Review
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The most striking characteristic of Prof. Bôcher's exposition is the closeness with which he follows the historical order of the development of the theory. Here, we think, he has probably been wise, though the method is one that has its disadvantages as well as its advantages. The advantages are obvious; the historical order is always the most interesting and the easiest to follow, and the reader is better able to preserve his sense of proportion when different sides of the theory are presented to him in turn. The chief disadvantage is that, when the author's space is as strictly limited as it is here, and when so much of it is devoted to the work of the pioneers, the more systematic modern theories are apt to receive less than their due. And in this tract we certainly wish that less space had been given to Volterra and more to Hilbert and his followers, and in particular that room had been found for some general account of the connection between Integral Equations and Differential Equations, and of the application of this theory to expansions such as Fourier's—to us at any rate the most remarkable and interesting of all its applications.

Still, it would be unreasonable to expect everything in 70 pages, and the wonder is that Prof. Bôcher has been able to give us so much as he has without compressing his argument beyond the limits of intelligibility. He is, indeed, considering the amount of information he contrives to give us in so short a space, extraordinarily lucid and readable throughout, and almost succeeds in making a difficult subject seem easy; and it is in many ways an excellent thing that he should have given us so full an account of the work of Abel and the other precursors of the theory—work which those who are familiar only with the writings of the German school might be in danger of forgetting.

May we make one remark of a technical character with reference to the formula (p. 6, f.n.)

$$\frac{d}{dx}\int_a^x \psi(x-\xi)\phi(\xi)\,d\xi=\int_a^x \psi(x-\xi)\phi'(\xi)\,d\xi,$$

where $\psi(x-\xi)$ is a function such as $(x-\xi)^{-s}$ $(0<s<1)$ and $\phi(a)=0$? The simplest method of procedure seems to be to put $x-\xi=u$ before differentiating, when the result follows at once. This method also leads to the more general formula

$$\frac{d}{dx}\int_a^x \psi(x-\xi)\phi(\xi,x)\,d\xi=\psi(x-a)\phi(a,x)+\int_a^x \psi(x-\xi)\left\{\frac{\partial\phi}{\partial\xi}+\frac{\partial\phi}{\partial x}\right\}d\xi.$$

G. H. Hardy.

Les systèmes d'équations aux dérivées partielles. By C. Riquier. Pp. xxvii + 590. Price 20 fr. 1910. (Paris, Gauthier-Villars.)

This book treats the theory of systems of partial differential equations from Cauchy's standpoint, the author's main object being to establish the existence of a set of solutions which can be expressed in the form of power series. The initial conditions by which the solutions are specified are of the same general type as those used in the existence theorems of Cauchy and Mme. Kowalewsky.

By following up an idea introduced by Méray in 1880, M. Riquier has succeeded in defining a class of systems of partial differential equations, called completely integrable, for which the solutions in the form of power series are specified *uniquely* by the initial conditions and for which the series converge when the arbitrary functions and constants occurring in the initial conditions and equations are subject to certain inequalities.

A useful distinction was drawn by Méray between principal variables and derivatives on the one hand, and parametric variables and derivatives on the other. The distinction may be illustrated by taking a single equation,

$$\frac{\partial^2 u}{\partial t^2}=\psi\left(\frac{\partial^2 u}{\partial x^2},\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial t},\ u\right)$$

with the initial conditions

$$u=f(x),\quad \frac{\partial u}{\partial t}=g(x)\ \text{ for }\ t=0.$$

In this case t is called a principal variable, x a parametric variable. The reason for this is that the initial values of a large number of derivatives depending on x may be calculated at once from the initial conditions; these derivatives are called parametric. The initial values of the other derivatives must be calculated from the differential equation and from the system of equations which are derived from it by differentiation. These derivatives are the principal ones.

The system of equations which is obtained from the original system by repeated differentiations with regard to the different variables is called the prolongment of the original system. In proving the existence theorem it is necessary to replace these equations by an equivalent system resolved with regard to the principal derivatives. A condition that this should be possible is that certain Jacobians should not vanish. The possession of this property of resolubility is the essential characteristic of a passive system of partial differential equations. A second property is that the resolution should be unique. A completely integrable system of equations is a passive system in which certain inequalities are satisfied and for which the existence theorem can be completely established. The first four chapters of the book are devoted to the general theory of functions of several variables. Chapter VIII. deals with the properties of implicit functions, and contains proofs of some general theorems which are often required, for instance: If the $(\mu+1)$ functions

$$f_1(t_1, \dots t_n),\quad f_2(t_1, \dots t_n), \dots f_\mu(t_1, \dots t_n),\quad f(t_1, \dots t_n),$$
$$\mu < n,$$

can be expanded as power series in the neighbourhood of certain initial values, and one at least of the Jacobians of the first μ functions has a value different from zero, and all the Jacobians of the $\mu+1$ functions are zero, then f can be expressed in terms of $f_1, f_2, \dots f_\mu$.

The author gives some applications of his theory to the transformation of quadratic differential forms, and considers systems of partial differential equations which are reducible to completely integrable forms and other systems which are reducible to total differential equations.

The book contains much that is new, but it is of an advanced nature, and deals with a branch of the subject of partial differential equations which is barely mentioned in other treatises. We can recommend it to an expert but not to a mathematician with only a moderate knowledge of partial differential equations and the modern theory of functions. H. Bateman.

Leçons sur la théorie de la croissance. E. Borel. 1910. (Gauthier-Villars.)

It is now twelve years since the appearance of M. Borel's *Leçons sur la théorie des fonctions*, the first of the admirable series of monographs on the Theory of Functions published under his direction. The object of the series was to provide connected and reasonably compact accounts of modern developments in analysis that have not yet found their way into the standard books. They were to occupy a position half-way between the *Traité d'Analyse* and the original memoir. Beyond question the series has been a most brilliant success, and many of its volumes are quite indispensable to any serious student of analysis: one has only to think of M. Lebesgue's *Leçons sur l'intégration*, of M. Baire's *Fonctions discontinues*, of M. Lindelöf's *Calcul des résidus*, or, above all, of M. Borel's own *Fonctions entières* and *Séries divergentes*. We are therefore naturally reluctant to use any language save that of praise in connection with an addition to the series. But we must confess that we are disappointed with M. Borel's latest volume.

Its cardinal defect is that it falls asunder into two almost entirely disconnected parts. In the first 70 pages we have the pure *Théorie de la croissance*—the theory of orders of greatness and smallness, the Infinitärcalcül of Paul du Bois-Reymond—or rather, we have a little of it; and there is no question that it is a fascinating subject, and that much of what M. Borel tells us is very interesting. Then we have two long chapters of analytical and arithmetical applications, and these chapters, too, contain interesting results, and a few that we had not seen before. But they are choked by a mass of work with which almost everybody who reads the book is bound to be perfectly familiar. Is this really the place for yet another account of the ordinary theory of the Gamma-function (pp. 88-104) or the elements of the theory of continued fractions (pp. 127 *et seq.*)? Surely we are entitled to expect something a little more exciting from a writer of M. Borel's reputation and originality. "Les Chapitres d'applications," says M. Borel, "ont été rédigés de manière que leurs résultats essentiels puissent être compris indépendamment du système de notation exposé dans les premiers