_{yx}(a, b)$ for all modes of approach other than those for which $\left|\frac{k}{h}\right|$ has zero for one of its limits.

For

$$m(a, b; a+h, b+k) = \frac{m_{a+h}(b, b+k) - m_a(b, b+k)}{h},$$

where

$$m_x(b, b+k) = \frac{f(x, b+k) - f(x, b)}{k}.$$

Therefore

$$\begin{aligned} m(a, b; a+h, b+k) &= \frac{d}{dx} m_x(b, b+k) \text{ for some value of } x > a \text{ and } < a+h, \\ &= \frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k} \end{aligned} \quad (1)$$

where

$$0 < \theta < 1,$$

and θ is an otherwise unknown function of h and k .

But since, by hypothesis, $\frac{d}{dx} f_x$ is a continuous function of the ensemble (x, y) at the point (a, b) , and $\frac{d}{dy} f_x$ exists there, we may apply the lemma.

But if $\left|\frac{k}{h}\right|$ has not zero for one of its limits, $\left|\frac{k}{\theta h}\right|$ certainly has not zero for one of its limits, since, whatever function θ is of h and k ,

$$\left|\frac{k}{\theta h}\right| > \left|\frac{k}{h}\right|.$$

Hence the lemma holding for the function $f_x(x, y)$

$$\frac{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)}{k}$$

has a unique limit, and this is $f_{yx}(a, b)$. Hence, by (1) the same is true of $m(a, b; a+h, b+k)$ for all the specified modes of approach of h and k to zero.

Theorem 12.—If f_{xx} and f_{yy} are both continuous functions of the

ensemble (x, y) at the point (a, b) , and f_{xy} and f_{yx} both exist at the point (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b),$$

and $m(a, b; a+h, b+k)$ has a unique double limit equal to both these repeated partial differential coefficients.

For, since f_{xx} is a continuous function of (x, y) , and f_{yx} exists at the point (a, b) , it follows by the preceding theorem that $m(a, b; a+h, b+k)$ has a unique limit for all modes of approach for which $\left|\frac{k}{h}\right|$ has not the limit zero, and this limit is $f_{yx}(a, b)$.

Again, since f_{yy} is a continuous function of (x, y) , and f_{xy} exists at the point (a, b) , it follows that $m(a, b; a+h, b+k)$ has a unique double limit for all modes of approach for which $\left|\frac{h}{k}\right|$ has not the limit zero, and that this limit is $f_{xy}(a, b)$.

Hence $f_{yx}(a, b) = f_{xy}(a, b) = \text{unique limit of } m(a, b; a+h, b+k) \text{ for all modes of approach which do not make either } \left|\frac{k}{h}\right| \text{ or } \left|\frac{h}{k}\right| \text{ have the limit zero.}$

But for modes of approach which make $\left|\frac{k}{h}\right|$ have zero for a limit, $m(a, b; a+h, b+k)$ has, by the above, the limit $f_{xy}(a, b)$, and for modes of approach which make $\frac{h}{k}$ have the limit zero, it has $f_{yx}(a, b)$ for limit. Hence for all modes of approach $m(a, b; a+h, b+k)$ has the unique limit $f_{yx}(a, b) = f_{xy}(a, b)$.

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