Journal of The Franklin Institute

Devoted to Science and the Mechanic Arts

Vol. 182

AUGUST, 1916

No. 2

EXPERIMENTAL RESEARCHES ON THE SKIN EFFECT IN STEEL RAILS.*

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THE following research was conducted during the years 1914-15-16, at the Massachusetts Institute of Technology, under an appropriation from the American Telephone and Telegraph Company. In the earlier part of 1914, it was conducted under the directorship of Prof. Harold Pender. The research originated in a thesis at the Massachusetts Institute of Technology in 1913-14, by Mr. R. W. Weeks, in the Department of Electrical Engineering.

Objects of the Research.—The research was made to determine, by direct measurement, the impedance offered, to alternating currents of the strengths likely to be employed in electric-railway work, by standard steel rails of various shapes and sizes; also, to ascertain how this impedance varied with frequency and with current-strength, and whether the impedance could be reduced to a simple engineering theory, so as to admit of being predicted, within reasonable limits, from physical data concerning the steel used in the rails. It is believed that some progress is here presented in each of these directions.

Types and Dimensions of Rails Tested.—The tests were extended to 11 sample track rails and 2 sample contact rails, 33 feet

* Communicated by Dr. Kennelly, April 14, 1916.

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(10.07 metres) in length, ranging in linear mass from 60 to 100 pounds avoirdupois per yard (30 to 50 kilogrammes per metre), with corresponding cross-sections from 6 to 10 square inches (38.5 to 64.5 square centimetres). The cross-sections of these rails are represented to scale in Figs. 5 to 10.

Frequencies and Strengths of Alternating Testing Current.—Three frequencies have been used in the tests; namely, 25, 45, and 60 cycles per second. The bulk of the work was, however, carried on at or near two frequencies only; i.e., 25 and 60 cycles per second.

The current-strengths employed in the test rails varied from 10 to 800 ampères r. m. s.

FIG. 1.

Diagram of test circuit.

Arrangement of Test Circuit.—In order to make the measurements as nearly under practical railway conditions as possible, the test circuit was so constructed that the rail and its parallel return conductor occupied the same relative positions as one rail of a railroad track and its overhead trolley wire (No. 0000 A. W. G., diameter 0.46 inch, or 1.17 centimetre); i.e., with the trolley wire 2 feet 4½ inches (71.8 centimetres) to one side of the rail and 22½ feet (6.86 metres) above it. The arrangement of the test circuit is shown in Fig. 1. AB is the rail under test, occupying a horizontal position raised about 1 foot (30 centimetres) above

the ground on brick pillars. The rest of the rectangular loop $BCDEFA$ was composed of straight copper conductors (wire or cable). The extensions AF and BC from the ends of the rail were made so as to enable the electro-magnetic field in the neighborhood to be both more nearly uniform and more readily calculated. The circuit was erected in an alley between two brick buildings, remote from iron. A photograph of the rail in position appears in Fig. 2.

FIG. 2.

Rail in position for test.

Sources of Alternating-current Supply.—For the frequency of $25 \text{ } \omega$, a General Electric 3-phase 50-kva. alternator, 6-pole 514 r. p. m.—230 volts—126 ampères was used. For the frequency of $60 \text{ } \omega$, a Mordey single-phase, 40-kva. alternator was used, with 9 poles and 9 pancake coils, for supplying about 135 ampères r. m. s., at about 300 volts. The arrangement of circuits used is shown in Fig. 3. The transformer T was a General Electric Company's 15-kva., $60 \text{ } \omega$, single-phase transformer from 220 to 18 volts. The secondary terminals were connected directly to the test circuit through two four-terminal manganin resistors, and the

primary terminals to the generator through an adjustable rheostat having very little inductance. At the frequency of $25 \text{ } \omega$, and maximum rail current, the primary voltage impressed on the transformer was about 230 volts, with 130 ampères or 29.9 kva. with 12.5 kw. At the frequency of $60 \text{ } \omega$, the corresponding primary voltage on the transformer was about 330 volts, with 125 ampères. Occasionally, tests were also made at $60 \text{ } \omega$, with currents from

FIG. 3.

General testing connections from generator.

the shop mains, supplied by a General Electric Company's 14-pole, 100-kva., three-phase alternator.

The wave-form of voltage and current supplied from the Mordey generator was nearly sinusoidal, as was shown by numerous oscillograms (see Fig. 4). The wave-form of $25 \text{ } \omega$ voltage, from the 50-kva. generator, showed appreciable harmonics; while the wave-form of its current showed only a moderate triple-frequency harmonic. The wave-form of $60 \text{ } \omega$ voltage and current from the shop mains had yet more marked harmonics.

Method of Measurement.—Since no exact theory of skin effect in steel rails has yet been worked out, it was very desirable to apply more than one method of measurement, in order to secure a satisfactory experimental check upon the results. Much time had to be expended in finding two different methods of testing which would give concordant results, with steel rails, to the requisite degree of precision.

The following were the four methods tried:

FIG. 4.

Mordey a.c. waves.

Method 1.—By astatic electro-dynamometer of fixed and moving coils, measurements were made of rail r. m. s. voltage and of rail r. m. s. current, as well as the active power expended in the rail. By an astatic dynamometer is meant one whose moving system comprises two oppositely wound coils, so arranged as to be substantially unaffected by local stray magnetic fields, either alternating or continuous. Denoting the fixed coils by F and the moving coils by M , the voltage across the rail potential points was measured with F and M in parallel, M having a suitable non-inductive resistance in its branch. The circuit current was meas-

ured by observing the voltage across the manganin strip potential points in a similar manner. The active power in the rail was measured by connecting F to the manganin strip, and M to the rail potential points, through a suitable non-inductive resistance. The dynamometer was calibrated by known continuous current passing through both coils. This method was found to be unsatisfactory. Not only does the impedance of both F and M vary to the harmonics in the traversing currents, but the mutual inductance between F and M also comes into consideration, and this varies with the deflection and angular position of M with respect to F .

Method 2.—*Oscillographic measurements* were made of both the voltage between rail potential points—hereafter called the “rail voltage”—and the current in the rail circuit. Fig. 4 shows one of the oscillograms obtained with 160 ampères in the rail circuit at $60 \sim$ and about 0.72 volt across the rail, the current lagging about 71 degrees behind the voltage. The current wave is seen to be nearly sinusoidal; while the voltage wave shows faintly the presence of harmonics. From these two curves it is theoretically possible to compute the impedance of the rail, and, therefore, its skin effect, when its resistance to continuous currents is known. This method, although useful and convenient as a check, was found to be unsatisfactory in detail, owing to lack of precision in the oscillograms, and also to variations in the constants of the oscillographic vibrators.

Method 3.—*The Drysdale-Tinsley alternating-current potentiometer* * was used at the frequency of $60 \sim$, measuring alternately the vector voltage across the rail and the manganin shunt. The ratio of these two measurements gives the vector impedance of the rail between potential points. This method has the advantage that, since the vibration-galvanometer of the detector circuit is tuned to the fundamental frequency, the harmonics in either voltage or current are virtually excluded, and also that the two measured values, E/α volts and I/β ampères, give as their quotient $Z/\gamma = \frac{E}{I} / \alpha - \beta$ ohms. This reduces the computation of the results to a minimum. Referring to Fig. 3, the alternating-current vector potential difference was measured between points A and B on the rail, and then between points ab on the manganin strip.

A disadvantage of the alternating-current potentiometer

* Bibliography No. 17.

method is its sensitiveness to slight changes of frequency, in the vibration galvanometer and in the phase-splitting device. Under shop conditions of generator driving, it is often hard to keep the frequency constant within satisfactory limits.

Method 4.—The *Sumpner dynamometer* * was used to measure the current I ampères in the rail, and also to measure both active and reactive components of the rail voltage, which may be called E_R and E_x volts respectively. Then the effective resistance of the rail is $R' = E_R / I$ ohms, while the effective inductance is $L' = E_x / (\omega I)$ henrys. An outline of the theory of the use of this instrument is given in Appendix I.

In all four of the above methods, the p. d. on the tested rail was taken from two potential taps near the ends, and 30 feet (914 cm.) apart. Only two methods, viz., numbers 3 and 4, were found to give uniformly consistent results to the desired degree of precision. Tests made of the impedance of a rail by these two methods—*i.e.*, by Drysdale potentiometer and by Sumpner dynamometer—were found not to differ by more than 3 per cent. within the entire range, including discrepancies attributable to temperature variations, and they usually agreed within 1 per cent. Check tests were also made by both methods on the impedance of copper-rod and copper-strand conductors in place of the steel rail, with satisfactory agreement between them. Having thus obtained, after many trials, two fairly consistent methods of measurement, the Sumpner dynamometer method was finally selected as the best adapted to the local testing conditions, reserving the alternating-current potentiometer for occasional checks. Although this dynamometer method is theoretically subject to errors of extra skin effect, due to harmonics in the rail current and voltage waves, this error was not found to be serious. In the first place, the waves of rail current and voltage have not been very distorted, and, in the second place, the average dynamometer action is zero between a current of fundamental frequency in one coil and a current of any harmonic frequency in the other. Moreover, the harmonic component actions, being subject to differential distortion in phase by the reactances in two-coil circuits, are likely to be diminished in effect.

*The Sumpner dynamometer method has the disadvantages (1) that it is not a null method and (2) that it is not free from the effects of harmonics (*Phil. Mag.*, vol. 20, 1910, p. 309).

The Manganin Strip.—In order to supply a working p. d. in phase with the rail current, a manganin resistance strip was employed in the rail circuit, marked *S* in Fig. 3, and provided with two pairs of potential terminals, one across 2788 microhms, and the other across 933 of these microhms. This strip is made of two separate parallel plates of manganin 58.3 cm. (23 inches) long, 10.2 cm. (4 inches) broad, and 1.3 mm. (0.05 inch) thick, separated by mica insulation 1 mm. (0.04 inch) thick; so that the resistor is in the form of a very flat loop with the going and return conductors separated only by a sheet of mica. This strip was supported horizontally in a bath of oil, to keep it cool, when carrying currents above 400 ampères.

Direct-current Resistance Measurements.—In order to determine the skin-effect resistance ratio, it was necessary to measure the direct-current resistance of each rail at various times during a series of observations, and also the temperature coefficient of the rail's resistivity. This was done by direct-current potentiometer measurements in the usual way.

Temperature Measurements of the Rails.—The temperature of the rail was observed by thermometer, both before and after the measurement of impedance, at each current-strength in a test series. The thermometer was laid upon the rail, and had its bulb protected with a pad of putty. The rail currents above 100 ampères appreciably affected the temperature of the rail. Rail currents of about 750 ampères, during the time of application necessary for the measurements, sometimes raised the rail temperatures to 20° C. above that of the surrounding air. In order to avoid unduly large changes of temperature during a test, the highest current-strength was first applied, for a sufficient length of time to reach substantially constant temperature before the first measurement was made, after which the rail was allowed to cool off slowly as the testing current strengths were successively reduced.

Method of Connecting the Main Leads to the Ends of the Rail.—The rail ends had a hole of about 2 cm. ($\frac{13}{16}$ inch) diameter about 7.5 cm. (3 inches) from the ends. The web of the rail was scraped and cleaned near these holes, and copper lugs clamped firmly against the web by steel bolts passing through the holes. The contact resistance at these junctions was sufficiently low to prevent the junctions from getting unduly hot.

Situation of Potential Leads with Respect to Rail Circuit.—In measuring, by potentiometer, the direct-current potential difference between a pair of terminals on a rail, the disposition of the potential leads with respect to the rail circuit is of no material consequence, assuming that proper insulation is maintained. In measuring alternating-current potential differences, however, the disposition of the potential leads with respect to the rail circuit becomes of great importance, since the mutual inductance between these circuits will cause an induced electromotive force to be developed in the potential circuit over and above that due to the vector IZ drop sought, where I is the current and Z the rail impedance. At first thought it might be supposed that the simplest way to obtain the pure IZ drop in the rail would be to carry the insulated potential wires close along the web of the rail, and thence, by a twisted pair, from the middle. Actual trial, however, showed that the mutual inductance correction in the latter case was not only large, but uncertain, being practically beyond computation. By placing the potential leads in the form of a rectangle, as shown in Fig. K, Appendix III, the mutual inductance correction was rendered definite and susceptible of computation. By suitably selecting the sides of the rectangle, the resulting mutual-inductance correction can, by trial, be made small. The potential leads were then maintained permanently in this position during the progress of the work. The principal formula for deriving the correction is worked out in Appendix III.

Any error made in the mutual-inductance correction would clearly not affect the skin-effect resistance ratio R'/R , but would affect the values of the inductance L' . The effect of the error would be to alter the value of L' by a constant quantity throughout. If only the changes in L' with frequency and current are studied, such an error becomes eliminated. It was thought desirable in this research to determine the correction (Appendix III), so as to arrive at the best available values for L' .

Corrections for Temperature.—As has already been mentioned, the temperature of the rails tested was appreciably raised when the rail current exceeded 100 ampères, and even with short applications of 750 ampères the rail temperature was frequently raised 20° C. The resistivity temperature coefficient of the steel forming the rail was in each case measured by taking a series of continuous-current rail-resistance measurements, by potenti-

TABLE I.
Data Concerning Steel Rails Tested.

Rail Number Type	96A	100	163	175	161	162	170	600	174	-51	702 N.Y.C.	type D. con- tact
	P.R.R. stand- ard and track	N.Y. N.H. stand- ard and track	A.R.A. type A and track	N.Y.C. R.R. stand- ard and track	A.R.A. type B and track	A.R.A. type B and track	A.R.A. type A and track	A. S.C.E. stand- ard and track	track	S.C.E. stand- ard and track	R.R. stand- ard and con- tact	
Linear mass.....	100	100	100	100	100	00	00	60	70	80	60.3	80.5
kg./m.....	50	50	50	50	50	45	45	30	35	40	34.0	43.3
Cross-sectional area.....	98	98	98	98	98	88	88	588	686	784	670	848
sq. in.	63.2	63.2	63.2	63.2	63.2	56.9	56.9	18.8	44.3	50.5	43.8	54.6
cm.	22.6	22.6	22.6	22.6	22.6	21.3	21.3	7.6	18.7	21.7	18.2	21.0
Perimeter of cross-section.....	57.4	57.4	57.4	57.4	57.4	53.0	53.0	47.8	47.5	55.1	38.6	53.4
in.	4.41	4.41	4.41	4.44	4.40	4.40	4.40	1.91	1.91	2.21	1.54	2.14
cm.	0.19	0.19	0.19	0.20	0.19	0.19	0.19	0.23	0.18	0.21	0.11	0.17
Approximate relative distribution of sectional area	0.40	0.35	0.40	0.30	0.41	0.41	0.40	0.37	0.40	0.37	0.30	0.35
Base	30	30	30	30	30	30	30	30	30	30	30	30
feet	914	914	914	914	914	914	914	914	914	914	914	914
Length of rail between potential contact points												
Resistance of tested length to continuous currents at 20° C. microhms.	305.6	319.2	322.3	312.3	223.9	339.7	334.0	428.8	420.5	367.7	281.2	210.5
Resistivity at 20° C. ohm-cm. per 100 ft.	211.0	223.0	210.0	155.0	203.0	268.0	178.0	178.0	203.0	203.0	135.0	135.0
Temperature coefficient of resistivity at 20° C. %	0.00272	0.00272	0.00272	0.00272	0.00272	0.00336	0.00331	0.00350	0.00345	0.00395	0.00474	0.00458
Maximum skin effect	0.25	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
Resistance ratio R/R_0	1.08	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05
V_{90}	1.52	1.59	1.59	1.59	1.55	1.56	1.54	1.56	1.51	1.54	1.59	1.64
Ratio, V_{90}	1.54	1.54	1.54	1.54	1.54	1.54	1.54	1.54	1.54	1.54	1.54	1.54
Square foot of ratio of actual test frequencies	6.83	7.02	7.02	7.02	7.02	6.6	6.6	4.2	4.85	6.1	2.7	2.48
Effective area of cross-section at maxi- mum skin effect, square centimetres	10.4	11.2	11.2	11.2	11.2	10.2	10.2	6.5	7.35	9.45	4.0	4.07
Effective depth of penetration in mm. at	1.8	1.8	1.8	1.8	1.8	1.7	1.7	0.9	1.0	1.1	0.7	0.76
Maximum skin effect	1.8	1.8	1.8	1.8	1.8	1.7	1.7	1.35	1.53	1.7	1.04	0.76
Observed i. m. s. current in rail at maxi- mum skin effect, amperes	700	800	800	800	800	600	700	450	505	850	280	160
zinc skin effect, amperes	750	800	800	800	800	670	700	480	540	860	270	140
Computed mean corresponding mag- netic intensity around perimeter	15.3	16.2	16.3	16.2	16.2	13.8	14.15	11.8	13.75	19.4	9.1	3.70
H_{90}	10.4	10.2	10.2	10.2	10.2	8.4	14.15	12.6	14.5	18.2	8.8	3.30
Permeability μ max.												
at H.....												
Computed skin-effect resistance ratio R/R_0												
V_{90}												
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TABLE II.
Observations on and Results for Sample Rail No. 174, Maryland Steel Company, April 27, 1915.
 Cross-section 6.86 sq. in. (44.3 sq. cm.) 70 pounds per yard (35 kg. per m.) Test frequency 60 $\times$.

Moving coil across strip			Moving coil across rail			Rail current, r. m. s. amperes	$\frac{R_{20}}{R_{20}}$	Total rail inductance, millihenrys
Rail temperature, $^{\circ}$ C.	Deflection, D_s , cm.	Capacitance in series, microfarad	Deflection, D_R , cm.	Capacitance in series, microfarad	Deflection, D_X , cm.	Total resistance in series, including M (ohms)		
33.5	43.68	0.0804	45.98	0.0200	43.42	214900	767.5	0.01622
38.5	44.72	0.0905	46.03	0.0220	40.22	182400	732.5	0.01609
40.7	42.80	0.1002	46.10	0.0250	42.92	171400	680.0	0.01625
39.7	41.93	0.1202	45.52	0.0301	39.95	151400	615.4	0.01635
37.8	43.92	0.1505	44.77	0.0351	40.60	125000	562.6	0.01639
35.2	47.23	0.2008	41.15	0.0402	41.00	100100	595.2	0.01646
32.9	43.05	0.2208	42.28	0.0503	42.25	80060	459.8	0.01637
30.0	42.30	0.2711	46.68	0.0703	44.83	60060	411.2	0.01629
26.7	43.60	0.3529	41.07	0.0804	42.78	50060	366.2	0.01633
11.9	45.33	0.0604	47.47	0.0503	42.57	100100	302.0	0.01600
13.3	41.23	0.0804	47.72	0.0804	40.87	70060	249.7	0.01586
13.7	44.08	0.1303	44.78	0.1303	45.65	40060	202.6	0.01526
14.0	41.57	0.2220	42.57	0.2723	44.02	22060	150.7	0.01464
14.0	43.98	0.6038	45.40	0.9060	39.10	9060	94.1	0.01375
0	I	II	III	IV	V	VI	VII	IX

The temperature of the rail during the test varied from 40° C. with the high currents to 12° C. with the low currents. The direct-current resistance of this rail at 20° C. was found, by potentiometer, to be 420.5 microhms per foot = 46 microhms per metre, corresponding to a resistivity of 20.350 abohm-cm.; and the temperature coefficient of resistivity, from and at 20° C., is 0.00345 per degree C.

ometer, at different observed temperatures. The mean values of the temperature coefficient so obtained are indicated in Table I. They range from 0.00265 in the harder steels, to 0.00474 in the soft, higher-conductivity steel of the contact rails. In order to be strictly comparative, the measurements should all be taken with the rails at one and the same temperature of reference, since the skin effect varies appreciably with the resistivity. Since, however, it was not practicable to keep the rails at such a reference temperature, either on different days or with different current-strengths on the same day, the observed rail resistances R' have all been reduced to the reference temperature of 20° C. by means of the ordinarily used formula:

$$R'_{20} = R'_T / \{ 1 + a_{20} (T - 20) \} \quad \text{ohms (1)}$$

where T was the rail temperature in degrees Centigrade at the time of the test, and a_{20} the resistivity temperature coefficient as found for this reference temperature. The results thus reduced were found to be fairly consistent, although the full temperature correction, including skin-effect variation, would be much more complicated.

Discussion of Results.—Table II gives a sample set of observations on one rail, taken by the Sumpner dynamometer method. The fixed coils of the dynamometer were conducted to the manganin shunt throughout the test, in the manner indicated in Fig. E. The flux quadrature compensation was adjusted for each alternating-current strength, in the manner described in Appendix I. Column I gives the full double deflection in scale cm. between + and -. The temperature of the rail during the test varied from 40° C. with the high currents, to 12° C. with the low currents. The direct-current resistance of this rail at 20° C. was found, by potentiometer, to be 420.5 microhms = 14.02 microhms per foot = 46 microhms per metre, corresponding to a resistivity of 20,350 absohm-cm.; and the temperature coefficient of resistivity, from and at 20° C., is 0.00345 per degree C.

Column X gives the total self-inductance of 30 feet (9.14 metres) of the rail in millihenrys. This is the inferred self-inductance to infinity, on the assumption that it includes all of the internal and external flux linked with the rail and due to the rail current in a rectangle based on 30 feet (9.14 metres) between potential points and extending radially to infinity.

FIG. 5.

Curves of resistance-ratio and self-inductance rail No. 95-A. Study of skin effect in steel rails.

The results of the observations are presented in Figs. 5 to 10 inclusive. In all of these figures the abscissas are in r. m. s. ampères flowing through the rail under test. There are two sets of ordinates. The lower set is in each figure the skin-effect

FIG. 6.

Curves of resistance-ratio and self-inductance. Rail No. 100, curve No. 3. Rail No. 163, curve No. 2. Rail No. 175, curve No. 1. Study of skin effect in steel rails.

resistance ratio R'/R ; or the ratio of the apparent resistance of the rail in the presence of skin effect (reduced to 20° C.) to the resistance which it offers to continuous current from a storage battery at 20° C. The upper set of ordinates gives the inductance of the

FIG. 7.

Curves of resistance-ratio and self-inductance. Rail No. 161. Study of skin effect in steel rails.

rail between contact points (914 cm.) in millihenrys, as referred to infinite flux radius. The total linear inductance of the rail L_1' is obtainable from these values by dividing with 914.

It will be seen that the skin-effect resistance ratio R'/R com-

FIG. 8.

Curves of resistance-ratio and self-inductance. Rail No. 162 —. Rail No. 170 —. Study of skin effect in steel rails.

mences with feeble rail current, at a relatively low value, increases when the current is increased, and, in nearly all cases, reaches a maximum value within the limits of 800 r. m. s. ampères. Similar remarks apply to the rail inductance L_1' . The skin-effect

FIG. 9.

Curves of resistance-ratio and self-inductance. Rail No. 600 ———, Rail No. 174 — — —, Rail No. 251 — · — ·. Study of skin effect in steel rails.

resistance ratio is greater at 60 ω than at 25 ω , as might be expected; whereas the rail inductance L_1' is greater at 25 ω than at 60 ω , although the rail reactance $L_1'\omega$ would always be greater at 60 ω than at 25 ω .

FIG. 10.

Curves of resistance-ratio and self-inductance. Rail No. 702—○—○. Rail No. 871—△—△. Study of skin effect in steel rails.

The rise of skin-effect resistance to a maximum at a certain value of the rail current, is attributable to the variable permeability of the steel, which reaches a maximum at a certain value of the magnetizing force H , depending on the magnetic quality

of the steel, the effective skin penetration depth being changed accordingly. This phenomenon of a maximum impedance in steel prisms to a particular current-strength has been known experimentally at least as far back as 1893.*

Fig. 10 shows the resistance ratios and inductances of two soft steel contact rails. It will be noticed that the resistance ratio of No. 871 rises to 22; *i.e.*, that its alternating-current linear resistance at 60 ω is 22 times its direct-current resistance at the same temperature (20° C.). At 25.5 ω , the resistance ratio of this rail is 13.5. Not only are these resistance ratios relatively large, but they occur at relatively small rail currents (150 r. m. s. ampères) and corresponding values of H at the perimeter (3 gilberts/cm.). This is attributable to the relatively large values of conductivity (7×10^{-5} abmho/cm.) and permeability (2500 gausses per gilbert/cm.). The equivalent penetration depth of the alternating current at this maximum permeability and at 25.5 ω is only 0.076 cm.; so that very little of the steel is utilized electrically.

On the upper right hand of each figure is the computed scale of linear inductance of the trolley wire and one track as deduced in Appendix II. The linear inductance of the system exceeds in one case 5 millihenrys per mile.

Fig. 11 shows the total alternating-current resistance R' in 30 feet (914 cm.) of each of the track rails at 25.5 ω to the scale of microhms ordinates. The scale is also repeated in ohms per mile and ohms per kilometre of a track of two rails in parallel, assuming perfect bonding but no copper reinforcement, also negligible leakance to ground. It will be observed that rail No. 161 has a higher linear resistance than any of the others, in spite of the fact that its conductivity is relatively high. The high alternating-current resistance is attributable to a small equivalent penetration depth ($\delta = 0.107$ cm. at 25.5 ω), which in turn is due to the large product of γ and μ . To direct currents, this rail would offer the lowest resistance among all the track rails tested.

It is shown in Appendix IV that the resistance-ratio R'/R for a rail can be deduced from the measured direct-current values of its conductivity γ and permeability μ , for any assigned frequency, subject to a certain range of inaccuracy due to errors of permeability and shape, the actual effective alternating-current permeability dif-

* Bibliography No. 2.

fering to some extent from that observed by direct-current permeameter, and the shape of the rail involving edge effects owing to departure from a simple cylinder. The permeability and shape correction factors which must be applied to the computed values of R'/R , in order to arrive at the actual values, are at present

FIG. 11.

13-A-14
A.S.D. 3-10-16

Comparison of R' at 25.5 ω . Rails number 161, 96A, 163, 100, 175. Study of skin effect in steel rails.

beyond direct calculation, but may be estimated from experimental observations. Thus, in Fig. 12, the observed and computed values of R'/R are given for the contact rail No. 871 at the two test frequencies 60 ω and 25 ω . The computed values are obtained by (65), from measured direct-current conductivity γ and direct-current permeameter μ of a rod 38 cm. (15 inches)

long and 1.25 cm. (0.49 inch) in diameter cut from the head of this rail. It will be seen that, at or near 150 ampères r. m. s., the observed values of R'/R reach a maximum, and are in fairly close agreement with the calculated values; so that the correction factor

FIG. 12.

Rail 871. Curves of resistance ratio calculated from infinite strip formula and D-C values of permeability. Study of skin effect in steel rails.

for 25 ω and 125 ampères is 1.0. At higher current-strengths, however, the observed resistance ratio rises above the computed resistance ratio and the correction factor becomes greater than unity. For 750 ampères of rail current at 60 ω , it is $14.8/12 =$

1.233, and at 25ω it is $9.38/7.65 = 1.23$. For this rail with its particular values of conductivity and permeability, the correction factor varies between 0.9 and 1.23, according to the strength of current. Since the resistance ratio R'/R is a maximum for that current-strength and value of H at which the effective permeability is a maximum, it is ordinarily of greater importance to determine the correction factor at this maximum point, as it is likely to be the condition of greatest power loss in practice.

FIG. 13.

Curves of resistance ratio calculated from infinite strip formula and D-C values of permeability.
Study of skin effect in steel rails.

Fig. 13 shows corresponding curves of observed and computed resistance ratios for low-permeability track rails 175 and 100, at 60 ω and 25 ω . Here the maximum values are not exceeded at the maximum current of 800 ampères, and the correction factor is appreciably over 1.0 throughout. Thus on rail 175, at 60 ω , this factor is 1.23 at 100 ampères, and attains 1.47 at 350 ampères. At 25 ω , the maximum value of the correction factor is 1.45 at 350 ampères. The samples of steel from which the calculated values were obtained were rods cut from both sides of the head and also from the flange. The values of γ and μ obtained in rods from these different positions in the section of one and the same rail were found to be in satisfactory agreement.

Fig. 14 shows corresponding curves of observed and computed resistance ratios for the medium-permeability track rail No. 161. Here the maximum ratios are reached near to 350 ampères. The maxima of the computed curves agree fairly well with the maxima of the observed curves, and this is a property found in all of the

FIG. 14.

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Rail 161 curves of resistance ratio calculated from infinite strip formula and D-C values of permeability. Study of skin effect in steel rails.

cases investigated. The correction factor for 400 ampères and 60 $\sim$ is $15.6/12.1 = 1.30$, and at 25 $\sim$ it is $10.05/7.75 = 1.30$. The computed values in this case were obtained from observations of γ and μ on one rod 38 cm. long and 1.25 cm. in diameter, cut from the head of the rail.

Measurements of Conductivity.—The sample steel rod was milled from the rail and then turned in the lathe. It was finally finished to the required diameter (1.25 cm.) by filing in the lathe. The conductivity of this rod was determined at a series of different temperatures between 10° C. and 25° C., by direct-current potentiometer measurement, using such strengths of storage-battery current as served to bring the temperature of the rod up to the required amount. Both the resistivity and its temperature coefficient found in this way agreed satisfactorily with the values obtained previously in the rail tests as already described.

Measurements of Permeability.—The sample steel rod from the rail under investigation was tested in a Burrows permeameter,* which gave satisfactory results. In one case the 38-cm. rod, after being tested in this permeameter, was cut down in the lathe to a diameter of 0.6 cm. and tested in a Koepsel permeameter.† The permeability of the material as obtained from the Koepsel permeameter was a few per cent. less throughout than that obtained in the Burrows permeameter, which difference might be attributed to the relatively greater effect of hardening machine treatment on the sample of smaller diameter. It should be observed from formula (65) that 1 per cent. error in the measurement of permeability only affects the resistance ratio R'/R by 0.5 per cent.

Technic Recommended for Predetermining Rail Resistance Ratios.—According to the results here reported, it is sufficient for the determination of the maximum skin-effect resistance ratio R'/R in a steel rail to measure the values of γ and μ in a sample rod of the same material, which, by formula (65), gives the computed value of R'/R for a given frequency and series of alternating-current rail-current strengths, assuming that the measured direct-current permeabilities do not appreciably differ from the actual effective alternating-current permeabilities at these currents, including the effects of hysteresis, and also that the rail has a simple circular or cylindrical cross-section. In order, then, to correct for the errors of permeability and shape, a correction factor should be applied to the computed ratio. This correction factor appears in the results here reported to have varied in track rails, at maximum permeability, from 1.2 to 1.3. It is probably advisable to use the latter value for security. In other words, the

* Bibliography No. 18.

† Bibliography No. 30.

computed value of R'/R should be increased by 30 per cent. for the unknown permeability and shape errors. The result may then be taken as the maximum skin-effect resistance ratio of the rail, at the temperature and frequency considered. At all other rail currents, the resistance-ratio and the resistance per kilometre will be less, at constant rail temperature.

Best Shape of Rail Section for Conducting Purposes.—In steel track rails, mechanical considerations are necessarily paramount for determining the shape of the cross-section. For conducting rails, however, where the stresses to be supported are much smaller, consideration may well be given to the best form of cross-section for conducting purposes. Formula (68) shows that, provided there is sufficient depth beneath the surface in which the electric and magnetic fluxes can deploy—say, 4δ for a single surface of slab, or 8δ between opposite surfaces of a strap—the effective alternating-current conductance, in the presence of skin effect, depends upon the ratio P/S , or the perimeter per square centimetre of cross-section. This would point to the use of a flat strap as the best type of conductor. The edge effect in such a strap, however, militates against its use, to say nothing of its clumsiness and oxidizability. For these reasons it is preferable to employ a steel tube or hollow cylinder. In practice, in order to improve the mechanical brush-contact surface, the tube should be flattened, at least on the contact side, and sharp edges in the contour avoided. If the section is simply tubular, of thickness exceeding 4δ cm., the interior surface of the tube is prevented from carrying any appreciable alternating current. By slitting the tube, however, or by rolling a plate into the form of a nearly closed box beam, 8δ or more in wall thickness, both the internal and external surfaces of the hollow conductor become available.

To test this plan, a steel tube was employed 150 cm. long and of the cross-section shown in Fig. 15. The resistance ratio of this tube was found to be 7.66, for 60 ω , at 53 ampères r. m. s.* When a thin, longitudinal slit (1.19 mm. wide) was milled along this tube, the resistance ratio fell to 5.26, with the maximum at about 90 ampères. When the slot was widened to 3.18 mm., the ratio was observed to fall to 5.04 at 80 ampères. When the tube

* The resistance ratio computed by formula (65), from a sample rod of this steel, had a maximum at 53 ampères of only 6.45, representing a correction factor of 1.18 for hysteresis and permeability alone.

FIG. 15.

Skin effect resistance ratio for steel tube at 60°C. Outside diam. 2.84 cm.—6.35mm. wall.

was cut in half, by making an opposite longitudinal slot, the value fell to 4.46 at 48 ampères. The values of current at which the maxima are obtained correspond very fairly with the reciprocals of the effective perimeters in each case. Inserting the slot not

double the effective alternating-current conductance, but did increase it in the ratio of $7.66/5.04 = 1.52$, the discrepancy being presumably attributable to magnetic leakage near or across the slit. It would appear, therefore, that the two half tubes, connected in parallel, would be a better joint conductor than the slotted tube, and would exceed the conductance of the unslotted tube in the ratio $7.66/4.46 = 1.72$.

It is hoped to investigate further this question of most economical conducting cross-section.

It should be pointed out that for a given shape of cross-section the effective alternating-current conductance of a rail depends upon both γ and μ , and not merely on one of these quantities. An increase in permeability may more than offset the advantage of an increase in conductivity γ . An ideal steel would have a very high conductivity associated with a very low permeability.

Chemical Analysis of Samples.—Table III gives the chemical analysis of the steel in the various rails tested.

TABLE III.

Chemical Contents of Rails Used in the Research on Skin Effect in Steel Rails.

Rail No.	C	P	Percentage Si	Element Mn	Cr	Ni	S
* 96A	0.65	0.090	0.06	0.68	0.35	0.55	0.065
*100	0.55	0.090	0.08	1.08	0.30	0.60	0.095
*163	0.70		0.07	0.97	0.47	0.46	0.05
*175	0.60	0.124	0.08	0.83	0.30	0.45	0.07
*161-1	0.42	0.004	0.04	0.50	0.05	0.15	0.08
*161-2	0.68		0.09	0.65	0.31	0.51	0.095
*162	0.38	0.032	0.06	0.62	0.31	0.40	0.05
*170	0.40	0.004	0.05	0.43	0.085	0.12	0.04
600	0.59	0.032	0.122	0.67			0.044
174	0.37	0.065	0.032	0.40			0.070
251	0.74	0.035	0.063	0.34			0.072
702	0.11	0.113	0.020	0.36			0.091
871	0.17	0.132	0.025	0.51			0.099

* These tests were made by students in chemistry and are not equally reliable.

In conclusion, the authors desire to express their indebtedness to the courtesy of the Lackawanna Steel Company and the Maryland Steel Company for the use of the steel rails tested, also to Dr. G. A. Campbell for valuable suggestions on the manuscript.

SUMMARY OF CONCLUSIONS.

1. The maximum observed skin-effect resistance ratio among ten track rails, at 25ω , varied between 5.35 and 10.1, and in two contact rails between 10.92 and 13.4.

2. The maximum observed skin-effect resistance ratio was found to vary substantially as the square root of the impressed frequency between the limits of 25ω and 60ω .

3. The effective skin depth of alternating-current penetration, at 25ω , among all the twelve rails tested, varied between 0.76 mm. and 1.8 mm.

4. The values of the mean superficial r. m. s. magnetic intensity H at which the maximum skin effect developed, were between the limits of 3.3 and 16.4 gilberts per centimetre. These were but little influenced by frequency, and were in good agreement with the values of H for maximum μ , as obtained by direct-current permeameter.

5. The best workshop method of measuring the skin effect in track rails was, in this case, found to be based on the use of a particular form of dynamometer.

6. The effective alternating-current conductance of a rail is, to a first approximation, inversely as the square root of the frequency, and, at a given frequency, is directly proportional to the perimeter, and to the square root of the ratio $\frac{\gamma}{\mu}$.

7. It is therefore, in general, useless to increase the conductivity γ of the steel in a rail if the permeability μ is thereby increased in the same or a greater ratio.

8. From the results here reported, it would seem that the skin-effect resistance ratio of a rail, to a given alternating-current frequency and current-strength, can be approximately predetermined from measurements of the conductivity γ and permeability μ of the steel, by applying an experimentally determined factor to cover edge effect and other discrepancies. This factor, which may be called the "edge-effect coefficient," appeared to be not more than 1.3 at the maximum skin-effect ratio.

9. The best form of rail for current-carrying capacity should be one in which the effective perimeter is a maximum while allowing sufficient depth of surface. Among the worst forms are probably a thick solid prism and a cylinder. A hollow cylinder may be greatly improved by slitting it, so as to admit current to its interior surface.

APPENDIX I.

METHOD OF EMPLOYING THE SUMPNER DYNAMOMETER IN IMPEDANCE MEASUREMENTS.

The Sumpner dynamometer had one fixed coil *A* with 4000 turns of relatively fine insulated wire having 90 ohms resistance, and another fixed coil *C*, with 200 turns of coarser wire, having about 0.75 ohm resistance. The moving coil *M* has about 50 turns with about 60 ohms resistance, including leads and suspension. The fixed coils being provided with a laminated iron core, the moving coil swings in a relatively strong magnetic field, due to the mmf. of the fixed coils.

FIG. A.

Sumpner-dynamometer test connections.

Fig. *A* indicates the connections employed with the instrument connected across the manganin strip *S* to measure the p. d. across the terminals *a*, *b*, and therefore the current in the rail circuit. The current flowing through the magnet coils from the terminals *a*, *b*, under the impressed voltage $E_s = IS$, may be assumed, for the present, to produce an alternating magnetic flux in the air-gap, 90° in phase behind E_s . The branch circuit containing the moving coil *M* includes a small condenser, of pure capacity reactance so large that the resistance and inductive reactance in *M* are negligible with respect thereto. The current in *M* will therefore lead the impressed voltage E_s by substantially 90° , and will therefore be in phase with the air-gap flux-density B_s . The moving torque will therefore be

$$\tau_s = k I_{Ms} B_s \quad \text{dyne perp. cms.*} \quad (2)$$

* Unit torque, in the C. G. S. system, is one dyne acting perpendicularly at a radius of one cm.

where k is a constant for the instrument, which can be determined from the use of a known voltage.

The current in the moving coil will be

$$I_{Ms} = jE_s C_s \omega \quad \text{ampères} \angle \quad (3)$$

and the flux density in the air-gap, assumed radial, is

$$B_s = -jk_1 E_s / \omega \quad \text{gausses} \angle \quad (4)$$

k , being a constant for the magnetic circuit of the instrument. Consequently

$$\tau_s = k k_1 E_s^2 C_s \quad \text{dyne} \perp \text{cm.} \quad (5)$$

and if the moving system has a unifilar suspension providing a resisting torque proportional to the scale deflection D cms.,

$$\tau_s = a D_s \quad \text{dyne} \perp \text{cm.} \quad (6)$$

where a is a constant depending on the elastic torsion of the suspension and on the scale. Equating the opposing torques,

$$k k_1 E_s^2 C_s = a D_s \quad \text{dyne} \perp \text{cm.} \quad (7)$$

or

$$E_s = \sqrt{\frac{a D_s}{k k_1 C_s}} = K \sqrt{\frac{D_s}{C_s}} \quad \text{volts} \quad (8)$$

where $K = \sqrt{\frac{a}{k k_1}}$;

and

$$\frac{D_s}{C_s} = E_s^2 \cdot \frac{1}{K^2} \quad (9)$$

The vector diagram of the case is presented in Fig. (B). OE_s is the alternating-current r. m. s. voltage across the terminals ab of

FIG. B.

Vector diagram of Sumpner-dynamometer phase relations.

the shunt, taken as at standard phase. OB is the flux density in the air-gap, assumed to be just 90° behind OE_s . OI_m represents the current in the moving coil M , advanced 90° ahead OE_s , by reason of the capacity reactance in its branch circuit.

If now the terminals mm' of the moving coil are transferred to the potential terminals AB of the rail, the flux density in the air-gap remains the same as before; but the current in the moving coil

FIG. C.

FIG. D.

Vector diagram of Sumpner-dynamometer phase relations.

is changed, both in magnitude and phase. Referring to Fig. C, let OE_{AB} be the vector voltage on the rail, advanced θ° ahead of the current I , according to the relation

$$E_{AB} = IZ/\theta = I(R' + jL'\omega) \quad \text{volts} \quad (10)$$

The current in the moving coil OI_{MC} will be thrown 90° ahead of OE_{AB} owing to the presence of the condenser c_R in its circuit. The torque will then be

$$\tau_R = kI_{MC} \cos \theta \cdot B_s \quad \text{dynes T cm.} \quad (11)$$

$$\begin{aligned} &= kE_{AB} c_R \omega \cos \theta \cdot k_1 E_s / \omega \\ &= kk_1 E_{AB} E_s \cos \theta \cdot c_R \quad \text{dynes T cm.} \quad (12) \end{aligned}$$

or proportional to the product of Oc , OB on the diagram. This torque produces a deflection

$$D_R = \tau_R / a \quad \text{cm.} \quad (13)$$

so that

$$\frac{D_R}{c_R} = \frac{kk_1}{a} \cdot E_{AB} \cdot E_s \cdot \cos \theta = E_{AB} \cdot E_s \cdot \cos \theta \frac{I}{K^2} \quad \text{cm.} \quad (14)$$

Dividing (14) by (9), we obtain

$$\frac{D_R}{c_R} \left/ \frac{D_s}{c_s} \right. = \frac{E_{AB} \cdot E_s \cos \theta}{E_s^2} = \frac{E_{AB} \cos \theta}{E_s} = \frac{IR'}{IS} = \frac{R'}{S} \quad \text{numeric} \quad (15)$$

If now a suitable non-reactive resistance R_M ohms is substituted for the condenser C_1 the current which will flow in the moving coil will be

$$I_{MX} = \frac{E_{AB}}{R_M} = \frac{IZ/\theta}{R_M} \quad \text{ampères} \quad (16)$$

In Fig. D, OE_s is the voltage at standard phase across the shunt S , and OE_{AB} the voltage across the terminals AB leading the current by the angle θ . Then the current in the moving coil will be in phase with OE_{AB} and is indicated by OI_{MR} . The component Od , on the axis OY , will be $OI_{MR} \sin \theta$, and the torque will be

$$\tau_X = kI_{MX} \sin \theta B_s \quad \text{dynes} \perp \text{cm.} \quad (17)$$

$$= k k_1 \cdot \frac{E_{AB}}{R_M} \cdot \sin \theta \cdot \frac{E_s}{\omega} \quad \text{dynes} \perp \text{cm.} \quad (18)$$

$$= aD_X$$

where D_X is the scale deflection in this case.

Equating torques

$$D_X R_M = \frac{1}{K^2} E_{AB} \cdot \sin \theta \cdot \frac{E_s}{\omega} \quad \text{cm-ohms} \quad (19)$$

Dividing (19) by (9), we have

$$\frac{D_X \cdot R_M}{D_s/c_s} = \frac{E_{AB}}{E_s} \cdot \frac{\sin \theta}{\omega} = \frac{L'}{S} \quad \frac{\text{henrys}}{\text{ohms}} \quad (20)$$

It is therefore evident that, assuming the flux in the air-gap to be in quadrature with the voltage impressed on the alternating-current magnet, the quantities I , R'/S , and D'/S are found in terms of the three deflections, D_s , D_R , and D_X , and the corresponding impedances in the moving-coil circuits.

In the actual instrument, however, the magnetic flux and flux density B_s are not more than about 85° behind the impressed voltage in phase, as is shown by the fact that a deflection on the scale is produced when the magnet coil and the moving coil are connected in parallel to one and the same voltage E_s . In order to correct for this deviation, and bring the resultant magnetic flux density into strict quadrature with E_s , a condenser C_2 is

inserted in the 4000-turn coil *A*, as shown in Fig. E. This condenser has from 1 to 6 microfarads capacitance, according to the frequency and to the magnitude of E_s , which affects the hysteresis in the core. Its magnitude is adjusted by trial in such a manner

FIG. E.

Sumpner-dynamometer test connections.

that when the condenser C_1 is replaced by a resistance in Fig. E, no deflection is produced.

APPENDIX II.

THE LINEAR INDUCTANCE OF A SIMPLE SYMMETRICAL ELECTRIC-RAILROAD CONDUCTING SYSTEM, COMPRISING A LONG, STRAIGHT, UNIFORM TROLLEY WIRE, SUPPORTED AT A FIXED HEIGHT ABOVE THE CENTRE OF A TRACK FORMED BY TWO SIMILAR AND PARALLEL STEEL RAILS.

As a first step, the simpler and more familiar case may be considered of a single long trolley wire *T* (Fig. F) of radius ρ_0 cm., supported at a constant interaxial vertical distance d cm. above the equivalent centre of a single steel rail *R*. The permeability of the trolley wire is μ_0 . The extra linear inductance of the trolley wire—i.e., the inductance per cm., in so far as depends on flux collapsing upon the surface of the wire when its current is brought to zero*—is, by Neumann's formula,

$$L_{0e} = 2 \left(\log \frac{2l}{\rho_0} - 1 \right) \frac{\text{external abhenrys}}{\text{wire cm.}} \quad (21)$$

* "The Self and Mutual Inductance of Linear Conductors," by E. B. Rosa, *Bulletin of the Bureau of Standards*, vol. 4, No. 2, p. 301, September, 1907.

where l is the length of the trolley wire in cm., and $\log h$ signifies the hyperbolic or Napierian logarithm. This may be written

$$L_{oe} = 2 \log h \frac{2l}{\epsilon \rho_0} \quad \frac{\text{external abhenrys}}{\text{wire cm.}} \quad (22)$$

where $\epsilon = 2.718 \dots$ the Napierian base. The linear internal

Cross-section of single rail and an overhead trolley wire.

inductance of a round wire of uniform permeability ($\mu_0 = 1$ for copper) is also well known to be

$$L_{oi} = \frac{\mu_0}{2} \quad \frac{\text{internal abhenrys}}{\text{wire cm.}} \quad (23)$$

The total linear inductance of the trolley wire, extending outwards to an indefinite distance from its axis, is therefore

$$L_o = L_{oi} + L_{oe} = 2 \log h \frac{2l}{\epsilon \rho_0} + \frac{\mu_0}{2} \quad \frac{\text{abhenrys}}{\text{wire cm.}} \quad (24)$$

Moreover, the linear mutual inductance of the trolley wire and the rail, considering the latter as substantially equivalent to a parallel wire of dimensions small compared with the distance l , is *

$$M_{01} = 2 \left(\log h \frac{2l}{d} - 1 \right) = 2 \log h \frac{2l}{\epsilon d} \quad \frac{\text{abhenrys}}{\text{wire cm.}} \quad (25)$$

* *Bulletin of the Bureau of Standards*, vol. 4, p. 306.

The linear inductance of the wire in the presence of the rail return conductor becomes, therefore,

$$L_{c1} = \frac{\mu_0}{2} + 2 \left(\log h \frac{2l}{\epsilon \rho_0} - \log h \frac{2l}{\epsilon d} \right) \quad \frac{\text{abhenrys}}{\text{wire cm.}} \quad (26)$$

$$= \frac{\mu_0}{2} + 2 \log h \frac{d}{\rho_0} \quad \frac{\text{abhenrys}}{\text{wire cm.}} \quad (27)$$

This trolley-wire linear inductance to rail return can be computed with satisfactory precision, since any error in assigning the equivalent centre of the rail has ordinarily but little effect on the result.

Again, the linear internal inductance of the rail being written L_{1i} abhenrys per cm., the linear external inductance of the rail may be taken as

$$L_{1e} = 2 \log h \frac{2l}{\epsilon \rho_1} \quad \frac{\text{abhenrys}}{\text{rail cm.}} \quad (28)$$

where ρ_1 is the equivalent radius of the rail, and a quantity not easy to assign precisely in practice. The linear total inductance of the rail is then outwards to infinity

$$L_1 = L_{1i} + 2 \log h \frac{2l}{\epsilon \rho_1} \quad \frac{\text{abhenrys}}{\text{rail cm.}} \quad (29)$$

and, taking the mutual inductance to the trolley wire M_{01} of (25) into account, the linear total inductance of the rail in the presence of the return trolley wire is

$$L_{10} = L_{1i} + 2 \log h \frac{d}{\rho_1} \quad \frac{\text{abhenrys}}{\text{rail cm.}} \quad (30)$$

The linear total inductance of the loop formed by trolley wire and rail together is then

$$L = L_{01} + L_{10} = \frac{\mu_0}{2} + L_{1i} + 2 \log h \frac{d}{\rho_0} + 2 \log h \frac{d}{\rho_1} \quad \frac{\text{abhenrys}}{\text{loop cm.}} \quad (31)$$

If the linear total inductance of the rail outwards to infinity (L_1) can be measured experimentally for a particular rail length l cm., then the corresponding value to the trolley wire return conductor is

$$L_{10} = L_1 + 2 \log h \frac{d\epsilon}{2l} \quad \frac{\text{abhenrys}}{\text{rail cm.}} \quad (32)$$

The linear loop inductance can thus be determined for any assigned value of the trolley-wire vertical distance d , if the actual linear inductance L_1 of the rail can be correctly measured outwards to infinity for a known rail length of l cm.

If now we consider the practical case of a trolley wire supported over a line midway between two parallel rails (Fig. G), distant b cms. apart on equivalent centres, then it is easily shown that if the mutual inductance between the two rails with respect to the trolley wire return is defined as:

$$M_{12} = 2 \log h \frac{d}{b} \quad \frac{\text{abhenrys}}{\text{rail cm.}} \quad (33)$$

the linear total inductance of the system, formed by the trolley wire with the two rails as joint return conductors, is

$$L = L_{01} + \frac{L_{10}}{2} + \frac{M_{12}}{2} = \frac{\mu_0}{2} + 2 \log h \frac{d}{\rho_0} + \frac{L_{11}}{2} + \log h \frac{d}{\rho_1} + \log h \frac{d}{b} \quad \frac{\text{abhenrys}}{\text{system cm.}} \quad (34)$$

It may be observed that as the two rails are brought closer together, M_{12} is increased; and, finally, if they be merged together

FIG. G.

Cross-section of track and trolley wire. The distance d should be measured from centre of trolley wire to the centre of either rail.

into a single rail, $M_{12} = L_{10}$, so that the last formula reduces to (31) for the case already found of a trolley wire and one rail. The value of M_{12} can always be computed from (33) with satisfactory practical precision. If, therefore, we measure the linear inductance L_1 of a given length l of the rail for infinite flux radius, we can readily compute the system linear inductance for assigned values of d and b by (33) and (34).

Appendix III indicates the procedure in regard to the geometrical arrangement of the rail circuits for enabling L to be measured. In the presence of skin effect, L_1 becomes reduced to an effective value L_1' abhenrys per rail length l , for infinite flux radius.

APPENDIX III.

THEORY OF CORRECTION FOR MUTUAL INDUCTANCE BETWEEN POTENTIAL LEADS AND RAIL CIRCUIT.

In order to determine the mutual inductance between the rail potential leads and the rectangular loop formed by the rail circuit, it was necessary to solve certain preliminary problems.

First.—A straight wire OO' of any small cross-section lies

FIG. H.

Diagram of geometrical relations between parallel conductors.

parallel to a shorter wire oo' , also of small cross-section, the geometrical distances are A , B , A' , and d cms. respectively, as shown in Fig. H. Required the mutual inductance between OO' and oo' ; i.e., to find the flux emitted by 1 absampère in OO' which enters a rectangle $Yoo'Y'$ containing oo' , the sides Y and Y' being indefinitely long. Any point P in the plane of this rectangle has rectangular coördinates x and y .

The field intensity H_P at P due to a current of I absampères in OO' is

$$H_P = \frac{I}{y} \left\{ \frac{x}{\sqrt{x^2 + y^2}} + \frac{l-x}{\sqrt{(l-x)^2 + y^2}} \right\} \frac{* \text{gilberts}}{\text{cm.}} \quad (35)$$

* E. B. Rosa, *Bulletin*, vol. 4, No. 2, Bureau of Standards, 1907, p. 303, formula (1).

and the corresponding flux density in a medium of permeability μ is:

$$B_P = \frac{\mu I}{y} \left\{ \frac{x}{\sqrt{x^2 + y^2}} + \frac{l-x}{\sqrt{(l-x)^2 + y^2}} \right\} \quad \text{gausses} \quad (36)$$

The total flux ϕ in the rectangle $Yoo'Y'$ will be

$$\phi = \int_A^{l-A'} \int_d^\infty B_P \cdot dy \cdot dx \quad \text{maxwells} \quad (37)$$

Taking one integration at a time,

$$\frac{d\phi}{dy} = \int_A^{l-A'} B_P \cdot dx = \int_A^{l-A'} \frac{\mu I}{y} \left\{ \frac{x}{\sqrt{x^2 + y^2}} + \frac{l-x}{\sqrt{(l-x)^2 + y^2}} \right\} dx \quad \frac{\text{maxwells}}{\text{cm. along } Y} \quad (38)$$

$$= \frac{\mu I}{y} \left[\sqrt{x^2 + y^2} - \sqrt{(l-x)^2 + y^2} \right] \frac{l-A'}{A} \quad \frac{\text{maxwells}}{\text{cm. along } Y} \quad (39)$$

$$= \frac{\mu I}{y} \left\{ \sqrt{(l-A')^2 + y^2} - \sqrt{A^2 + y^2} - \sqrt{A'^2 + y^2} + \sqrt{(l-A)^2 + y^2} \right\} \quad \frac{\text{maxwells}}{\text{cm. along } Y} \quad (40)$$

and

$$\phi = \mu I \int_d^\infty \left\{ \frac{\sqrt{(l-A')^2 + y^2}}{y} + \frac{\sqrt{(l-A)^2 + y^2}}{y} - \frac{\sqrt{A'^2 + y^2}}{y} - \frac{\sqrt{A^2 + y^2}}{y} \right\} dy \quad \text{maxwells} \quad (41)$$

Observing that

$$\int \frac{\sqrt{c^2 + y^2}}{y} dy = \sqrt{c^2 + y^2} - c \log \frac{c + \sqrt{c^2 + y^2}}{y} \quad (42)$$

c being any constant,

$$\begin{aligned} \phi = \mu I \left[\sqrt{(l-A')^2 + y^2} - (l-A') \log \frac{(l-A') + \sqrt{(l-A')^2 + y^2}}{y} \right. \\ + \sqrt{(l-A)^2 + y^2} - (l-A) \log \frac{(l-A) + \sqrt{(l-A)^2 + y^2}}{y} \\ - \sqrt{A'^2 + y^2} + A' \log \frac{A' + \sqrt{A'^2 + y^2}}{y} \\ \left. - \sqrt{A^2 + y^2} + A \log \frac{A + \sqrt{A^2 + y^2}}{y} \right]_d^\infty \quad \text{maxwells} \quad (43) \end{aligned}$$

All of the terms sum to zero at the upper limit (∞) so that

$$\begin{aligned}
 -\phi = \mu I \left\{ \sqrt{(l-A')^2 + d^2} - (l-A') \log \frac{(l-A') + \sqrt{(l-A')^2 + d^2}}{d} \right. \\
 + \sqrt{(l-A)^2 + d^2} - (l-A) \log \frac{(l-A) + \sqrt{(l-A)^2 + d^2}}{d} \\
 - \sqrt{A'^2 + d^2} + A' \log \frac{A' + \sqrt{A'^2 + d^2}}{d} \\
 \left. - \sqrt{A^2 + d^2} + A \log \frac{A + \sqrt{A^2 + d^2}}{d} \right\} \quad \text{maxwells} \quad (44)
 \end{aligned}$$

The quantity within the final brackets taken with opposite sign is, therefore, the required mutual inductance M abhenrys in a medium of unit permeability.

If we take $A' = A$ this becomes

$$\begin{aligned}
 -M = 2 \left\{ \sqrt{(l-A)^2 + d^2} - (l-A) \log \frac{(l-A) + \sqrt{(l-A)^2 + d^2}}{d} \right. \\
 \left. - \sqrt{A^2 + d^2} + A \log \frac{A + \sqrt{A^2 + d^2}}{d} \right\} \quad \text{abhenrys} \quad (45)
 \end{aligned}$$

If again $A = A' = 0$,

$$-M = 2 \left\{ \sqrt{l^2 + d^2} - d - l \log \frac{l + \sqrt{l^2 + d^2}}{d} \right\} \quad \text{abhenrys} \quad (46)$$

which agrees with formula (98) of Rosa and Grover's "Formulæ for Mutual- and Self-Inductance." *

FIG. K.

System of main and potential circuits in rail tests.

In the system of circuits indicated in Fig. K the working circuit is the rectangle $FABCDE$ and the potential-lead circuit is the inner rectangle nearly in the same plane $ABhgA$. It is required to deter-

* *Bulletin*, vol. 8, No. 1, Bureau of Standards, 1912, p. 151. See also Bibliography 35.

mine by computation the electromotive force induced by each element of the work circuit on the potential circuit, and to find the algebraic sum of the same. The vector subtraction of this sum from the voltage measured at the terminals jk is the vector IZ voltage drop in the rail between the potential points AB .

$$E_{jk} = I \{ (R'_1 + jL'_1\omega) + j\omega (M_{1,4} - M_{4,6} + M_{231,6} - 2M_{2,1} - M_{8,5} + M_{8,9} + M_{9,5} - M_{9,7}) \} = I \{ (R'_1 + jL'_1\omega) + j\omega M_0 \} \quad \text{abvolts } \angle \quad (47)$$

Owing to the symmetry of the rectangles, $M_{8,5} = M_{9,7}$ and $M_{8,7} = M_{9,5}$.

$$\begin{aligned} M_0 &= M_{1,4} - M_{4,6} + M_{231,6} - 2M_{2,1} + 2M_{9,5} - 2M_{8,5} \\ &= 2576 - 3738 + 3697 - 2468 + 256 - 510 \\ &= 6529 - 6716 = -187 \end{aligned} \quad \text{abhenrys} \quad (48)$$

$$\text{so that } R'_1 + j\omega (L'_1 - 187) = R_1 + jL''_1\omega = \frac{E_{jk}}{I} \quad \text{abohms } \angle \quad (49)$$

$$\text{and } L'_1 = L''_1 + 187 \quad \text{abhenrys} \quad (50)$$

In the above computation the term $M_{2,1}$ and its counterpart $M_{3,1}$ are obtained from the end-effect formula (102) of Rosa and Grover in the publication above referred to.

APPENDIX IV.

OUTLINE OF THE THEORY OF SKIN EFFECT IN STEEL RAILS.

The working theory of the extra effective resistance in steel rails due to skin effect, or the imperfect penetration of alternating currents, is based on two provisional assumptions:

1. That the surface of a rail may be regarded as of negligible curvature for the degree of precision required; so that the skin-effect formulas applying to flat strips, as first developed by Lord Rayleigh,* may be utilized. At the curves in the contour of the rail it suffices to assume, as a first approximation, that the magnetic flux densities and electric current densities are distributed in layers of corresponding curvature beneath the surface.

2. That the magnetic permeability μ of the substance of the rail may be taken as substantially constant at any one r. m. s. rail current-strength, in spite of the fact that, at different depths below the surface, the magnetic flux density B dwindles to zero and the

* Bibliography No. 1.

permeability μ is subject to a certain range of variation. This means that we may take a certain average effective value of μ , which may correspond to the more complex integrated value of μ at different depths. Since at the frequency of 25ω the flux density must fall to one-half within less than 2 mm. of depth below the surface, the effects of changes in permeability at greater depths than 2 mm. rapidly disappear. This subject has recently been investigated by Lombardi.*

Let $ABCD$ (Fig. L) represent a cross-section of a thick sheet of steel $2X$ cm. in total thickness. Let $abcd$ be a 1-cm. portion

FIG. L.

Diagram of cross-section of conducting sheet.

of the section; so that the area of $abcd$ is $2X$ sq. cm. Let the electric current carried in the sheet be directed towards the observer, or be perpendicular to the plane of the paper. Then the magnetic flux in the metal must lie across the current or in the plane of the paper.

At a point P , distant x cm. from the midplane OO' of the sheet, or $\xi = X - x$ cm. below the upper surface AB , let the alternating-current density be i_x r.m.s. absampères per sq. cm. Then we know †

* Bibliography No. 33.

† Bibliography 32, 36.

that in the case of a very wide sheet this current density varies in simple proportion to the cosine of the hyperbolic angle αx where

$$\alpha = \sqrt{j4\pi\gamma\mu\omega} = \sqrt{4\pi\gamma\mu\omega}/45^\circ = \sqrt{2\pi\gamma\mu\omega} + j\sqrt{2\pi\gamma\mu\omega} = \alpha_2 + j\alpha_2 \text{ cm}^{-1} \quad (51)$$

and $j = \sqrt{-1}$, $\pi = 3.14159 \dots$

γ = conductivity of the sheet (abmhos per cm.)
 = $1/\rho$ if ρ is the resistivity of the sheet (abohm-cm.)

μ = permeability of the sheet $\frac{\text{gausses}}{\text{gilberts per cm.}}$

ω = angular velocity of alternating current $\frac{\text{radians}}{\text{second}}$

= $2\pi f$, and f is the impressed frequency $\frac{\text{cycles}}{\text{second}}$

Similarly, the flux density B_x gausscs at P varies in simple proportion to the sine of the same hyperbolic angle αx , which as shown by (51) is a semi-imaginary quantity, having equal real and imaginary components, each equal to $\alpha_2 x$.

When the sheet has such thickness that with the proper values of γ , μ , and ω the maximum hyperbolic angle αX has a modulus of, say, 6 or more radians, with therefore components of 4.2 or more, it is known that,* owing to the cosine relations of such large hyperbolic angles,

$$i_x = i_x e^{-\alpha x} = i_x e^{-\alpha_2 x} \angle \alpha_2 x \quad \frac{\text{absampères}}{\text{sq. cm.}} \quad (52)$$

where i_x is the current density at the surface, and e is the Napierian base, so that the current density falls off as we descend into the sheet according to a simple exponential law, at the same time lagging increasingly in phase.

These conditions are indicated graphically in Fig. M. Here the horizontal vector OA , of unit length, represents the alternating-current density at the surface of the strap, in r. m. s. absampères per sq. cm. The diagram is drawn for the particular case where $f = 25 \infty$ or $\omega = 157.1$ rad/sec., $\mu = 506.6$, $\rho = 20,000$ abohms-cm. or $\gamma = 0.5 \times 10^{-4}$ abmho/cm.; so that $\alpha = 7.071/45^\circ = 5 + j5$ hyperbolic radians per cm. The current density in layers of successive depth will therefore fall off both in magnitude and in phase at the rate $e^{-7.071x/45^\circ} = e^{-5x} \angle 5x$ radians. If we give to the depth x the successive values 0.05, 0.1, 0.15, 0.2 cm., etc., we obtain from this expression the successive vectors OB , OC , OD , OE ; and the

* Bibliography 23, 16.

curve $ABCDE$ is an equiangular spiral about the origin O , the tangent at any point on this curve making an angle of 45° with the radius vector.

It will be observed that at a depth of 3 mm. ($\xi = 0.3$), the current density is 22 per cent. of the density at the surface, and lags nearly 90° with respect thereto. At 6 mm., or 3δ , the current density has fallen to less than 5 per cent., and is nearly in opposite phase to the density at the surface.

FIG. M.

Graph of magnitude and phase of alternating current density at various depths below surface.

The total vector r. m. s. current carried along the sheet per cm. of its breadth may be obtained from the integral

$$I = \int_0^X i \cdot d\xi = i_X \int_0^X \xi^{-\alpha\xi} \cdot d\xi = \frac{i_X}{\alpha} (1 - \epsilon^{-\alpha X}) \quad \frac{\text{absampères}}{\text{cm. breadth}} \angle \quad (53)$$

When, as in the condition assumed, αX exceeds 6, $\epsilon^{-\alpha X}$ is numerically less than 0.015; so that we may take the integral current in the half of the strap above the midplane OO' , Fig. L, as

$$I = \frac{i_X}{\alpha} = \frac{i_X}{7.071} \angle 45^\circ = i_X (0.1 - j0.1) \quad \frac{\text{absampères}}{\text{cm. breadth}} \angle \quad (54)$$

We may also obtain a close geometrical approximation to the total integral current by taking the vector OB , the density at the middle of the first 0.1 cm. layer ($\xi = 0.05$). Multiplying this by 0.1, we find the vector current Ob in this layer. Next taking the vector OD at the middle of the second 0.1 cm. layer ($\xi = 0.15$), and multiplying it by 0.1, we obtain a vector increment bd , to be added vectorially to Ob , in order to obtain the total vector current in the first two such layers. Proceeding in this way, the total vector r. m. s. current per cm. of breadth of the sheet above the

midplane OO' , Fig. L, will be $OI = i_X 0.1414 \sqrt{45^\circ}$ or $i_X (0.1 - j 0.1)$ absampères per cm. It is evident that if, as in the case assumed, the current has room to penetrate (αX not less than 6.0, say), the total vector current carried by the sheet will lag just 45° behind the superficially impressed electromotive force and the current at the surface.

If we denote by η_X the electric intensity impressed on the sheet in abvolts per linear cm. acting at the surface, and independent of that consumed in overcoming the external reactance, or producing magnetic flux in the external air, then in each cm. of breadth of the strap we have

$$I = \frac{\eta_X}{Z'} \quad \frac{\text{absampères}}{\text{breadth cm.}} \angle \quad (55)$$

and at the surface the current density is

$$i_X = \frac{\eta_X}{\rho} = \eta_X \gamma \quad \frac{\text{absampères}}{\text{sq. cm.}} \angle \quad (56)$$

The linear impedance of the strap per cm. of breadth and above the midplane OO' is

$$Z' = \alpha \rho = \alpha_2 \rho + j \alpha_1 \rho \quad \frac{\text{absohms}}{\text{breadth cm.}} \angle \quad (57)$$

The alternating-current resistance R' in the presence of skin effect is then

$$R' = \alpha_2 \rho \quad \frac{\text{absohms}}{\text{breadth cm.}} \quad (58)$$

But the continuous-current resistance R of the strap per cm. breadth and above the midplane OO' is

$$R = \frac{\rho}{X} \quad \frac{\text{absohms}}{\text{breadth cm.}} \quad (59)$$

The resistance R' would be offered to continuous currents by a certain depth δ cm., such that

$$\frac{\delta}{X} = \frac{R}{R'} = \frac{1}{\alpha_2 X} \quad \text{numeric} \quad (60)$$

or

$$\delta = X \cdot \frac{R}{R'} = \frac{X}{\left(\frac{R'}{R}\right)} = \frac{1}{\alpha_2} \quad \text{cm.} \quad (61)$$

This equivalent skin depth δ , first arrived at by Rayleigh,* which

* Bibliography 1.

would have the same resistance to continuous currents that the strap above the midplane, offers to alternating currents, is thus equal to the half thickness X divided by the skin-effect resistance-ratio R'/R , and is the reciprocal of a_2 . In the case presented in Fig. M, where $a_2 = 5$, the value of δ is $\frac{1}{5} = 0.2$ cm.; that is, 2-mm. skin, on either surface of the plate, would offer the same continuous current resistance. Referring to Fig. M, if we multiply the surface current density OA by $\delta = 0.2$, we obtain a vector current Oa , which leads the total current OI by 45° . The projection of Oa on the vector direction of this total current coincides with OI . A skin depth δ , carrying the r. m. s. current density at the surface, would therefore give a total current, the component of which, in

FIG. N.

Rail No. 174

The shaded superficial layer indicates the effective equivalent skin of depth, $\delta = 0.15$ cm.—as deduced from measurements—at 25.5 ∞ and maximum permeability.

phase with the total sheet current, is equal in magnitude to the latter, assuming plenty of available depth and $aX = 6$ or more. In such cases, the current density at depth δ , attenuates to $1/\epsilon\theta$ and lags 1 radian behind the surface density.

Fig. N represents the cross-section of an ordinary type of steel rail, the effective skin depth δ being 0.15 cm. The actual value of δ in this rail at 25.5 ∞ was found to be 1.55 mm. at μ_{max} . The web of this rail, having a thickness of about 9δ , is fairly well utilized; but the head and the base are only partially utilized for conductance at 25 ∞ . The linear resistance of this rail with continuous currents, at, say, 20° C., is supposed to have been ascertained as R absohms per linear cm. The impedance of the rail at the same temperature to currents of a given impressed fre-

quency will depend on the effective permeability μ , which, in turn, depends upon the average magnetic intensity H_X at the surface,

$$H_X = \frac{4\pi I}{P} \quad \text{r. m. s. gilberts per cm.} \quad (62)$$

where P is the perimeter of the rail section in cm. At any assigned number of r. m. s. ampères in the rail, the average value of H_X will

be this number multiplied by $\frac{1.257}{P}$. The linear resistance of the

rail at this current strength will be, say, R' absohms per cm. With continuous currents, the whole area of cross-section S sq. cm. will be effective for carrying current, but with the alternating current the effective area S' will be reduced in the ratio $\frac{R}{R'}$, so that,

$$S' = S \frac{R}{R'} \quad \text{sq. cm.} \quad (63)$$

This effective area S' of alternating-current-carrying capacity is nearly equal to the effective skin depth δ multiplied by the rail perimeter P , or

$$\delta = \frac{S'}{P} = \frac{S}{P} \cdot \frac{R}{R'} = \frac{1}{\alpha_2} \quad \text{cm.} \quad (64)$$

Consequently

$$\frac{R'}{R} = \frac{S}{P} \cdot \alpha_2 = \frac{S}{P} \sqrt{2\pi\gamma\mu\omega} = \frac{S}{P} 2\pi \sqrt{\gamma\mu f} \quad \text{numeric.} \quad (65)$$

Since $R = \rho/S = 1/S\gamma$, it follows that

$$R' = \frac{2\pi}{P} \sqrt{\frac{\mu f}{\gamma}} \quad \frac{\text{absohms}}{\text{cm.}} \quad (66)$$

or calling $G' = 1/R'$ the linear conductance in the presence of skin effect,

$$G' = \frac{P}{2\pi} \sqrt{\frac{\gamma}{\mu f}} \quad \text{abmho-cm.} \quad (67)$$

Similarly, if G is the linear conductance of the rail to continuous currents, or in the absence of skin effect; then the skin-effect conductance ratio, or the fraction of utilization of the rail section, is by (64),

$$\frac{G'}{G} = \frac{P}{2\pi S \sqrt{\gamma\mu f}} = \frac{P}{S \sqrt{2\pi\gamma\mu\omega}} = \frac{P}{S\alpha_2} \quad \text{numeric} \quad (68)$$

If, therefore, we can find γ and μ for the quality of steel employed in the rail, taking into account the temperature and the value of H_X for the current I , we can compute the skin-effect resistance ratio $\frac{R'}{R}$ at that current. Thus, with $\alpha_2 = 5$, $S = 65$ sq. cm., and

$P = 60$ cm., $\frac{R'}{R} = 5 \times 1.083 = 5.42$. A curve of $\frac{R'}{R}$ may then be plotted for different alternating current strengths carried by the rail, based on the known curve connecting μ and H .

Errors in the Method.

There are two very noticeable sources of error in the above computation; namely, (1) permeability error, (2) shape error.

Permeability Error.—If the permeability μ in formula (65) is taken at the value observed with continuous-current permeameter measurements on a sample bar, for the value of H at the rail surface, corresponding to a given current strength through the rail by (62), it cannot be expected to lead to a correct result for the skin-effect resistance ratio R'/R , since the value of alternating H dwindles at the increasing depths below the rail surface, and the permeability will correspondingly vary. The direction and extent of this error will depend upon the shape of the $\mu - H$ curve for the steel, upon the rate of dwindling as we descend below the surface, and as to whether the maximum value of the permeability is exceeded at the value of H reached on the surface. Since, however, in ordinary steels, the equivalent skin depth δ at electric-railroad frequencies seldom exceeds 2 mm. and the conductance in this skin depth usually greatly preponderates over the sum of all the conductances at depths underlying δ , the errors attributable to this variation in μ at different depths appears by actual measurement to be relatively small, especially in the neighborhood of maximum permeability at the surface.

The use of the permeameter value of μ in (65) is also in error, because the ordinary permeameter measures the permeability of the steel sample bar longitudinally, or in the direction of its axis; whereas alternating currents in the rail set up magnetic fluxes near the surface directed cylindrically, and in planes transverse to the axis. Strictly speaking, the transverse permeability and not the longitudinal permeability should be used. It has been found by Miller * that in the case of steel wires and rods there were in some cases appreciable differences between the longitudinal and transverse permeabilities. However, the error attributable to the differences reported seems to be small by comparison with the outstanding errors. It should be noted that the resistance ratio

* Bibliography No. 34 (Bureau of Standards).

FIG. Q.

Iron-filing diagram of alternating-current magnetic flux surrounding an active rail.

from these results that the resistance-ratio of cylinders cut from these steel rails agrees fairly well with that deduced from the longitudinal resistivity and permeability of the same rod at the same temperature.

Shape Error.—Since the ordinary steel rail is everywhere so thick by comparison with the equivalent skin-depth δ at 25 ω that the current density has ample room for attenuation according to (52), the only important error to be considered is the departure from the simple cylindrical form. If the rail were externally cylindrical, the magnetic flux at the surface would be conformal to it at every point. In the actual I or T shape, the magnetic flux tends to pass through the surrounding air from one projection to another, as is shown in Fig. Q, where an iron-filings picture was obtained on one side of a rail (No. 161), when an alternating current of 275 ampères r. m. s. at 60 ω was passing along the rail. It will be seen that the magnetic flux cuts across corners, as though from polar projections, and in passing through the surface of the rail generates alternating electromotive forces in the skin. These electromotive forces, if allowed to close their circuits locally, would produce local eddy currents and consume power, tending to increase the apparent linear resistance R' of the rail. Since, however, the electromotive forces are generated symmetrically throughout the entire length of the rail, we may expect that they will markedly increase the current density at the edges of projections in the contour, and so increase the apparent resistance. This is the same effect as was observed in the case of a steel strap, 3.8 cm. wide and 0.64 cm. thick. The skin-effect resistance ratio R'/R for this strap was found to be 46 per cent. greater than was deducible from its longitudinal resistivity and permeability. It may be called the "edge effect," and is essentially the same phenomenon as occurs in a cylinder; *i.e.*, the current leaves the central region, and crowds into the most distant regions of the cross-section.

The magnitude of the edge-effect coefficient in a steel rail is too complex to be attempted by calculation at the present time. It can, however, be determined experimentally for a given rail section and material, provided we assume that the R'/R of the same rail perimeter, in the form of a cylinder, would be the same as that computed from the longitudinal permeability and resistivity by (65). In other words, if we ignore the permeability and hysteresis errors as trivial by comparison with the shape error and edge effect, the observed value of R'/R for a rail, when divided by the computed value of R'/R , gives the edge-effect coefficient.

LIST OF SYMBOLS EMPLOYED.

α_{20}	Temperature coefficient of resistivity for rail; referred to 20° C. (Temperature $^{-1}$).
$\alpha = \sqrt{4\pi\gamma\mu\omega} \angle 45^\circ$	Propagation constant for a flat strip, a positive semi-imaginary (cm. $^{-1} \angle$).
$\alpha_2 = \sqrt{2\pi\gamma\mu\omega}$	Imaginary or real component of a semi-imaginary propagation constant (cm. $^{-1}$).
B	Flux density (gausses $\angle$).
B_s	Flux density in air gap of Sumpner dynamometer due to potential drop in S (gausses $\angle$).
B_P	Flux density at point P of infinite sheet (gausses $\angle$).
B_x	Flux density at distance x from midplane of sheet (gausses $\angle$).
b	Distance between centres of rails (cm.).
$\gamma = 1/\rho$	Electric conductivity of material (abmhos per cm.).
C_1	Condenser in series with moving coil of Sumpner dynamometer.
C_S	Capacitance of condenser C_1 when M is connected across the shunt S (farads).
C_R	Capacitance of condenser C_1 when M is connected across the rail (farads).
δ	Thickness of equivalent skin (cm.).
D_S	Deflection of dynamometer when M is connected across the shunt S (cm.).
D_R	Deflection of dynamometer when M is connected across the rail (cm.).
D_X	Deflection of dynamometer when M is connected across the rail and R_M substituted for C_1 (cm.).
d	Vertical distance between equivalent centre of rail and the trolley wire (cm.).
E_{AB}	Drop of potential in rail (volts $\angle$).
E_{jk}	Voltage as measured at potential leads jk (volts $\angle$).
E_s	Drop of potential in shunt S (volts).
$e = 2.71828 \dots$	Napierian base.
η	r. m. s. electric intensity impressed on sheet and acting at surface (abvolts per linear cm. $\angle$).
f	Frequency (cycles per second).
G	Linear conductance of a rail in the absence of skin effect (abmhos/cm.).
G'	Linear conductance of a rail in the presence of skin effect (abmhos/cm.).
H	R. m. s. values of magnetic intensity (gilberts per cm. $\angle$).
H_P	R. m. s. values of magnetic intensity at point P (gilberts per cm. $\angle$).
H_X	Average r. m. s. magnetic intensity at surface of sheet (gilberts per cm. $\angle$).
I	R. m. s. current in rail (ampères $\angle$); strap absamp/cm.

I_M	R. m. s. current in moving coil of dynamometer and condenser C , due to potential drop in shunt S (ampères $\angle$).
I_{MC}	R. m. s. current in moving coil of dynamometer and condenser C , due to potential drop in rail (ampères $\angle$).
I_{MX}	R. m. s. current in moving coil of dynamometer and resistance R_M , due to potential drop in rail (ampères $\angle$).
i_x	R. m. s. current density at ξ cm. from surface of sheet (absampères per sq. cm. $\angle$).
i_X	R. m. s. current density at surface of sheet (absampères per sq. cm. $\angle$).
$j = \sqrt{-1}$.	
L'	Total inductance of rail in presence of skin effect (henrys).
L_{oe}	External linear inductance of wire to infinity (abhenrys per cm.).
L_{oi}	Internal linear inductance of wire (abhenrys per cm.).
L_o	Total linear inductance, outwards to infinity, of trolley wire (abhenrys per cm.).
L_{ri}	Internal linear inductance of rail (abhenrys per cm.).
L_{re}	External linear inductance of rail, outwards to infinity (abhenrys per cm.).
L_1	Total linear inductance of rail, outwards to infinity (abhenrys per cm.).
L_{10}	Total linear inductance of rail in presence of trolley return wire (abhenrys per cm.).
L	Total linear inductance of loop of trolley and rail (abhenrys per loop cm.).
L'_1	Total inductance of rail in presence of trolley wire and skin effect (abhenrys).
L''_1	Apparent total inductance of rail as measured by dynamometer (abhenrys).
l	Length of trolley wire (cm.).
$\log h$	Napierian or hyperbolic logarithm.
M	Moving coil of Sumpner dynamometer.
M_{01}	Mutual linear inductance of trolley and rail (abhenrys per cm.).
M_{12}	Mutual linear inductance of two rails with respect to trolley wire (abhenrys per cm.).
μ	Permeability of rail as measured by direct-current methods ($\frac{\text{gausses}}{\text{gilberts per cm.}}$).
μ_0	Permeability of trolley wire ($\frac{\text{gausses}}{\text{gilberts per cm.}}$).
V_{60}, V_{25}	Values of resistance ratio R'/R , at 60 ω and at 25 ω respectively (numeric).
ξ	Distance of point P below surface of sheet (cm.).
P	Perimeter of rail (cm.).
$\pi = 3.14159 \dots$	
R'_T	Effective resistance of rail at T° C. (ohms).

R'_{20}	Effective resistance of rail at 20° C. (ohms).
R'	Effective a. c. resistance of rail as measured (ohms). Also its effective linear α c. resistance (absohms cm.).
R	Direct-current resistance of rail (ohms). Also its linear resistance (absohms / cm.).
R'/R	Skin-effect resistance ratio.
R_M	Resistance substituted for C_1 in moving-coil circuit of dynamometer.
ρ	Resistivity of sheet or rail (absohms-cm.).
ρ_1	Equivalent radius of rail (cm.).
ρ_0	Radius of trolley wire (cm.).
r. m. s.	Root-mean-square value.
S	Area of rail (sq. cm.).
S'	Effective equivalent area of rail to alternating currents (sq. cm.).
S	Resistance of shunt in rail testing circuit (ohms).
T	Value of temperature (degrees Cent.).
τ_s	Torque due to potential drop in S (dyne perp. cm.).
τ_X	Torque due to potential drop in rail with R_M in circuit (dyne perp. cm.).
τ_R	Torque due to potential drop in rail with C_1 in circuit (dyne perp. cm.).
X	Half-thickness of sheet (cm.).
$\omega = 2\pi f$	Angular velocity of impressed alternating current (radians per second).
Z'	Impedance of sheet per cm. of breadth and above the mid-plane (absohms per breadth cm.).
ab or abs	Prefix denoting a C. G. S. magnetic unit.
$\angle$	Indication of a complex or plane-vector unit.

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