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I. *Notes on the Theory of Lubrication.*

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MODERN views respecting mechanical lubrication are founded mainly on the experiments of B. Tower†, conducted upon journal bearings. He insisted upon the importance of a complete film of oil between the opposed solid surfaces, and he showed how in this case the maintenance of the film may be attained by the dragging action of the surfaces themselves, playing the part of a pump. To this end it is "necessary that the layer should be thicker on the ingoing than on the outgoing side"‡, which involves a slight displacement of the centre of the journal from that of the bearing. The theory was afterwards developed by O. Reynolds, whose important memoir § includes most of what is now known upon the subject. In a later paper Sommerfeld has improved considerably upon the mathematics, especially in the case where the bearing completely envelops the journal, and his exposition ‖ is much to be recommended to those who wish to follow the details of the investigation. Reference may also be made to Harrison ¶, who includes the consideration of compressible lubricants (air).

* Communicated by the Author.

† Proc. Inst. Mech. Eng. 1883, 1884.

‡ British Association Address at Montreal, 1884; Rayleigh's Scientific Papers, vol. ii. p. 344.

§ Phil. Trans. vol. 177. p. 157 (1886).

‖ *Zeitschr. f. Math.* t. 50. p. 97 (1904).

¶ Camb. Trans. vol. xxii. p. 39 (1913).

In all these investigations the question is treated as two-dimensional. For instance, in the case of the journal the width—axial dimension—of the bearing must be large in comparison with the arc of contact, a condition not usually fulfilled in practice. But Michell* has succeeded in solving the problem for a plane rectangular block, moving at a slight inclination over another plane surface, free from this limitation, and he has developed a system of pivoted bearings with valuable practical results.

It is of interest to consider more generally than hitherto the case of two dimensions. In the present paper attention is given more especially to the case where one of the opposed surfaces is plane, but the second not necessarily so. As an alternative to an inclined plane surface, consideration is given to a broken surface consisting of two parts, each of which is parallel to the first plane surface but at a different distance from it. It appears that this is the form which must be approached if we wish the total pressure supported to be a maximum, when the length of the bearing and the closest approach are prescribed. In these questions we may anticipate that our calculations correspond pretty closely with what actually happens,—more than can be said of some branches of hydrodynamics.

In forming the necessary equation it is best, following Sommerfeld, to begin with the simplest possible case. The layer of fluid is contained between two parallel planes at $y=0$ and at $y=h$. The motion is everywhere parallel to x, so that the velocity-component u alone occurs, v and w being everywhere zero. Moreover u is a function of y only. The tangential traction acting across an element of area represented by dx is $\mu(du/dy)dx$, where μ is the viscosity, so that the element of volume $(dx\,dy)$ is subject to the force $\mu(d^2u/dy^2)\,dx\,dy$. Since there is no acceleration, this force is balanced by that due to the pressure, viz. $-(dp/dx)\,dx\,dy$, and thus

$$\frac{dp}{dx}=\mu\frac{d^2u}{dy^2}. \qquad (1)$$

In this equation p is independent of y, since there is in this direction neither motion nor components of traction, and (1), which may also be derived directly from the general hydrodynamical equations, is immediately integrable. We have

$$u=\frac{1}{2\mu}\frac{dp}{dx}y^2+\mathrm{A}+\mathrm{B}y, \qquad (2)$$

where A and B are constants of integration. We now

* *Zeitschr. f. Math.* t. 52. p. 123 (1905).

suppose that when $y=0$, $u=-\mathrm{U}$, and that when $y=h$, $u=0$. Thus

$$u=\frac{y^2-hy}{2\mu}\frac{dp}{dx}-\left(1-\frac{y}{h}\right)\mathrm{U}. \quad . \quad . \quad . \quad (3)$$

The whole flow of liquid, regarded as incompressible, between 0 and h is

$$\int_0^h u\,dy=-\frac{h^3}{12\mu}\frac{dp}{dx}-\frac{h\mathrm{U}}{2}=-\mathrm{Q},$$

where Q is a constant, so that

$$\frac{dp}{dx}=-\frac{6\mu\mathrm{U}}{h^3}\left(h-\frac{2\mathrm{Q}}{\mathrm{U}}\right). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we suppose the passage to be absolutely blocked at a place where x is negatively great, we are to make $\mathrm{Q}=0$ and (4) gives the rise of pressure as x decreases algebraically. But for the present purpose Q is to be taken finite. Denoting $2\mathrm{Q}/\mathrm{U}$ by H, we write (4)

$$\frac{dp}{dx}=-\frac{6\mu\mathrm{U}}{h^3}(h-\mathrm{H}). \quad . \quad . \quad . \quad . \quad (5)$$

When $y=0$, we get from (3) and (5)

$$\mu\frac{du}{dy}=\mu\mathrm{U}\frac{4h-3\mathrm{H}}{h^2}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which represents the tangential traction exercised by the liquid upon the moving plane.

It may be remarked that in the case of a simple shearing motion $\mathrm{Q}=\frac{1}{2}h\mathrm{U}$, making $\mathrm{H}=h$, and accordingly

$$dp/dx=0, \qquad du/dy=\mathrm{U}/h.$$

Our equations allow for a different value of Q and a pressure variable with x.

So far we have regarded h as absolutely constant. But it is evident that Reynolds' equation (5) remains approximately applicable to the lubrication problem in two dimensions even when h is variable, though always very small, provided that the changes are not too sudden, x being measured circumferentially and y normally to the opposed surfaces. If the whole changes of direction are large, as in the journal-bearing with a large arc of contact, complication arises in the reckoning of the resultant forces operative upon the solid parts concerned; but this does not interfere with the applicability of (5) when h is suitably expressed as a function of x. In the present paper we confine ourselves to the case where one surface (at $y=0$) may be treated as absolutely plane. The second surface is supposed to be limited at $x=a$ and at $x=b$, where h is equal to h_1 and h_2 respectively, and the pressure at both these places is taken to be zero.

For the total pressure, or load, (P) we have

$$P=\int_a^b p\,dx=-\int_a^b x\frac{dp}{dx}dx,$$

on integration by parts with regard to the evanescence of p at both limits. Hence by (5)

$$\frac{P}{6\mu U}=\int_a^b\frac{x\,dx}{h^2}-H\int_a^b\frac{x\,dx}{h^3}. \quad . \quad . \quad . \quad . \quad (7)$$

Again, by direct integration of (5),

$$0=\int_a^b\frac{dx}{h^2}-H\int_a^b\frac{dx}{h^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

by which H is determined. It is the thickness of the layer at the place, or places, where p is a maximum or a minimum. A change in the sign of U reverses also that of P.

Again, if $\bar{x}$ be the value of x which gives the point of application of the resultant force,

$$\bar{x}\,.\,P=\int_a^b p\,x\,dx=\tfrac{1}{2}\int_a^b x^2\frac{dp}{dx}dx,$$

so that

$$\frac{\bar{x}\,.\,P}{3\mu U}=\int_a^b\frac{x^2dx}{h^2}-H\int_a^b\frac{x^2dx}{h^3}. \quad , \quad . \quad . \quad . \quad (9)$$

By (7), (8), (9) $\bar{x}$ is determined.

As regards the total friction (F), we have by (6)

$$\frac{F}{\mu U}=4\int_a^b\frac{dx}{h}-3H\int_a^b\frac{dx}{h^2}. \quad . \quad . \quad . \quad . \quad (10)$$

Comparing (7) and (10), we see that the ratio of the tota friction to the total load is *independent of* μ *and of* U. And, since the right-hand members of (7) and (10) are dimensionless, the ratio is also independent of the linear scale. But if the scale of h only be altered, F/P varies as h.

We may now consider particular cases, of which the simplest and the most important is when the second surface also is flat, but inclined at a very small angle to the first surface. We take

$$h = mx, \quad \ldots \ldots \ldots \quad (11)$$

and we write for convenience

$$b - a = c, \qquad h_2/h_1 = b/a = k, \quad \ldots \quad (12)$$

so that

$$m = (k-1)h_1/c. \quad \ldots \ldots \quad (13)$$

We find in terms of c, k, and h_1

$$H = \frac{2kh_1}{k+1} \quad \ldots \ldots \ldots \quad (14)$$

$$\frac{P}{6\mu U} = \frac{c^2}{(k-1)^2 h_1^2}\left\{\log_e k - \frac{2(k-1)}{k+1}\right\} \quad \ldots \quad (15)$$

$$\frac{\bar{x}}{\frac{1}{2}c} = \frac{k^2 - 1 - 2k\log k}{(k^2-1)\log k - 2(k-1)^2} \quad \ldots \ldots \quad (16)$$

$$\frac{F}{P} = \frac{h_1}{c}\,\frac{2(k^2-1)\log k - 3(k-1)^2}{3(k+1)\log k - 6(k-1)}. \quad \ldots \quad (17)$$

U being positive, the sign of P is that of

$$\log k - \frac{2(k-1)}{k+1}.$$

If $k > 1$, that is when $h_2 > h_1$, this quantity is positive. For its derivative is positive, as is also the initial value when k exceeds unity but slightly. In order that a load may be sustained, the layer must be thicker where the liquid enters.

In the above formulæ we have taken as data the length of the bearing c and the minimum distance h_1 between the surfaces. So far k, giving the maximum distance, is open. It may be determined by various considerations. Reynolds examines for what value P, as expressed in (15), is a maximum, and he gives (in a different notation) $k = 2{\cdot}2$. For values of k equal to 2·0, 2·1, 2·2, 2·3 I find for the coefficient of c^2/h_1^2 on the right of (14) respectively

·02648, ·02665, ·02670, ·02663.

In agreement with Reynolds the maximum occurs when $k = 2{\cdot}2$ nearly, and the maximum value is

$$P = 0{\cdot}1602\frac{\mu U c^2}{h_1^2}. \quad \ldots \ldots \quad (18)$$

It should be observed—and it is true whatever value be taken for k—that P varies as the *square* of c/h_1.

With the above value of k, viz. 2·2,

$$H = 1{\cdot}27 h_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

fixing the place of maximum pressure.

Again, from (16) with the same value of k,

$$\bar{x} - a = 0{\cdot}4231\, c, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which gives the distance of the centre of pressure from the trailing edge.

And, again with the same value of k, by (17)

$$F/P = 4{\cdot}70 h_1/c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Since h_1 may be very small, it would seem that F may be reduced to insignificance.

In (18) .... (21) the choice of k has been such as to make P a maximum. An alternative would be to make F/P a minimum. But it does not appear that this would make much practical difference. In Michell's bearings it is the position of the centre of pressure which determines the value of k by (16). If we use (20), k will be 2·2, or thereabouts, as above.

When in (16) k is very large, the right-hand member tends to zero, as also does a/c, so that $\bar{x}-a$ tends to vanish, c being given. As might be expected, the centre of pressure is then close to the trailing edge. On the other hand, when k exceeds unity but little, the right-hand member of (16) assumes an indeterminate form. When we evaluate it, we find

$$\bar{x} - a = \tfrac{1}{2} c.$$

For all values of $k\,(> 1)$ the centre of pressure lies nearer the narrower end of the layer of fluid.

The above calculations suppose that the second surface is *plane*. The question suggests itself whether any advantage would arise from another choice of form. The integrations are scarcely more complicated if we take

$$h = m x^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

We denote, as before, the ratio of the extreme thicknesses (h_2/h_1) by k, and c still denotes $b-a$. For the total pressure we get from (15)

$$\frac{P}{6\mu U} = \frac{c^2}{(k^{1/n}-1)^2 h_1^2}\left\{\frac{3n-1}{(2n-1)(3n-2)}\,\frac{(k^{-2+1/n}-1)(k^{-3+2/n}-1)}{k^{-3+1/n}} - \frac{k^{-2+2/n}-1}{2n-2}\right\}, \quad . \quad . \quad . \quad (23)$$

from which we may fall back on (15) by making $n=1$.

For example, if $n=2$, so that the curve of the second surface is part of a common parabola, P is a maximum at

$$P=0{\cdot}163\frac{\mu Uc^2}{h_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

when $k=2{\cdot}3$. The departure from (18) with $k=2{\cdot}2$ is but small. In order to estimate the curvature involved we may compare $\frac{1}{2}(h_1+h_2)$ with the middle ordinate of the curve, viz.

$$\tfrac{1}{4}m(a+b)^2=\tfrac{1}{4}\{\sqrt{h_1}+\sqrt{(2{\cdot}3h_1)}\}^2=1{\cdot}58\,h_1,$$

which is but little less than

$$\tfrac{1}{2}(h_1+h_2)=\tfrac{1}{2}h_1(1+2{\cdot}3)=1{\cdot}65\,h_1.$$

It appears that curvature following the parabolic law is of small advantage.

I have also examined the case of $n=\infty$. It is perhaps simpler and comes to the same to assume

$$h=e^{\beta x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The integrals required in (7), (8) are easily evaluated. Thus

$$\int\frac{dx}{h^2}=\frac{e^{-2\beta a}-e^{-2\beta b}}{2\beta}=\frac{k^2-1}{2\beta k^2h_1^2},$$

$$\int\frac{dx}{h^3}=\frac{e^{-3\beta a}-e^{-3\beta b}}{3\beta}=\frac{k^3-1}{3\beta k^3h_1^3},$$

making

$$H=\frac{3kh_1(k^2-1)}{2(k^3-1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In like manner

$$\int\frac{x\,dx}{h^2}=\frac{k^2(1+2\beta a)-1-2\beta b}{4\beta^2k^2h_1^2},$$

$$\int\frac{x\,dx}{h^3}=\frac{k^3(1+3\beta a)-1-3\beta b}{9\beta^2k^3h_1^3}.$$

Using these in (7), we get on reduction

$$P=\frac{3\mu U}{\beta^2k^2h_1^2}\left\{\frac{k^2-1}{6}+\frac{\beta(k^2-k^3)(b-a)}{k^3-1}\right\},$$

or, since $\beta c=\log k$,

$$P=\frac{3\mu U\,.\,c^2}{k^2(\log k)^2h_1^2}\left\{\frac{k^2-1}{6}-\frac{k^2(k-1)\log k}{k^3-1}\right\}. \quad . \quad . \quad (27)$$

If we introduce the value of β, the equation of the curve may be written

$$h = k^{x/c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

When we determine k so as to make P a maximum, we get $k=2\cdot3$, and

$$P=0\cdot165\frac{\mu U c^2}{h_1^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

again with an advantage which is but small.

In all the cases so far considered the thickness h increases all the way along the length, and the resultant pressure is proportional to the square of this length (c). In view of some suggestions which have been made, it is of interest to inquire what is the effect of (say) r repetitions of the same curve, as, for instance, a succession of inclined lines ABCDEF (fig. 1). It appears from (8) that H has the

Fig. 1.

same value for the aggregate as for each member singly, and from (5) that the increment of p in passing along the series is r times the increment due to one member. Since the former increment is zero, it follows that the pressure is zero at the beginning and end of each member. The circumstances are thus precisely the same for each member, and the total pressure is r times that due to the first, supposed to be isolated. But if we imagine the curve spread once over the entire length by merely increasing the scale of x, we see that the resultant pressure would be increased r^2 times, instead of merely r times. Accordingly a repetition of a curve is very unfavourable. But at this point it is well to recall that we are limiting ourselves to the case of two dimensions. An extension in the third dimension, which would suffice for a particular length, might be inadequate when this length is multiplied r times.

The forms of curve hitherto examined have been chosen with regard to practical or mathematical convenience, and it remains open to find the form which according to (5) makes P a maximum, subject to the conditions of a given length and a given minimum thickness (h_1) of the layer of liquid. If we suppose that h becomes $h+\delta h$, where δ is the symbol of the calculus of variations, (8) gives,

$$2\int\frac{\delta h}{h^3}dx-3\mathrm{H}\int\frac{dh}{h^4}dx+\delta\mathrm{H}\int\frac{dx}{h^3}=0, \quad . \quad . \quad (30)$$

and from (7)

$$\frac{\delta\mathrm{P}}{6\mu\mathrm{U}}=\int\frac{\delta h(-2h+3\mathrm{H})x\,dx}{h^4}-\delta\mathrm{H}\int\frac{x\,dx}{h^3}, \quad . \quad . \quad (31)$$

the integrations being always over the length. Eliminating $\delta\mathrm{H}$, we get

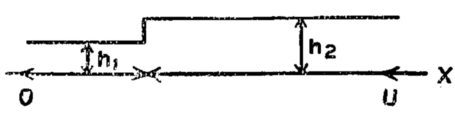
$$\frac{\delta\mathrm{P}}{12\mu\mathrm{U}}=-\int\frac{\delta h}{h^4}\left\{x-\frac{\int h^{-3}x\,dx}{\int h^{-3}dx}\right\}\left\{h-\tfrac{3}{2}\mathrm{H}\right\}x\,dx. \quad . \quad (32)$$

The evanescence of $\delta\mathrm{P}$ for all possible variations δh would demand that over the whole range either

$$x=\frac{\int h^{-3}x\,dx}{\int h^{-3}dx}, \quad \text{or} \quad h=\tfrac{3}{2}\mathrm{H}. \quad . \quad . \quad . \quad (33)$$

But this is not the requirement postulated. It suffices that the coefficient of δh on the right of (32) vanish over that part of the range where $h>h_1$, and that it be negative when $h=h_1$, so that a *positive* δh in this region involves a decrease in P, a negative δh here being excluded *a priori*. These conditions may be satisfied if we make $h=h_1$ from $x=0$ at the edge where the layer is thin to $x=c_1$, where c_1 is finite, and $h=\frac{3}{2}\mathrm{H}$ over the remainder of the range from c_1 to c_1+c_2, where $c_1+c_2=c$, the whole length concerned (fig. 2). For the moment we regard c_1 and c_2 as prescribed.

Fig. 2.



For the first condition we have by (8)

$$\tfrac{2}{3}h_2=\mathrm{H}=\frac{c_1/h_1^2+c_2/h_2^2}{c_1/h_1^3+c_2/h_2^3},$$

so that

$$c_2/c_1=k^2(2k-3), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

determining k, where as before $k=h_2/h_1$. The fulfilment of (34) secures that $h=\frac{2}{3}\text{H}$ over that part of the range where $h=h_2$. When $h=h_1$, $h-\frac{2}{3}\text{H}$ is negative; and the second condition requires that over the range from 0 to c_1

$$\frac{\int h^{-3}x\,dx}{\int h^{-3}\,dx}-x$$

be positive, or since c_1 is the greatest value of x involved, that

$$\int h^{-3}x\,dx-c_1\int h^{-3}dx=+. \quad . \quad . \quad . \quad (35)$$

The integrals can be written down at once, and the condition becomes

$$k^3<c_2{}^2/c_1{}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

whence on substitution of the value of c_2/c_1 from (34),

$$k(2k-3)^2>1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

If k be such as to satisfy (37) and c_2/c_1 be then chosen in accordance with (34) and regarded as fixed, every admissible variation of h diminishes P. But the ratio c_2/c_1 is still at disposal within certain limits, while c_1+c_2 $(=c)$ is prescribed. In terms of k and c by (34)

$$c_1=\frac{c}{1+2k^3-3k^2}, \quad c_2=\frac{c(2k^3-3k^2)}{1+2k^3-3k^2}, \quad . \quad . \quad (38)$$

and by (7)

$$\frac{\text{P}}{\mu\text{U}}=\frac{1}{h_1{}^2}\left\{c_1{}^2(3-2k)+\frac{2c_1c_2+c_2{}^2}{k^2}\right\}=\frac{c^2}{h_1{}^2}\,\frac{2k-3}{1+2k^3-3k^2}=\frac{c^2}{h_1{}^2}f(k), \quad . \quad . \quad . \quad (39)$$

The maximum of $f(k)$ is 0·20626, and it occurs when $k=1·87$. The following shows also the neighbouring values:

k.	$f(k)$.	$k(2k-3)^2$.
1·86	0·20624	0·964
1·87	0·20626	1·024
1·88	0·20617	1·086

It will seen that while $k=1·86$ is inadmissible as not satisfying (37), $k=1·87$ is admissible and makes

$$\text{P}=·20626\frac{\mu\text{U}c^2}{h_1{}^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

no great increase on (18). It may be repeated that k is the ratio of the two thicknesses of the layer (h_2/h_1), and that by (34)

$$c_2/c_1 = 2\cdot588. \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

This defines the form of the upper surface which gives the maximum total pressure when the minimum thickness and the total length are given, and it is the solution of the problem as proposed. But it must not be overlooked that it violates the supposition upon which the original equation (5) was founded. The solution of an accurate equation would probably involve some rounding off of the sharp corners, not greatly affecting the numerical results.

The distance $\bar{x}$ of the centre of pressure from the narrow end is given by

$$\bar{x} = 0\cdot4262c, \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

differing very little from the value found in (20). From (10) with use of (38) we get

$$\frac{\mathrm{F}}{\mu \mathrm{U}} = \frac{4c(k-1)^2}{h_1(1+2k^3-3k^2)} = \frac{4c}{(2k+1)h_1}, \quad . \quad . \quad (43)$$

and

$$\frac{\mathrm{F}}{\mathrm{P}} = \frac{4h_1(k-1)^2}{c(2k-3)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

If $k = 1\cdot87$,

$$\mathrm{F}/\mathrm{P} = 4\cdot091h_1/c, \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

a little less than was found in (21). The maximum total pressure and the corresponding ratio F/P are both rather more advantageous in the arrangement now under discussion than for the simply inclined line. But the choice would doubtless depend upon other considerations.

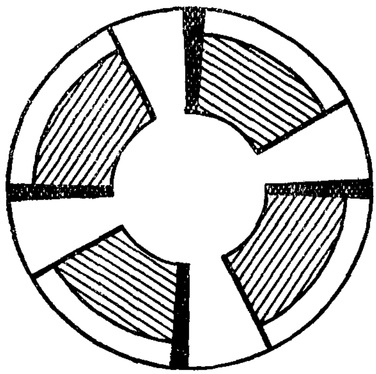
The particular case treated above is that which makes P a maximum. We might inquire as to the form of the curve for which F/P is a minimum, for a given length and closest approach to the axis of x. In the expression corresponding with (32), instead of a product of two linear factors, the coefficient of δh will involve a quadratic factor of the form

$$\mathrm{B}xh + \mathrm{C}h^2 + \mathrm{D}x + \mathrm{E}h + \mathrm{F}, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

so that the curve is again hyperbolic in the general sense. But its precise determination would be troublesome and probably only to be effected by trial and error. It is unlikely that any great reduction in the value of F/P would ensue.

Fig. 3 is a sketch of a suggested arrangement for a footstep. The white parts are portions of an original plane

Fig. 3.



surface. The 4 black radii represent grooves for the easy passage of lubricant. The shaded parts are slight depressions of uniform depth, such as might be obtained by etching with acid. It is understood that the opposed surface is plane throughout.

P.S. Dec. 13.—In a small model the opposed pieces were two pennies ground with carborundum to a fit. One of them—the stationary one—was afterwards grooved by the file and etched with dilute nitric acid according to fig. **3**, sealing-wax, applied to the hot metal, being used as a "resist." They were mounted in a small cell of tin plate, the upper one carrying an inertia bar. With oil as a lubricant the contrast between the two directions of rotation was very marked.

Opportunity has not yet been found for trying polished glass plates, such as are used in optical observations on "interference." In this case the etching would be by hydrofluoric acid *, and air should suffice as a lubricant.

* Compare 'Nature,' vol. lxiv. p. 385 (1901); Scientific Papers, vol. iv. p. 546.