THE TORQUE BETWEEN THE TWO COILS OF AN
ABSOLUTE ELECTRODYNAMOMETER.¹

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AN absolute electro-dynamometer employing two single layer coils, similar to an instrument described by the writer and Dr. Karl E. Guthe in a paper read before Section B of the American Association for the Advancement of Science, is at present being used by Professor Henry S. Carhart and the writer in an absolute determination of the electromotive force of standard cells. The former instrument was employed by Patterson and Guthe² in a determination of the electrochemical equivalent of silver, and by Carhart and Guthe³ in a determination of the electromotive force of the Clark cell. Both instruments were designed to take advantage of a simple formula given by Gray⁴ for the case in which the centers of the axes of both coils are coincident and both coils have lengths $\sqrt{3}$ times their radii respectively.

The following derivation of a general formula is along different lines from that of Gray, but leads to the same conclusion. I have thought it desirable to consider the effect which the asymmetry of the leading wires might produce and also the effect of the earth's magnetic field, with a view to the elimination of these effects from the final result.

It will be assumed in advance that the magnetic field at points near the axis of a single layer coil of fine wire carrying a current is equivalent to that produced by a current sheet of the same effective radius and of the same number of ampere turns; and that the torque produced on the suspended coil, wound with a copper rib-

¹ Read at the Washington meeting of the American Physical Society, April 22, 1904.

² Patterson and Guthe, *Proc. A. A. A. S.*, 1898, p. 154, and *PHYSICAL REVIEW*, Dec., 1898, Vol. VII., p. 257.

³ Carhart and Guthe, *Proc. A. A. A. S.*, 1899, p. 103, and *PHYSICAL REVIEW*, Nov.-Dec., 1899, Vol. IX., p. 288.

⁴ A. Gray, *Theory and Pract. Absolute Meas. in Elect. and Mag.*, Vol. II., part I., p. 275.

bon, is equal to that which would be obtained with a corresponding current sheet as suspended coil.

First let us find the strength of the magnetic field at a point P (Fig. 1) on the axis OP of a circular turn of wire carrying a current of intensity I . The point P is at a distance r from the wire, and ξ from the center O of the circle which has a radius R . The strength f of the magnetic field at the point P is

$$f = \frac{2\pi R^2 I}{(R^2 + \xi^2)^{3/2}}. \quad (1)$$

The magnetic potential V_s at the point P is

$$V_s = - \int_0^\xi f d\xi = - \frac{2\pi I \xi}{\sqrt{R^2 + \xi^2}}, \quad (2)$$

The potential at the point O is taken as zero.

The potential V_0 at P (Fig. 2) due to the same current I passing

Fig. 1.

Fig. 2.

through n turns of fine wire wound on a cylinder of length L , with an effective radius R , is

$$V_0 = \frac{n}{L} \int_{x-\frac{L}{2}}^{x+\frac{L}{2}} V_s d\xi = - \frac{2\pi n I}{L} \left[\sqrt{R^2 + \left(x + \frac{L}{2}\right)^2} - \sqrt{R^2 + \left(x - \frac{L}{2}\right)^2} \right].$$

As the right hand term of this equation does not develop into a series which is convenient, we shall introduce the value of the half diagonal D obtained from the relation

$$D^2 = R^2 + \frac{L^2}{4}.$$

We obtain :

$$V_0 = - \frac{2\pi n I D}{L} \left(\sqrt{1 + \frac{x(x+L)}{D^2}} - \sqrt{1 + \frac{x(x-L)}{D^2}} \right). \quad (3)$$

When (3) is expanded and the collected terms arranged by powers of x , we obtain

$$\begin{aligned}
 V_0 = -\pi nI \left[2 \frac{x}{D} - \left(1 - \frac{1}{4} \frac{L^2}{D^2} \right) \frac{x^3}{D^3} + \left(\frac{3}{4} - \frac{5}{8} \frac{L^2}{D^2} + \frac{7}{64} \frac{L^4}{D^4} \right) \frac{x^5}{D^5} \right. \\
 - \left(\frac{5}{8} - \frac{35}{32} \frac{L^2}{D^2} + \frac{63}{128} \frac{L^4}{D^4} - \frac{35}{512} \frac{L^6}{D^6} \right) \frac{x^7}{D^7} \\
 + \left(\frac{35}{64} - \frac{105}{64} \frac{L^2}{D^2} + \frac{693}{512} \frac{L^4}{D^4} - \frac{429}{1024} \frac{L^6}{D^6} + \frac{715}{16384} \frac{L^8}{D^8} \right) \frac{x^9}{D^9} \quad (4) \\
 - \left(\frac{63}{128} - \frac{1155}{512} \frac{L^2}{D^2} + \frac{3003}{1024} \frac{L^4}{D^4} - \frac{6435}{4096} \frac{L^6}{D^6} + \frac{12155}{32768} \frac{L^8}{D^8} \right. \\
 \left. \left. - \frac{4199}{131072} \frac{L^{10}}{D^{10}} \right) \frac{x^{11}}{D^{11}} + \text{etc.} \right].
 \end{aligned}$$

For convenience in handling equation (4) we may indicate the coefficients of the successive odd powers of x/D by M , N , Q , S , U and W as follows :

$$M = 2, \quad N = 1 - \frac{1}{4} \frac{L^2}{D^2}, \quad Q = \frac{3}{4} - \frac{5}{8} \frac{L^2}{D^2} + \frac{7}{64} \frac{L^4}{D^4},$$

$$S = \frac{5}{8} - \frac{35}{32} \frac{L^2}{D^2} + \frac{63}{128} \frac{L^4}{D^4} - \frac{35}{512} \frac{L^6}{D^6},$$

$$U = \frac{35}{64} - \frac{105}{64} \frac{L^2}{D^2} + \frac{693}{512} \frac{L^4}{D^4} - \frac{429}{1024} \frac{L^6}{D^6} + \frac{715}{16384} \frac{L^8}{D^8},$$

and

$$\begin{aligned}
 W = \frac{63}{128} - \frac{1155}{512} \frac{L^2}{D^2} + \frac{3003}{1024} \frac{L^4}{D^4} - \frac{6435}{4096} \frac{L^6}{D^6} + \frac{12155}{32768} \frac{L^8}{D^8} \\
 - \frac{4199}{131072} \frac{L^{10}}{D^{10}}.
 \end{aligned}$$

Equation (4) then becomes

$$\begin{aligned}
 V_0 = -\pi nI \left[M \frac{x}{D} - N \frac{x^3}{D^3} + Q \frac{x^5}{D^5} - S \frac{x^7}{D^7} + U \frac{x^9}{D^9} \right. \\
 \left. - W \frac{x^{11}}{D^{11}}, \text{ etc.} \right]. \quad (5)
 \end{aligned}$$

To find the potential V at a point off the axis, coördinates x , ρ , we shall make use of Laplace's equation, which for a magnetic field symmetrical about the axis of the coil, becomes¹

¹ Mascart et Joubert, L'Elect. et le Mag. (Edit., 1896), I., p. 396.

$$\frac{\partial^2 V}{\partial x^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial \rho^2} = 0.$$

For points near the axis, including all points within the coil, V may be expanded in an equation of the form

$$V = V_0 + A_2 \rho^2 + A_4 \rho^4 + A_6 \rho^6 + \text{etc.},$$

and values of A_2, A_4 , etc., may be found which separately satisfy Laplace's equation. Because of symmetry, no odd powers of ρ can appear in the series.

The convergence of the series is rapid when applied to our electro-dynamometer, for which the largest value of ρ is only about one fourth of R and one fifth of D .

When the values of A_2, A_4 , etc., have been substituted the equation becomes

$$V = V_0 - \frac{\rho^2}{2^2} \frac{\partial^2 V_0}{\partial x^2} + \frac{\rho^4}{(2.4)^2} \frac{\partial^4 V_0}{\partial x^4} - \frac{\rho^6}{(2.4.6)^2} \frac{\partial^6 V_0}{\partial x^6} \\ + \text{etc.} + \frac{(-1)^n \rho^{2n}}{2^{n2} n!} \frac{\partial^{2n} V_0}{\partial x^{2n}} + \text{etc.} \quad (6)$$

If the value of V_0 , equation (5), is substituted in (6) we obtain :

$$V = -\pi n l \left[M \frac{x}{D} - N \frac{x^3}{D^3} + Q \frac{x^5}{D^5} - S \frac{x^7}{D^7} + U \frac{x^9}{D^9} - W \frac{x^{11}}{D^{11}} + \text{etc.} \right. \\ + \frac{3}{2} N \frac{x \rho^2}{D^3} - 5 Q \frac{x^3 \rho^2}{D^5} + \frac{21}{2} S \frac{x^5 \rho^2}{D^7} - 18 U \frac{x^7 \rho^2}{D^9} + \frac{55}{2} W \frac{x^9 \rho^2}{D^{11}} - \text{etc.} \\ + \frac{15}{8} Q \frac{x \rho^4}{D^5} - \frac{105}{8} S \frac{x^3 \rho^4}{D^7} + \frac{189}{4} U \frac{x^5 \rho^4}{D^9} - \frac{495}{4} W \frac{x^7 \rho^4}{D^{11}} + \text{etc.} \\ + \frac{35}{16} S \frac{x \rho^6}{D^7} - \frac{105}{4} U \frac{x^3 \rho^6}{D^9} + \frac{495}{4} W \frac{x^5 \rho^6}{D^{11}} - \text{etc.} \\ + \frac{315}{128} U \frac{x \rho^8}{D^9} - \frac{5575}{128} W \frac{x^3 \rho^8}{D^{11}} + \text{etc.} \\ + \frac{693}{256} W \frac{x \rho^{10}}{D^{11}} - \text{etc.} \\ \left. + \text{etc.} \right] \quad (7)$$

In equation (7) we may substitute the relation $\rho^2 = y^2 + z^2$, thus replacing cylindrical by cartesian coördinates. We take y horizontal and z vertical both at right angles to the axis and to x . The components of the magnetic field are

$$X = -\frac{\partial V}{\partial x}, \text{ horizontal and parallel to } x,$$

$$Y = -\frac{\partial V}{\partial y}, \text{ horizontal and parallel to } y,$$

$$Z = -\frac{\partial V}{\partial z}, \text{ vertical and parallel to } z.$$

As in our electro-dynamometer, the suspended coil is always used with its axis at right angles to that of the fixed coil; we shall consider the torque of one coil on the other for this position only.

First, let us derive the expression for the torque on a single turn of the suspended coil. The equation for this turn is $x^2 + z^2 = r^2$, in which r is the radius. A conductor carrying a current tends to move across a magnetic field. The force on an element of the conductor equals the product of the strength of the magnetic field, the intensity of the current and the projection of the element on a plane normal to the field. The moment of this force, or the torque, about the axis of suspension (passing through the center of the coil) for an element whose coördinates are x, y, z , is derived as follows: The turn of effective radius r lies in a plane normal to the y axis. Assume that the element is taken in a position for which the coördinates are all positive, and that the direction of the current is taken clockwise, positive with respect to x , negative with respect to z . The products of the current and the projections of the element are: $-Idz$, for the y, z plane; $-Idz$, for the x, z plane; $+Idx$, for the x, y plane. A consideration of the magnetic field, within and due to the fixed coil, shows that it is symmetrical about the axis; that it converges for negative values of x and diverges for positive values, and that it has its largest value for the plane $x = 0$. It follows that the components of the field are all positive for a point for which x, y and z are all positive; that all the components of the torque are clockwise (looking downward), and that every element, wherever situated, will

give a clockwise torque, as the factors always change signs in pairs. The expression for the torque on the element (with projections dx , dz) is

$$dT_s = I[-(xX + yY)dz + xZdx]. \quad (8)$$

As $xdy = -\sqrt{r^2 - x^2}dz$, we obtain

$$dT_s = I\left[\frac{x^2X + xyY}{\sqrt{r^2 - x^2}} + xZ\right]dx. \quad (8')$$

As the value for each quadrant is the same, the torque T_s for one term is four times the integral of (8') over one quadrant, or

$$T_s = 4I \int_{x=0}^{x=r} \left[\frac{x^2X + xyY}{\sqrt{r^2 - x^2}} + xZ\right]dx. \quad (9)$$

The torque due to the whole coil of length l and n' turns is

$$T = \frac{n'}{l} \int_{y=-\frac{l}{2}}^{y=+\frac{l}{2}} T_s dy. \quad (10)$$

Performing the operations indicated in (8') we find

$$\begin{aligned} x^2X &= \pi n I \left\{ \frac{M}{D} x^2 - \frac{N}{D^3} \left[3x^4 - \frac{3}{2} x^2 \rho^2 \right] \right. \\ &\quad + \frac{Q}{D^5} \left[5x^6 - 15x^4 \rho^2 + \frac{15}{8} x^2 \rho^4 \right] \\ &\quad - \frac{S}{D^7} \left[7x^8 - \frac{105}{2} x^6 \rho^2 + \frac{315}{8} x^4 \rho^4 - \frac{35}{16} x^2 \rho^6 \right] \\ &\quad + \frac{U}{D^9} \left[9x^{10} - 126 x^8 \rho^2 + \frac{945}{4} x^6 \rho^4 - \frac{315}{4} x^4 \rho^6 + \frac{315}{128} x^2 \rho^8 \right] \\ &\quad \left. - \frac{W}{D^{11}} [\text{etc.}] \right\} \\ xyY &= \pi n I \left\{ \frac{N}{D^3} \left[3x^2 y^2 \right] - \frac{Q}{D^5} \left[10x^4 y^2 - \frac{15}{2} x^2 y^2 (y^2 + z^2) \right] \right. \\ &\quad + \frac{S}{D^7} \left[21x^6 y^2 - \frac{105}{2} x^4 y^2 (y^2 + z^2) + \frac{105}{8} x^2 y^2 (y^2 + z^2)^2 \right] \\ &\quad \left. - \frac{U}{D^9} \left[36x^8 y^2 - 189x^6 y^2 (y^2 + z^2) + \frac{315}{2} x^2 y^2 (y^2 + z^2)^2 \right] \right\} \end{aligned}$$

$$\begin{aligned}
& -\frac{315}{16}x^2y^2(y^2+z^2)^3 \Big] + \frac{W}{D^{11}} [\text{etc.}] - \text{etc.} \Big\} \\
xZ = \pi nI & \left\{ \frac{N}{D^3} [3x^2z] - \frac{Q}{D^5} \left[10x^4z - \frac{15}{2}x^2z(y^2+z^2) \right] \right. \\
& + \frac{S}{D^7} \left[21x^6z - \frac{105}{2}x^4z(y^2+z^2) + \frac{105}{8}x^2z(y^2+z^2)^2 \right] \\
& - \frac{U}{D^9} \left[36x^8z - 189x^6z(y^2+z^2) + \frac{315}{2}x^4z(y^2+z^2)^2 \right. \\
& \left. \left. - \frac{315}{16}x^2z(y^2+z^2)^3 \right] + \frac{W}{D^{11}} [\text{etc.}] - \text{etc.} \right\}
\end{aligned}$$

For the integration indicated in equation (9) z and ρ are functions of x , ($z^2 = r^2 - x^2$ and $\rho^2 = y^2 + r^2 - x^2$). Making these substitutions we obtain

$$\begin{aligned}
x^2X = \pi nI & \left\{ \frac{M}{D} x^2 - \frac{N}{D^3} \left[\frac{9}{2}x^4 - \frac{3}{2}(y^2+r^2)x^2 \right] \right. \\
& + \frac{Q}{D^5} \left[\frac{175}{8}x^6 - \frac{75}{4}(y^2+r^2)x^4 + \frac{15}{8}(y^2+r^2)^2x^2 \right] \\
& - \frac{S}{D^7} \left[\frac{1617}{16}x^8 + \frac{2205}{16}(y^2+r^2)x^6 + \frac{735}{16}(y^2+r^2)^2x^4 \right. \\
& \left. - \frac{35}{16}(y^2+r^2)^3x^2 \right] + \frac{U}{D^9} \left[\frac{56915}{128}x^{10} - \frac{27027}{32}(y^2+r^2)x^8 \right. \\
& + \frac{31185}{64}(y^2+r^2)^2x^6 - \frac{2835}{32}(y^2+r^2)^3x^4 \\
& \left. + \frac{315}{128}(y^2+r^2)^4x^2 \right] - \frac{W}{D^{11}} [\text{etc.}] + \text{etc.} \Big\}; \\
xyY = \pi nI & \left\{ \frac{N}{D^3} [3y^2x^2] - \frac{Q}{D^5} \left[\frac{35}{2}y^2x^4 - \frac{15}{2}y^2(y^2+r^2)x^2 \right] \right. \\
& + \frac{S}{D^7} \left[\frac{693}{8}y^2x^6 - \frac{315}{4}y^2(y^2+r^2)x^4 + \frac{105}{8}y^2(y^2+r^2)^2x^2 \right] \\
& \left. - \frac{U}{D^9} \left[\frac{6435}{16}y^2x^8 - \frac{9009}{16}y^2(y^2+r^2)x^6 + \frac{3465}{16}y^2(y^2+r^2)^2x^4 \right] \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{315}{16}y^2(j^2+r^2)^3x^2 \Big] + \frac{W}{D^{11}} [\text{etc.}] - \text{etc.} \Big\}; \\
xZ = \pi n I \sqrt{r^2 - x^2} & \left\{ \frac{N}{D^3} 3x^2 - \frac{Q}{D^5} \left[\frac{35}{2}x^4 - \frac{15}{2}(j^2+r^2)x^2 \right] \right. \\
& + \frac{S}{D^7} \left[\frac{693}{8}x^6 - \frac{315}{4}(j^2+r^2)x^4 + \frac{105}{8}(j^2+r^2)^2x^2 \right] \\
& - \frac{U}{D^9} \left[\frac{6435}{16}x^8 - \frac{9009}{16}(j^2+r^2)x^6 + \frac{3465}{16}(j^2+r^2)^2x^4 \right. \\
& \left. \left. - \frac{315}{16}(j^2+r^2)^3x^2 \right] + \frac{W}{D^{11}} [\text{etc.}] - \text{etc.} \right\}.
\end{aligned}$$

For convenience in performing the integrations, let us tabulate the values of the integrals of the form

$$4 \int_0^r \frac{x^{2n} dx}{\sqrt{r^2 - x^2}} = \frac{(2n-1)(2n-3) \cdots 3}{2n(2n-2) \cdots 4} \pi r^{2n}.$$

Value of n	1	2	3	4	5	6
Value of the integral	πr^2	$\frac{3}{4}\pi r^4$	$\frac{5}{8}\pi r^6$	$\frac{35}{64}\pi r^8$	$\frac{63}{128}\pi r^{10}$	$\frac{321}{512}\pi r^{12}$

In the same manner for the integral

$$4 \int_0^r x^{2n} \sqrt{r^2 - x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} 2\pi r^{2n+2}$$

we obtain

Value of n	1	2	3	4	5
Value of the integral	$\frac{\pi}{4}r^4$	$\frac{\pi}{8}r^6$	$\frac{5}{64}\pi r^8$	$\frac{7}{128}\pi r^{10}$	$\frac{21}{512}\pi r^{12}$

The integrals involving X and Y are of the former form and that involving Z is of the latter form. We obtain

$$\begin{aligned}
4I \int_0^r \frac{x^2 X + xyY}{\sqrt{r^2 - x^2}} dx = \pi^2 r^2 n I^2 & \left[\frac{M}{D} + \frac{N}{D^3} \left(\frac{9}{2}y^2 - \frac{15}{8}r^2 \right) \right. \\
& + \frac{Q}{D^5} \left(\frac{75}{8}y^4 - \frac{225}{16}y^2 r^2 + \frac{95}{64}r^4 \right) + \frac{S}{D^7} \left(\frac{245}{16}y^6 \right. \\
& \left. \left. - \frac{3885}{64}y^4 r^2 + \frac{4095}{128}y^2 r^4 - \frac{1435}{1024}r^6 \right) \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{U}{D^9} \left(\frac{2835}{128} y^8 - \frac{20475}{128} y^6 r^2 + \frac{105525}{512} y^4 r^4 - \frac{108675}{2048} y^2 r^6 \right. \\
& \quad \left. + \frac{22365}{16384} r^8 \right) + \frac{W}{D^{11}} \left(\frac{7623}{256} y^{10} - \frac{349965}{1024} y^8 r^2 \right. \\
& \quad \left. + \frac{832755}{1024} y^6 r^4 - \frac{4244625}{8192} y^4 r^6 + \frac{2595285}{32768} y^2 r^8 \right. \\
& \quad \left. - \frac{45903363}{131072} r^{10} \right) + \text{etc.} \Big],
\end{aligned}$$

$$\begin{aligned}
4I \int_0^r xZdx = \pi^2 r^2 n I^2 \Big[\frac{N}{D^3} \frac{3}{4} r^2 \\
- \frac{Q}{D^5} \left(\frac{5}{16} r^4 - \frac{15}{8} y^2 r^2 \right) + \frac{S}{D^7} \left(\frac{105}{512} r^6 - \frac{105}{32} y^2 r^4 \right. \\
+ \frac{105}{32} y^4 r^2 \Big) - \frac{U}{D^9} \left(\frac{315}{2048} r^8 - \frac{4725}{1024} y^2 r^6 \right. \\
\left. + \frac{1575}{128} y^4 r^4 - \frac{315}{64} y^6 r^2 \right) + \frac{W}{D^{11}} (\text{etc.}) - \text{etc.} \Big].
\end{aligned}$$

Combining the above integrals we obtain the torque on a single suspended turn,

$$\begin{aligned}
T_s = \pi^2 r^2 n I \Big[\frac{M}{D} + \frac{N}{D^3} \left(\frac{9}{2} y^2 - \frac{9}{8} r^2 \right) + \frac{Q}{D^5} \left(\frac{75}{8} y^4 - \frac{225}{16} y^2 r^2 \right. \\
+ \frac{75}{64} r^4 \Big) + \frac{S}{D^7} \left(\frac{245}{16} y^6 - \frac{3675}{64} y^4 r^2 + \frac{3675}{128} y^2 r^4 - \frac{1225}{1024} r^6 \right) \\
+ \frac{U}{D^9} \left(\frac{2835}{128} y^8 - \frac{19845}{128} y^6 r^2 + \frac{99225}{512} y^4 r^4 - \frac{99225}{2048} y^2 r^6 \right. \\
\left. + \frac{19845}{16384} r^8 \right) + \frac{W}{D^{11}} (\text{etc.}) - \text{etc.} \Big].
\end{aligned}$$

For convenience in performing the integration indicated in (10), it is desirable to tabulate integrals of the form

$$\frac{1}{l} \int_{-\frac{l}{2}}^{+\frac{l}{2}} y^{2p} dy = \left(\frac{l}{2} \right)^{2p} \frac{1}{2p+1}.$$

Value of p	0	1	2	3	4	5
		l^2	l^4	l^6	l^8	l^{10}
Value of the integral	1	$\frac{1}{12}$	$\frac{1}{80}$	$\frac{1}{448}$	$\frac{1}{2304}$	$\frac{1}{11264}$

We then obtain

$$\begin{aligned}
 T = \frac{n'}{l} \int_{-\frac{l}{2}}^{+\frac{l}{2}} T_s dy = \pi^2 r^2 n n' I^2 \left[\frac{M}{D} + \frac{N}{D^3} \left(\frac{3}{8} l^2 - \frac{9}{8} r^2 \right) \right. \\
 + \frac{Q}{D^5} \left(\frac{15}{128} l^4 - \frac{75}{64} l^2 r^2 + \frac{75}{64} r^4 \right) \\
 + \frac{S}{D^7} \left(\frac{35}{1024} l^6 - \frac{735}{1024} l^4 r^2 + \frac{1225}{512} l^2 r^4 - \frac{1225}{1024} r^6 \right) \\
 + \frac{U}{D^9} \left(\frac{315}{32768} l^8 - \frac{2835}{8192} l^6 r^2 + \frac{19845}{8192} l^4 r^4 - \frac{33075}{8192} l^2 r^6 \right. \\
 \left. + \frac{19845}{16384} r^8 \right) + \frac{W}{D^{11}} (\text{etc.}) + \text{etc.} \left. \right].
 \end{aligned}$$

When the values of M , N , Q , etc., have been substituted and the expression simplified, we have

$$\begin{aligned}
 T = \pi^2 r^2 n n' I^2 \left[\frac{2}{D} + \frac{3}{32} \frac{1}{D^5} (4D^2 - L^2)(l^2 - 3r^2) \right. \\
 + \frac{15}{8192} \frac{1}{D^9} (48D^4 - 40D^2L^2 + 7L^4)(l^4 - 10l^2r^2 + 10r^4) \\
 + \frac{35}{524288} \frac{1}{D^{13}} (320D^6 - 560D^4L^2 + 252D^2L^4 - 33L^6)(l^6 - 21l^4r^2 \\
 + 70l^2r^4 - 35r^6) + \frac{315}{536870912} \frac{1}{D^{17}} (8960D^8 - 26880D^6L^2 \\
 + 22176D^4L^4 - 6864D^2L^6 + 715L^8)(l^8 - 36l^6r^2 + 252l^4r^4 \\
 \left. - 105l^2r^6 + 126r^8) - \text{etc.} \right].
 \end{aligned}$$

The series expressing the value of T is rapidly convergent if l and r are small in comparison with D . Any term of the series after the first may be reduced to zero by assuming such a relation between D and L or l and r as will make one of the factors zero. The relation $D = L/2$ would make every term after the first zero, but

this relation is equivalent to saying that the diagonal ($2D$) of the coil equals its length, a relation clearly impossible, but one which is approximated when the coil becomes very long, and the field near the middle becomes nearly uniform. If $l^2 = 3r^2$ or $l = r\sqrt{3}$ the second term will vanish. If $12D^2 = 7L^2$ the next term will vanish. This relation is equivalent to $L = R\sqrt{3}$. If this relation between the length and radius of each of the two coils is chosen, the next term of the series which remains (that involving the sixth degree of l and r) is too small for consideration as are all following terms, and the series practically reduces to

$$T = \frac{2\pi^2 r^2 n n' I^2}{D} = \frac{4\pi^2 r^2 n n' I^2}{\sqrt{L^2 + 4R^2}}.$$

In deriving the formula, it has been assumed, that the coils are equivalent to current sheets. It is well to inquire whether or not this assumption may be allowed. The large (fixed) cylinder is wound with wire about 0.07 cm. diameter, including a silk insulation. The suspended cylinder is wound with a copper ribbon about .375 mm. thick and 1.25 mm. wide with spaces of approximately equal width. The cylinders are about 47 cm. and 10 cm. in diameter respectively. As the space between turns of the ribbon on the small cylinder is equal to the ribbon's width, and as the spaces on one side are diametrically opposite to a corresponding portion of the ribbon on the other side, it is evident that the average effect is that of a current sheet. We have assumed that it is proper to take as the effective diameter, the arithmetical mean between the extreme outer and inner diameters, for although the torque depends on the square of the radius for turns at various distances, we must recognize that the innermost layer of the ribbon is relatively shortened in comparison with layers further out. There is a consequent tendency to reduce the resistance and to increase the current density in the inner layers. It would seem probable therefore that the arithmetical mean diameter is fairly taken as the true mean.

The leading wires to the suspended coil might have a disturbing influence unless put in such places as to exert no torque. We provided against such torque by putting the leading wires in the plane normal to the axis of the large coil. The wires lead from

the ends of the suspended coil at opposite ends of the bottom element of the cylinder and they are carried either parallel to this bottom element or vertical, to two mercury cups (one over the other) directly below the centers of both coils. The wires to the mercury cups are twisted together, except for a short space at the cups, where of necessity they are separated. Want of symmetry here may be eliminated by reversing the connection between coils, and in our work it is always so eliminated. The winding of the fixed coil is wire of so small radius that the ripples in the magnetic field are not appreciable, and the arithmetical mean of outer and inner radius of the coil is taken for reasons similar to those mentioned in connection with the suspended coil. The leading wires to the fixed coil go to the ends of an element of the cylinder level with the axis and are twisted together except for a piece parallel and near this element. The latter wire can produce no torque about a vertical axis. Errors due to the effect of the earth's magnetic field are avoided by reversing the current through the whole instrument. For a set of observations the four possible permutations of current through the two coils are taken. The resulting average is practically free from errors due to asymmetry and stray fields.

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