XXV. Theory of the string galvanometer of Einthoven

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TABLE IV.
Threshold values of $H' \times \sqrt{f}$.

f .	$H' \times \sqrt{f}$.			
	C.E.M.	H.C.S.	F.B.P.	M.L.
11	395.	597.	1120.	525.
18	970.	970.	1440.	765.
25	965.	1040.	1260.	960.
32	775.	845.	950.	1120.
39	509.	750.	782.	863.
46	638.	699.	842.	961.
53	1260.	1290.	1705.	1760.
60	1200.	1273.	1508.	1185.
67	1265.	1330.	1680.	1318.
74	1150.		1980.	1740.
81	1573.		2230.	1960.
88	1500.		2500.	1980.
95	1860.		2600.	2040.
102	2250.		3120.	2190.
109	2500.		2990.	2140.
116	1965.		3220.	2060.
123	2230.		3410.	2000.

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XXV. *Theory of the String Galvanometer of Einthoven.*
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THE theory of the transverse vibration of stretched strings is very fully treated in Lord Rayleigh's 'Theory of Sound'†. The string galvanometer of Einthoven requires, however, special consideration because of the manner in which the impressed force is applied to the string. It is not applied at a single point of the string as is the case with a plucked string, a pianoforte wire, or a violin string, but throughout almost its entire length, and, since an electrical current forced through the string supplies the motive power, the nature of the circuit external to the galvanometer has an influence upon the motion of the string which must be taken into the account.

Imagine a stretched string, fixed at its two ends, having

* Communicated by the Author.

† Rayleigh, 'Theory of Sound,' vol. i. Chap. VI.

sufficient electrical conductivity to admit of carrying a current, immersed throughout almost its entire length in a powerful magnetic field, and we have the essential features of the string galvanometer. The direction of the field is, of course, approximately perpendicular to that of the string, and the transverse motion of each element of the string due to the mutual action between the current in it and the magnetic field is in a third direction approximately perpendicular to each of the other two directions.

In this paper the string will be treated as if it were immersed in a magnetic field the lines of which are parallel, but the force may vary in intensity from point to point along the string. As particular cases two different field distributions are considered, first, the uniform field which most nearly represents the galvanometer as at present constructed, and second, the best distribution for simplifying the resulting motion of the string. The open space at the centre occupied by the microscope and the ends of the string which project beyond the field are not specifically considered, as it will be apparent how these irregularities may be allowed for in an approximate way when the data are known.

Let PQ, fig. 1, be the element of any string of length ds ,

and confined to a plane. The element is acted upon by the tension T_1 at P and the opposite tension $T_1 + dT_1$ at Q, and the resultant of all external forces Fds in the plane of the string acting at any angle χ .

If the element of the string is at rest these three forces are in equilibrium, and resolving along the tangent, we find

$$\frac{dT_1}{ds} + F \cos \chi = 0, \quad . \quad . \quad . \quad (1)$$

and along the normal

$$\frac{T_1}{r} - F \sin \chi = 0, \quad . \quad . \quad . \quad (2)$$

where $r = \frac{ds}{d\theta}$ = the radius of curvature at P.

and its value is $\text{HId}s$. Hence in (1) and (2) $\chi = \frac{\pi}{2}$; HI takes the place of F , and we have

$$\frac{dT_1}{ds} = 0, \quad \text{and } T_1 = \text{a constant throughout the length of the string,} \quad (3)$$

and

$$r = \frac{T_1}{HI} \quad \text{or} \quad r \propto \frac{1}{H}. \quad (4)$$

The curve assumed by the string carrying a steady direct current when the field is uniform is, therefore, the arc of a circle, and since the deflexions employed in practice are very small compared with the length of the string, the radius of curvature is correspondingly large. Let OL, figure 2, represent the chord of the string which forms the arc OBL with

Fig. 2.

centre at O' , showing a small deflexion y_0 at the centre of the length. With O' as origin, and abscissæ parallel with OL , the equation of the circle is

$$x'^2 + y'^2 = r^2.$$

Transferring the origin to O, so that OL becomes the axis of x , by putting

$$x' = x - \frac{l}{2}$$

and

$$y' = y + p,$$

where $O'C = p$, the equation of the circle is

$$x^2 - lx + y^2 + 2py = 0. \quad (5)$$

And from the properties of the circle,

$$\overline{OC}^2 = y_0(2r - y_0) = \frac{l^2}{4}.$$

Hence the radius

$$r = \frac{l^2}{8y_0} + \frac{y_0}{2}. \quad (6)$$

This expression for the radius together with that already found in (4) gives the tension

$$T_1 = HI \left(\frac{l^2}{8y_0} + \frac{y_0}{2} \right).$$

In practice the deflexions are small and the second term may be omitted, giving

$$T_1 = \frac{HI l^2}{8y_0}. \quad (7)$$

This expression involves three quantities l , I , and y_0 , which may be accurately and easily measured. H and T_1 are not as readily measured directly. If, however, H is determined for one string and one strength of field, it remains the same for any other string. The tension cannot be measured directly with a fine quartz fibre, but if a fine wire is substituted for the fibre, the upper end being fastened to the same point as the fibre, and the lower end passed through a V-groove held in the same position as the lower end of the fibre, weights may be suspended from the wire and the tension thus directly measured. By this device the field strength is found from (7), and it has the advantage that the average value is measured in just the same position as the fibre occupies, so that when the fibre is replaced its tension is indirectly measured by a knowledge of H .

The order of magnitude of the deflexion and the radius of curvature may be obtained from an actual case. With a magnification of 900, and a deflexion of the shadow of the centre of the string of 3 cm., and length of string 14 cm., the deflexion $y_0 = \frac{3}{900} = .00333$ cm.; and $\frac{l^2}{8y_0} = 7350$ cm. The radius of the arc formed by the string is therefore 73.5 metres.

Solving (5) for y and using the positive sign before the radical, we find as the equation of the circular arc formed by the string

$$y = -p + (-x^2 + lx + p^2)^{\frac{1}{2}}.$$

Putting $z^2 = -x^2 + lx$, and expanding the radical into infinite series, we find

$$y = \frac{z^2}{2p} - \frac{z^4}{2 \cdot 4p^3} + \frac{1 \cdot 3z^6}{2 \cdot 4 \cdot 6p^5} - \dots$$

Since p is always a very large quantity compared with z , the maximum value of which is $\frac{l}{2}$ when $x = \frac{l}{2}$, all terms after the first in the series are of the second order of smallness and may be omitted, giving

$$y = \frac{1}{2p}(lx - x^2) \dots \dots \dots (8)$$

This is the equation of the parabola which most closely approximates the arc of the circle throughout the length of the string.

The equation (8) may be developed by Fourier's series* into the following, which will be used as the equation of the string when deflected by a steady direct current to determine later the constants in the equations of motion,

$$y = \frac{32}{\pi^3} y_0 \left[\sin \frac{\pi x}{l} + \frac{1}{3^3} \sin \frac{3\pi x}{l} + \frac{1}{5^3} \sin \frac{5\pi x}{l} + \dots \right], \quad (9)$$

where $\frac{\pi x}{l}$ is written instead of x in the previous equation (8).

The Differential Equations.

We will next form the differential equations applying to the string under the conditions of the string galvanometer. Let y denote the deflexion of an element of the string at the distance x from the end and time t , and Y the impressed force. The partial differential equation† of motion of the string is then

$$\frac{d^2 y}{dt^2} + k \frac{dy}{dt} = a^2 \frac{d^2 y}{dx^2} + Y \quad \dots \dots \dots (10)$$

*The developments of x and x^2 by Fourier's series from which (9) is directly derived are

$$\begin{aligned} x &= 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right) \\ x^2 &= \frac{2}{\pi} \left[\left(\frac{\pi^2}{1} - \frac{4}{1^3} \right) \sin x - \frac{\pi^2}{2} \sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3} \right) \sin 3x - \frac{\pi^2}{4} \sin 4x \right. \\ &\quad \left. + \left(\frac{\pi^2}{5} - \frac{4}{5^3} \right) \sin 5x - \dots \right]. \end{aligned}$$

† Rayleigh, 'Theory of Sound,' 1894 Edition, page 192.

where k is a constant damping factor, and

$$a^2 = \frac{T_1}{\rho} = \text{a constant,} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

T_1 being the tension and ρ the mass of string per unit of length.

Instead of using this form of the equation it is advantageous to express it in terms of the so-called *normal * coordinates*, which gives the equation the form

$$\ddot{\phi}_s + k\dot{\phi}_s + n_s^2 \phi_s = \frac{2}{l\rho} \Phi_s \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where k is the air damping factor, and

$$n_s = \frac{s\pi a}{l}; \quad \text{or } n_s^2 = \frac{s^2 \pi^2 T_1}{l^2 \rho}, \quad s \text{ being an integer.} \quad . \quad (13)$$

The deflexion of the string, y , in terms of the normal coordinates is

$$y = \phi_1 \sin \frac{\pi x}{l} + \phi_2 \sin \frac{2\pi x}{l} + \phi_3 \sin \frac{3\pi x}{l} + \dots \quad . \quad (14)$$

and the velocity

$$\dot{y} = \dot{\phi}_1 \sin \frac{\pi x}{l} + \dot{\phi}_2 \sin \frac{2\pi x}{l} + \dot{\phi}_3 \sin \frac{3\pi x}{l} + \dots \quad . \quad (15)$$

The customary expression for Φ_s in terms of the mechanical force is

$$\Phi_s = \int_0^l \rho Y \sin \frac{s\pi x}{l} dx,$$

where $\rho Y dx$ represents the force upon an element of the string. In our case this is replaced by $Hidx$, where i denotes any variable current, and hence

$$\Phi_s = i \int_0^l H \sin \frac{s\pi x}{l} dx \quad . \quad . \quad . \quad . \quad (16)$$

Assuming that the string is in a uniform field throughout its entire length, this integral is

$$\Phi_s = \frac{Hil}{s\pi} (1 - \cos s\pi) \quad . \quad . \quad . \quad . \quad (17)$$

As s takes in succession all integral values from 1 on, Φ_s vanishes for all even values, showing that the impressed force due to a current in the string generates no even harmonics.

* Thomson & Tait's 'Natural Philosophy,' first edition 1867, § 337.

If we write a constant h_s to abbreviate the expression

$$h_s = \frac{2H}{s\pi\rho} (1 - \cos s\pi), \quad . \quad . \quad . \quad . \quad . \quad (18)$$

we have

$$\frac{2}{l\rho} \Phi_s = h_s i,$$

and the equation (12) becomes

$$\ddot{\phi}_s + k\dot{\phi}_s + n_s^2 \phi_s = h_s i \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The Circuit Equation.

When the circuit of the string is completed outside of the galvanometer, the motion of the string of itself generates a current which opposes the motion, giving rise to the well-known electromagnetic damping. If R is the total resistance and L the inductance of the circuit, and e' the back electromotive force generated by the movement of the string, we have

$$e = Ri + L \frac{di}{dt} + e'.$$

Since the back electromotive force is proportional to the velocity we may write

$$e' = h_s' \dot{\phi}_s,$$

where h_s' is a constant quantity. The rate at which mechanical energy is supplied to the moving string is equal to the rate at which the electrical energy is converted into heat. The former is the impressed force, hi times the velocity $\dot{\phi}_s$, and the latter is the back electromotive force times the current. Hence

$$h_s i \dot{\phi}_s = h_s' i \dot{\phi}_s,$$

and consequently the constant h_s' is the same constant h_s before used in (18), and the prime may be suppressed. The circuit differential equation becomes

$$e = Ri + L \frac{di}{dt} + h_s \dot{\phi}_s = f(t) \quad . \quad . \quad . \quad . \quad (20)$$

The current, i , may now be eliminated from the pair of simultaneous differential equations (19) and (20), giving the following equation connecting the normal coordinate ϕ_s with the time

$$\frac{L}{h_s} \ddot{\phi}_s + \left(\frac{R}{h_s} + \frac{L}{h_s} k \right) \dot{\phi}_s + \left(\frac{R}{h_s} k + \frac{L}{h_s} n_s^2 + \frac{h}{\rho} \right) \phi_s + \frac{R}{h_s} n_s^2 \phi_s = e = f(t). \quad (21)$$

The solution of this equation might be obtained under certain conditions, but it is not the present purpose to examine the effects of inductance in the external circuit. Assuming, therefore, L to be small and negligible (21) is reduced to the following equation of the second order,

$$\ddot{\phi}_s + \left(k + \frac{h_s^2}{R\rho}\right)\dot{\phi}_s + n_s^2\phi_s = \frac{h_s}{R}e = \frac{h_s}{R}f(t) = h_s i \quad . \quad . \quad (22)$$

Upon comparing this with (19) it appears that the only difference is in the coefficient of $\dot{\phi}_s$. Denoting the new coefficient by k_s we have

$$k_s = k + \frac{h_s^2}{R\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

which means that the effect of closing the external circuit is to increase the coefficient of air damping k by an amount $\frac{h_s^2}{R\rho}$, which latter may be called the coefficient of electromagnetic damping.

The Impressed Electromotive Force.

We will now consider that the impressed electromotive force in the external circuit is a simple harmonic function of the time, since any form of periodic force may be resolved into a series of such simple waves. Let

$$e = E \cos \omega t = f(t) \quad . \quad . \quad . \quad . \quad . \quad (24)$$

With this value substituted in (22), the solution for the periodic portion, or particular integral not containing arbitrary constants, is

$$\phi_s = \frac{h_s I \sin \epsilon_s}{k_s \omega} \cos (\omega t - \epsilon_s) \quad . \quad . \quad . \quad (25)$$

where

$$\sin \epsilon = \frac{k_s \omega}{[(n_s^2 - \omega^2)^2 + k_s^2 \omega^2]^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (26)$$

$$\cos \epsilon = \frac{n_s^2 - \omega^2}{[(n_s^2 - \omega^2)^2 + k_s^2 \omega^2]^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (27)$$

$$\tan \epsilon = \frac{k_s \omega}{n_s^2 - \omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The values of n_s and h_s are given in (13) and (18) above, and $I = \frac{E}{R}$, the maximum value of the harmonic current.

For the complete solution the complementary function containing the arbitrary constants must be added to (25). This takes three forms according to the relative values of the physical constants n_s and k_s .

When $n_s > \frac{1}{2}k_s$ the transient portion takes the oscillatory form

$$\phi_s = A_s e^{-\frac{k_s t}{2}} \cos (n_s' t - \alpha_s) \quad . \quad . \quad . \quad (29)$$

where

$$n_s' = (n_s^2 - \frac{1}{4}k_s^2)^{\frac{1}{2}}, \quad . \quad . \quad . \quad (30)$$

A_s and α_s being the arbitrary constants to be determined according to the initial conditions of the motion.

When $n_s < \frac{1}{2}k_s$, the transient portion takes the exponential form,

$$\phi_s = A_s e^{\mu_s' t} + B_s e^{\mu_s'' t}, \quad . \quad . \quad . \quad (31)$$

where μ_s' and μ_s'' are the two roots of the equation

$$\mu^2 + k_s \mu + n_s^2 = 0, \quad . \quad . \quad . \quad (32)$$

namely

$$\mu_s' = -\frac{k_s}{2} + n_s', \quad . \quad . \quad . \quad (33)$$

and

$$\mu_s'' = -\frac{k_s}{2} - n_s', \quad . \quad . \quad . \quad (34)$$

n_s' being the same as in (30), but with the signs of the terms under the radical changed, so that n_s' is a real quantity.

When $n_s = \frac{1}{2}k_s$

$$\phi_s = (A_s + B_s t) e^{-\frac{k_s t}{2}} \quad . \quad . \quad . \quad (35)$$

As the velocities corresponding to the three cases just given are required for finding the arbitrary constants, they are given below.

When $n_s > \frac{1}{2}k_s$

$$\dot{\phi}_s = -A_s \left\{ \frac{k_s}{2} \cos (n_s' t - \alpha_s) + n_s' \sin (n_s' t - \alpha_s) \right\} e^{-\frac{k_s t}{2}} \quad . \quad (36)$$

When $n_s < \frac{1}{2}k_s$

$$\dot{\phi}_s = A_s' \mu_s' e^{\mu_s' t} + B_s \mu_s'' e^{\mu_s'' t} \quad . \quad . \quad . \quad (37)$$

When $n_s = \frac{1}{2}k_s$

$$\dot{\phi}_s = \left[B_s - (A_s + B_s t) \frac{k_s}{2} \right] e^{-\frac{k_s t}{2}} \quad . \quad . \quad . \quad (38)$$

Having obtained the expressions for the normal co-ordinates, we may write out the complete values of the deflexions, y , and the velocities, $\dot{y}$, at any point x and time t by the relations in (14) and (15) between the normal co-ordinates and these quantities. As a preliminary step we will write out in full the values of h_s , k_s , n_s , and ϵ_s as s takes in succession the values 1, 2, 3 &c.

If s is odd, by (18)

$$h_1 = \frac{4H}{\pi\rho} ; \quad h_3 = \frac{h_1}{3} ; \quad h_5 = \frac{h_1}{5} ; \quad \dots \quad h_s = \frac{h_1}{s} \quad \dots \quad (39)$$

If even

$$h_2 = h_4 = h_6 = \&c. = 0.$$

If s is odd

$$k_1 = k + \frac{h_1^2}{R\rho} ; \quad k_3 = k + \frac{h_1^2}{3^2 R\rho} ; \quad \dots \quad k_s = k + \frac{h_1^2}{s^2 R\rho} \quad (40)$$

If even

$$k_2 = k_4 = k_6 = \&c. = k.$$

By (13), (26), and (27)

$$n_1 = \frac{\pi}{l} \sqrt{\frac{T_1}{\rho}} ; \quad n_2 = \frac{n_1}{2} ; \quad \dots \quad n_s = \frac{n_1}{s} \quad \dots \quad (41)$$

$$\sin \epsilon_s = \frac{k_s \omega}{[(s^2 n_1^2 - \omega^2) + k_s^2 \omega^2]^{\frac{1}{2}}} ; \quad \cos \epsilon_s = \frac{s^2 n_1^2 - \omega^2}{[(s^2 n_1^2 - \omega^2) + k_s^2 \omega^2]^{\frac{1}{2}}} \quad (42)$$

The permanent periodic portion of the motion of any point of the string in a uniform magnetic field is therefore (s being odd), by (14), (25), and (42),

$$y = \frac{4HI}{\pi\rho} \left\{ \frac{\cos(\omega t - \epsilon_1)}{1[(n_1^2 - \omega^2)^2 + k_1^2 \omega^2]^{\frac{1}{2}}} \sin \frac{\pi x}{l} + \frac{\cos(\omega t - \epsilon_3)}{3[(3^2 n_1^2 - \omega^2)^2 + k_3^2 \omega^2]^{\frac{1}{2}}} \sin \frac{3\pi x}{l} \right. \\ \left. + \dots + \frac{\cos(\omega t - \epsilon)}{s[s^2 n_1^2 - \omega^2)^2 + k_s^2 \omega^2]^{\frac{1}{2}}} \sin \frac{s\pi x}{l} \right\} \quad \dots \quad (43)$$

It is to be remarked that the coefficients of the terms in this equation have particular numerical values because of the original assumption that the string is immersed in a uniform magnetic field. This causes the curve for constant current to be the arc of a circle. At each and every point of the string, corresponding to a single value of x , the motion of that point is evidently a simple harmonic function of the time, having a period agreeing with that of the impressed force, but differing from it in phase, since all the simple harmonic

waves represented by the separate terms have the same period and may be combined into a single wave of this period, and differing in phase.

If the impressed force were not a simple sine-wave but a complex wave, a similar expression to (43) might be written for each component wave in the complex impressed force. There would then be a sine-wave in the resulting motion of the string corresponding to each component in the impressed force, but each differing in phase from the impressed wave by a definite amount depending upon the physical constants. These phase-angles differ from each other among the waves of different frequencies, and, as a result, the record cannot be a strictly faithful reproduction of the impressed force; but the amount of departure from it may be small, and differs according to circumstances.

If we do not introduce the supposition of a uniform magnetic field, but suppose that H varies from point to point along the length of the string, the value of Φ_s in (16) is

$$\Phi_s = i \int_0^l H \sin \frac{s\pi x}{l} dx. \quad . \quad . \quad . \quad (44)$$

And if K denotes the integral, we have as the second member of (12)

$$\frac{2}{l\rho} \Phi_s = \frac{2K}{l\rho} i = h_s i \quad . \quad . \quad . \quad (45)$$

where h now stands for the more general expression

$$h_s = \frac{2}{l\rho} \int_0^l H \sin \frac{s\pi x}{l} dx. \quad . \quad . \quad . \quad (46)$$

If the impressed force is harmonic, as in (24), the general solution applying to any field distribution for the periodic part is

$$y = I \left\{ \frac{h_1}{k_1\omega} \sin \epsilon_1 \cos (\omega t - \epsilon_1) \sin \frac{\pi x}{l} + \frac{h_2}{k_2\omega} \sin \epsilon_2 \cos (\omega t - \epsilon_2) \sin \frac{2\pi x}{l} \right. \\ \left. + \frac{h_3}{k_2\omega} \sin \epsilon_3 \cos (\omega t - \epsilon_3) \sin \frac{3\pi x}{l} + \dots \right\} \quad . \quad (47)$$

Some consideration will be given to the best form of field distribution along the length of the string to give the simplest results, that in which the motion best represents the impressed force, and the force generates the least motion which does not occur in the original electromotive force, but this is deferred to a subsequent section.

Referring to the solution (43) for the case of a uniform field and harmonic impressed force, it is to be expected that

when the frequency of the forced motion is very small compared with the natural fundamental period, and the damping is not excessive, the curve of the string, at the instant of maximum electromotive force, should be the same as if a direct current flowed in the string. The equation shows this to be the case. Putting $t=0$, when the e.m.f. (24) is a maximum, we have

$$y = \frac{4HI}{\pi\rho} \left[\frac{n_1^2 - \omega^2}{(n_1^2 - \omega^2)^2 + k_1^2 \omega^2} \sin \frac{\pi x}{l} + \frac{3^2 n_1^2 - \omega^2}{3[(3^2 n_1^2 - \omega^2)^2 + k_3^2 \omega^2]} \sin \frac{3\pi x}{l} + \dots \right],$$

whence, making ω small compared with n_1 and neglecting $k_1 \omega$ as compared with n_1^2 , we have

$$y = \frac{4HI}{\pi\rho n_1^2} \left(\sin \frac{\pi x}{l} + \frac{1}{3^3} \sin \frac{3\pi x}{l} + \frac{1}{5^3} \sin \frac{5\pi x}{l} + \dots \right) \quad (48)$$

and by (7) and (13)

$$\frac{4HI}{\pi\rho n_1^2} = \frac{32}{\pi^3 y_0}.$$

Hence the form of the string agrees with (9), which is the approximate equation of the circular arc.

As the frequency of the impressed force increases, other things remaining the same, the form of the string at its maximum deflexion changes. For example, let us increase the impressed frequency in (43) so far as to reach the first critical

frequency, so that $\omega = n_1$. Then $\sin \epsilon_1 = 1$ and $\epsilon_1 = \frac{\pi}{2}$, and

$\cos(\omega t - \epsilon_1) = \sin \omega t$, which becomes zero when the time is zero and e.m.f. a maximum. The harmonics present are the only elements which prevent the string from taking the form of a straight line at this time. To find the curve at the

maximum deflexion, put $\omega t = \frac{\pi}{2}$. If the damping-factor is

so small that $k_3 \omega$ may be neglected in comparison with $3^2 n_1^2 - \omega^2$, and $k_5 \omega$ with $5^2 n_1^2 - \omega^2$, &c., all terms but the first in (43) are negligible, and

$$y = \frac{4HI}{\pi\rho k_1 \omega} \sin \frac{\pi x}{l} \quad (49)$$

The curve at maximum deflexion has now changed to that of a simple curve of sines from that of a circular arc.

*Case of return to zero position in a uniform
field when $n_s > \frac{k_s}{2}$.*

It is not necessary to multiply examples, and we pass to a consideration of the transient portion of the motion given above in (29), (31), and (35) in the normal coordinates.

When $n_s > \frac{1}{2}k_s$, and the magnetic field is uniform,

$$y = A_1 e^{-\frac{k_1 t}{2}} \cos(n_1' t - \alpha_1) \sin \frac{\pi x}{l} + A_2 e^{-\frac{k_2 t}{2}} \cos(n_2' t - \alpha_2) \sin \frac{2\pi x}{l} \\ + \dots + A_s e^{-\frac{k_s t}{2}} \cos(n_s' t - \alpha_s) \sin \frac{s\pi x}{l} \quad . \quad (50)$$

As a first particular case suppose the string has been deflected by a steady direct current, and then the circuit suddenly opened, so that we may put $R = \infty$, and $k_s = k$. If $n_1 > \frac{1}{2}k$, then $n_s > \frac{1}{2}k$, and the oscillatory case applies to all the harmonics which occur. It is now necessary to find the values of the constants $A_1, A_2, \dots, \alpha_1, \alpha_2, \dots$ according to the assumption that the initial position of the string is the circular arc given by (9), and that the velocity of projection is zero when $t = 0$. Putting $t = 0$ in (50), we find

$$y = A_1 \cos \alpha_1 \sin \frac{\pi x}{l} + A_2 \cos \alpha_2 \sin \frac{2\pi x}{l} + \dots + A_s \cos \alpha_s \sin \frac{s\pi x}{l}.$$

This is the same curve as (9) and the coefficients of $\sin \frac{\pi x}{l}$, $\sin \frac{2\pi x}{l}$, &c. may be equated, giving

$$\left. \begin{aligned} A_1 \cos \alpha_1 &= \frac{32}{\pi^3} y_0 \\ A_3 \cos \alpha_3 &= \frac{1}{3^3} \frac{32}{\pi^3} y_0 \\ &\dots \dots \dots \\ A_2 \cos \alpha_2 &= A_4 \cos \alpha_4 = A_6 \cos \alpha_6 = \&c. = 0. \end{aligned} \right\} \quad . \quad (51)$$

By writing out the velocities from (36), when $k_s = k$ we have

$$\dot{y} = \left\{ -A_1 \left[\frac{k}{2} \cos(n_1' t - \alpha_1) + n_1' \sin(n_1' t - \alpha_1) \right] \sin \frac{\pi x}{l} \right. \\ - A_2 \left[\frac{k}{2} \cos(n_2' t - \alpha_2) + n_2' \sin(n_2' t - \alpha_2) \right] \sin \frac{2\pi x}{l} \\ \left. - A_3 \left[\dots \dots \dots \&c. \dots \dots \right] e^{-\frac{kt}{2}} \right\} \quad . \quad (52)$$

Making $\dot{y}=0$, when $t=0$, we find

$$\tan \alpha_s = \frac{k}{2\left(s^2 n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}}}; \quad \frac{1}{\cos \alpha_s} = \frac{sn_1}{\left(s^2 n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}}} \quad \dots \quad (53)$$

The constants are now determined in terms of the fundamental physical constants in (51) and (53). y_0 is the deflexion of the centre of the string, and, substituting the constants in (50), we finally have the complete motion

$$\begin{aligned} y = \frac{32}{\pi^3} y_0 n_1 \left\{ \frac{1}{\left(n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}}} \cos \left[\left(n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}} t - \tan^{-1} \frac{k}{2\left(n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}}} \right] \sin \frac{\pi x}{l} \right. \\ + \frac{1}{3^2 \left(3^2 n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}}} \cos \left[\left(3^2 n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}} t - \tan^{-1} \frac{k}{2\left(3^2 n_1^2 - \frac{k^2}{4}\right)^{\frac{1}{2}}} \right] \sin \frac{3\pi x}{l} \\ \left. + \dots \dots \dots \right\} e^{-\frac{kt}{2}} \quad \dots \dots \dots \quad (54) \end{aligned}$$

It is evident that when $t=0$ the string takes the form of the arc of a circle, as the equation then reduces to (9). Since the galvanometer is arranged to observe the middle point only of the string, we may put

$$\sin \frac{\pi x}{l} = 1, \quad \sin \frac{3\pi x}{l} = -1, \quad \&c.$$

the terms alternating in sign. The equation shows that if the tension is large, so that n_1 is large compared with $\frac{k}{2}$, the coefficients within the brackets become as

$$1 : \frac{1}{27} : \frac{1}{125} : \frac{1}{343}, \quad \&c.,$$

and the second and third terms become appreciable in the record; but the chief departure from the simple law of logarithmic decrement is due to the second term. The negative sign of this term shows that the string shadow oscillates between logarithmic curves during the latter part of the time which have an initial amplitude greater than those corresponding to the initial deflexion of the string.

With small quartz fibres, say between .002 and .003 mm. in diameter, the air damping is so great that two or three waves are all that can be seen on the record, when the tension is almost as great as it is advisable to make it. The effect of

the second term is then confined to the first part of the curve affecting the shape of the first descent to zero. With very fine fibres, 0.001 mm., the tension cannot be safely increased to such a value as to make the string oscillate in air at the ordinary pressure.

*Case of return to zero position in a uniform
field when $n_s < \frac{k_s}{2}$.*

When the tension is such that $n_s \left(= \frac{s\pi}{l} \sqrt{\frac{T_1}{\rho}} \right)$ is less than $\frac{1}{2}k_s$, the form of solution is given by (31) from which we find

$$y = \left(A_1 e^{\mu_1' t} + B_1 e^{\mu_1'' t} \right) \sin \frac{\pi x}{l} + \left(A_2 e^{\mu_2' t} + B_2 e^{\mu_2'' t} \right) \sin \frac{2\pi x}{l} + \dots \quad (55)$$

and

$$\dot{y} = \left(A_1 \mu_1' e^{\mu_1' t} + B_1 \mu_1'' e^{\mu_1'' t} \right) \sin \frac{\pi x}{l} + \left(A_2 \mu_2' e^{\mu_2' t} + B_2 \mu_2'' e^{\mu_2'' t} \right) \sin \frac{2\pi x}{l} + \dots \quad (56)$$

Putting $t=0$ in (55) and equating coefficients to those for the approximate circle in (9), we find, when s is odd

$$A_s + B_s = \frac{1}{s^3} \frac{32}{\pi^3} y_0 \dots \dots \dots (57)$$

When even

$$A_s + B_s = 0.$$

Equating all the coefficients in (56) to zero and combining with (57) we have, when s is odd,

$$A_s = -\frac{1}{s^3} \frac{32}{\pi^3} \frac{\mu_s''}{\mu_s' - \mu_s''} y_0 \dots \dots \dots (58)$$

and
$$B_s = \frac{1}{s^3} \frac{32}{\pi^3} \frac{\mu_s'}{\mu_s' - \mu_s''} y_0 \dots \dots \dots (59)$$

When s is even $A_s = B_s = 0$, and hence

$$y = \frac{32}{\pi^3} y_0 \left\{ \frac{1}{1^3 (\mu_1' - \mu_1'')} \left(-\mu_1'' e^{\mu_1' t} + \mu_1' e^{\mu_1'' t} \right) \sin \frac{\pi x}{l}, \right. \\ \left. + \frac{1}{3^3 (\mu_3' - \mu_3'')} \left(-\mu_3'' e^{\mu_3' t} + \mu_3' e^{\mu_3'' t} \right) \sin \frac{3\pi x}{l} + \dots \right\} \quad (60)$$

The values of μ_s' and μ_s'' have been given in (33) and (34), by which $\mu_s' - \mu_s'' = 2n_s'$; hence

$$y = \frac{32}{\pi^3} y_0 \left\{ \frac{1}{1^3 \cdot 2n_1'} \left[\left(\frac{k_1}{2} + n_1' \right) e^{-\left(\frac{k_1}{2} - n_1' \right) t} - \left(\frac{k_1}{2} - n_1' \right) e^{-\left(\frac{k_1}{2} + n_1' \right) t} \right] \sin \frac{\pi x}{l} \right. \\ \left. + \frac{1}{3^3 \cdot 2n_3'} \left[\left(\frac{k_3}{2} + n_3' \right) e^{-\left(\frac{k_3}{2} - n_3' \right) t} - \left(\frac{k_3}{2} - n_3' \right) e^{-\left(\frac{k_3}{2} + n_3' \right) t} \right] \sin \frac{3\pi x}{l} + \dots \right\} \quad (61)$$

This equation gives the complete motion of the string in terms of the four physical constants T_1 , ρ , l , and k . If the external circuit is open $k_s = k$, and

$$n_s' = \left(\frac{k^2}{4} - n_s^2 \right)^{\frac{1}{2}} = \left(\frac{k^2}{4} - \frac{s^2 \pi^2 T_1}{l^2 \rho} \right)^{\frac{1}{2}}.$$

As the tension of the string is increased the values of n_s' become imaginary one after another, and the oscillatory case applies when s is greater than a fixed value, while the non-oscillatory solution applies to the fundamental value of $s (=1)$ and all others less than the fixed value.

The equation (61) reduces to (9), when $t=0$, and evidently represents the arc of a circle. The motion of the string shadow at the central point is not a simple case of logarithmic decrement where the successive ratios of the deflexions in equal time intervals are constant. Prof. Einthoven* remarks in referring to this point that a distinct difference was often found between the ratios of the deflexions at uniform time intervals, so that he was often obliged to calculate an average value. A knowledge of the constants above given should make it possible to calculate these discrepancies.

It is not necessary to cite other particular cases which are included under the general equation, but it is worth remarking that in every instance there are harmonics present in the motion of the string shadow which are directly traceable to the form of the curve assumed by the string when deflected by a steady direct current. It is undesirable to have these introduced into the record as they are, simply because of the construction of the instrument. If the form of curve is such that all the coefficients in the equation of the string (9), when developed by Fourier's series, vanish except the first one, this is evidently the best form for the curve of the string since all the higher harmonics are absent. Equation (54), for example, reduces to the first

* W. Einthoven, "Weitere Mitteilungen über das Saitengalvanometer," *Annalen der Physik*, 4 Folge, vol. xxi. p. 490 (1906).

term only and the form of the string is a simple curve of sines.

This result may be obtained by so altering the field distribution along the string as to make the string assume the sine form. The conditions for this result may be examined by the general differential equations of the string (1) and (2). The equation of the string should be

$$y = y_0 \sin \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The force applied to the element of the string is always perpendicular to the string, so $\cos \chi = 0$, and the string tension is constant at all points as was the case with the uniform field. By (4) we have

$$r = \frac{T_1}{I} \frac{1}{H}$$

and, as r the radius of curvature is now variable, it follows that H must vary inversely as r , the coefficient being constant. The general expression for the radius of curvature of any curve is

$$r = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Applying this to the sine curve (62),

$$\frac{1}{r} = \frac{\pi^2}{l^2} y_0 \frac{\sin \frac{\pi x}{l}}{\left[1 + \frac{\pi^2}{l^2} y_0^2 \cos^2 \frac{\pi x}{l} \right]^{\frac{3}{2}}}.$$

And since in practice $\frac{\pi^2}{l^2} y_0^2$ is a very small quantity compared with unity, the second term in the bracket may be dropped, giving

$$H = \frac{\pi^2 T_1}{l^2 I} y_0 \sin \frac{\pi x}{l} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

for the field distribution, which is as expected a curve of sines having the maximum field at the centre of the string.

This points to a possible improvement in the string galvanometer to be brought about by tapering the pole-pieces to increase the air-gap from the centre towards the

ends of the string. It is not suggested that the shape of the pole-pieces may be calculated, but it seems as if the result may be obtained empirically with sufficient approximation to reduce to an imperceptible amount the undesired harmonics. The field should then be used at the same field excitation at which the measurements were taken because a change in the exciting current will probably change the distribution. The total magnetic flux may be reduced a little but the tapering will tend to increase the intensity near the centre above the value it had for a uniform field, and diminish it near the ends, that is, to redistribute rather than greatly reduce the total field, and consequently the sensitivity would not be reduced as much as might be expected.

When there is inductance in the circuit of the galvanometer the differential equation (21) may be used, but its general solution involves the solution of a cubic equation. The solution is feasible in certain particular cases and it may be serviceable for some kinds of work with the galvanometer. If an harmonic current is assumed the permanent periodic portion is readily found; and this applies to the study of sounds when there is an induction-coil in circuit.

It is hoped that on a future occasion some records taken with the galvanometer may be given to compare the recorded and calculated curves, and to give numerical values of the constants.

XXVI. *Dynamical Theory of Diffraction.*

By Prof. D. N. MALLIK, B.A., Sc.D., F.R.S.E.*

1. IF T is the kinetic energy of a strained elastic medium of density σ , having displacements ξ, η, ζ at a point x, y, z , then

$$2T = \int \sigma (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) dx dy dz. \quad . \quad . \quad (1)$$

Again, if W = the potential energy of deformation, then, on MacCullaugh's theory, as well as Lord Kelvin's,

$$\delta W = \delta \int 2n(\omega_x^2 + \omega_y^2 + \omega_z^2) dx dy dz, \quad . \quad . \quad (2)$$

where ω_x , &c. are molecular rotations, and δ the operator of the calculus of variation (n being the rigidity).

On the other hand, the electromagnetic energy in a

* Communicated by the Author. Paper read before the First Indian Science Congress, Calcutta.