EXTENDED MEANING OF CONJUGATE SETS

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IN connexion with the paper by Prof. Eddington, which precedes this note, I should like to take the opportunity of pointing out that the definition of conjugate sets, in my paper to which he refers, may be made somewhat wider. If A and B are two variable quantities, not necessarily of the same kind, we can denote by $(A; B)$ some quantity which (1) is determinate when A and B have assigned meanings but is independent of particular values of A and B , and (2) satisfies the laws of arithmetic for multiplication, *i.e.* is such that

$$(A; B) = (B; A), \quad (A; B+C) = (A; B) + (A; C),$$

and, if p is a constant with regard to A and B ,

$$(A; pB) = p(A; B).$$

Usually, if A and B relate to a member of a class, $(A; B)$ will be something depending on the values of A and B for the class as a whole. If, for instance, as in the original paper, A and B were measurements containing errors, $(A; B)$ might be the mean product of such pairs of errors; or, if A and B were, say, the height and weight of an individual, $(A; B)$ might be the mean product of the deviations of these from their respective means. A meaning having been given to $(A; B)$, the definition of conjugate sets, for this meaning of $(A; B)$, is to be adapted accordingly. Let $A_\mu \equiv A_0, A_1, \dots, A_l$ be a set of $l+1$ variable quantities. Then the conjugate set $A^\mu \equiv A^0, A^1, \dots, A^l$ can be defined either by the condition that $(A^r; A_s)$ is 0 if $r \neq s$ or 1 if $r = s$, or, as in Prof. Eddington's paper, directly by the relation $A^\nu = g^{\mu\nu} A_\mu$, where $g_{rs} \equiv (A_r; A_s)$.

For finding improved values we require the further condition (3) that $(A; A)$ is positive unless $A = 0$ or a constant, in which case it is $= 0$. If w is any linear function of $A_0, A_1, \dots, A_l$, its improved value, which I will here call Iw , is defined by the condition that it is the sum of w and a

linear function of $l-j$ specified A 's (or specified linear functions of the A 's), called auxiliaries, the coefficients in this linear function being chosen so as to make $(Iw; Iw)$ a minimum.

With these extensions, Prof. Eddington's methods and results still apply, with the substitution of $(A; B)$ for the mean product of errors of A and B . His notation is adopted in the following supplementary paragraphs; w, x, y denoting definite linear functions of the A 's, and B_μ, C_ν , etc. sets of such functions.

(i) We can write w in the form $w = k_\mu A_\mu$. If also $x = k_\mu B_\mu$, then x is related to the B 's in the same way that w is related to the A 's. We can express this by saying that $w/A_\mu = x/B_\mu$; it being understood that a Greek letter in a denominator is what Prof. Eddington elsewhere calls a "dummy," *i.e.* that the quantity in which it occurs is one of the factors of an "inner product" such as $k_\mu A_\mu$. We may also have relations such as $A_\mu/B_\nu = C_\mu/D_\nu$ or $A_\mu/B_\nu = E_\nu/F_\mu$; in this latter case $A_\mu F_\mu = B_\nu E_\nu$. These relations can be inverted; *e.g.* if $A_\mu/B_\nu = C_\mu/D_\nu$, then $B_\nu/A_\mu = D_\nu/C_\mu$.

(ii) The second and third of the conditions stated under (2) of the first paragraph of this note may be combined in the form

$$(y; k_\mu A_\mu) = k_\mu (y; A_\mu).$$

If we write $w \equiv k_\mu A_\mu$, so that $k_\mu = w/A_\mu$, this becomes

$$w/A_\mu = (y; w)/(y; A_\mu).$$

(iii) If we define a conjugate set in the first of the two ways mentioned in the first paragraph, we easily obtain

$$w = (w; A^\mu) A_\mu = (w; A_\mu) A^\mu,$$

$$\text{or} \quad w/A_\mu = (w; A^\mu), \quad w/A^\mu = (w; A_\mu).$$

$$\text{Hence} \quad B_\nu = (B_\nu; A^\mu) A_\mu = (B_\nu; A_\mu) \dot{A}^\mu.$$

(iv) From $w = (w; A^\mu) A_\mu$ it follows that

$$(w; y) = (w; A^\mu)(A_\mu; y),$$

since $(w; A^\mu)$ is a constant as regards w and y . Hence

$$(B_\nu; C_\rho) = (B_\nu; A^\mu)(A_\mu; C_\rho).$$

(v) For two related sets, and their conjugates, we have four relations

of the form

$$\frac{A^\mu}{B_\nu} = \frac{B^\nu}{A_\mu} = (A^\mu; B^\nu) = \frac{Q}{A_\mu B_\nu},$$

where $Q \equiv A_\mu A^\mu$. This inner product Q is invariant for any particular original set, and may be expressed either as a quadratic form of the members of a related set, the coefficients being the $(\ ; \)$ of members of its conjugate set—*e.g.*

$$Q = (C^0; C^0) C_0 C_0 + 2(C^0; C^1) C_0 C_1 + (C^1; C^1) C_1 C_1 + \dots$$

—or in the more general form $(C^\mu; D^\nu) C_\mu D_\nu$.

(vi) Relations similar to those mentioned above may hold between sets of coefficients or of other constants. If, for instance, A_μ and B_ν are related in a specified way, and if w is a linear function of B_ν which we want to express in terms of A^μ , then, if we write

$$w = p_\mu A^\mu, \quad q_\nu \equiv (w; B_\nu),$$

we have $p_\mu = w/A^\mu = (w; A_\mu) = (w; A_\mu/B_\nu \cdot B_\nu) = A_\mu/B_\nu \cdot q_\nu$,

so that

$$p_\mu/q_\nu = A_\mu/B_\nu.$$

Thus the p 's are related to the q 's in the same way that the A 's are related to the B 's.

(vii) The relation between improved values and original values is such that

$$Ik_\mu A_\mu = k_\mu IA_\mu, \quad \text{or} \quad Iw/IA_\mu = w/A_\mu.$$

It may be noted that a set of $l+1$ improved values has no conjugate set, as the $l+1$ values are not independent but are connected by $l-j$ relations.

(viii) The problem of finding improved values may be expressed as a problem of finding parts of two conjugate sets in succession. Take $A_\mu \equiv A_F \& A_R$, where A_F are $j+1$ quantities whose improved values IA_F are required, and A_R are the $l-j$ auxiliaries. Let the set conjugate to $A_F \& A_R$ be $A^F \& A^R$: and let the set conjugate to $A^F \& A_R$ be $B_F \& B^R$. Then $B_F = IA_F$. For finding either or both of A^F and B_F we may replace A_R by any $l-j$ linear functions of A_R , and may add to members of A_F any linear functions of A_R .