The Theory of Order, as Defined by Boundaries

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Source: *The Mathematical Gazette*, Vol. 6, No. 94 (Oct., 1911), pp. 135-142

Published by: [The Mathematical Association](#)

Stable URL: <http://www.jstor.org/stable/3603246>

Accessed: 12-03-2016 14:49 UTC

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THE MATHEMATICAL GAZETTE.

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LONDON :
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY,
AND BOMBAY.

VOL. VI.

OCTOBER, 1911.

No. 94.

INTERNATIONAL CONGRESS OF MATHEMATICIANS, 1912.

THE MATHEMATICAL ASSOCIATION have undertaken to organise an exhibit, illustrative of Mathematical Education in Great Britain, for the Educational Section of the forthcoming International Congress of Mathematicians, to be held at Cambridge in 1912. It is specially desired to obtain specimens of ordinary mathematical work, both good and average, in various types of schools, as well as models and diagrams. All those willing to co-operate are requested to communicate with the Hon. Secretary of the Exhibition Committee, P. ABBOTT, Esq., Wildwood, West View, Highgate.

THE THEORY OF ORDER, AS DEFINED BY BOUNDARIES.

I. INTRODUCTION.

I PROPOSE in these papers to demonstrate, or rather to indicate the nature of the demonstration of, the proposition that—*All Geometry, whether projective or metrical, may logically be regarded as merely a particular concrete application of the general theory of Order, as defined by Boundaries.*

It is true that Geometry, as ordinarily understood, is something more than this; it is concerned with the actual subjective conceptions we entertain of Space, and so far provides matter of discussion to the Psychologist; and it is concerned with the actual objective measurements we make upon physical bodies in Space, and so far concerns the Physicist; while in both these respects it extends beyond the region of pure Logic into those of Epistemology and Philosophy generally. But so far as the pure Mathematician is concerned, every geometrical theorem is embraced in the Theory about to be discussed.

This Theory divides itself naturally into three parts:

- (i) The Cataloguing of Continuous Groups;
- (ii) Collations in the Catalogue;
- (iii) Theorems deducible from the above.

D

The first of these corresponds to the geometrical problem, how to determine a system of co-ordinate axes, but without assuming to begin with any of those spatial conceptions upon which the definitions of Geometry are ordinarily based. As the name I have given it implies, the problem is essentially that of giving to the units of thought names by which we may be enabled logically (*i.e.* by means of words) to discuss their Order relations. This is done, briefly, by reducing groups of the order n to groups of the first order or groups of the order $(n-1)$, and so reducing the problem of Order in general to that of serial, or one-dimensional order.

And it is with this latter problem that I deal in these papers, under the title *Collations in the Catalogue*. After this there remain, of course, all the problems of Geometry to be dealt with, upon some of which the Theory of Order is capable of throwing some interesting light. I do not, however, intend to discuss any of these problems here, nor shall I say more about those in part (i) than is necessary to make clear the definitions which are required for part (ii), which I proceed to give, in rather discursive form, indicating the words to be defined by printing them in *italics*.

Suppose we are given an assemblage of objects, or *units of thought*, which, in accordance with some convention, or *rule of contiguity*, may be thought of successively, or as I say, *passed in review*, in such wise that we are bound (in theory), after thinking of any one unit, to think next only of one of a certain sub-group of units which are said to be *contiguous* to it, and not of any other unit chosen at random in the assemblage. Such an assemblage, which may be so passed in review, I call an *Ordered Group*.

The relation of contiguity discussed here is a reciprocal one, *i.e.* if A is contiguous to B , then B is also contiguous to A . But unless the contrary is expressly stated, I assume that in "passing in review," after passing from A to B , we pass on from B to some new unit, and not at once back to A again. In general, therefore, an unit has at least two units contiguous to it, even in the simplest ordered group. Such a group, a serial group, or *group of the first order*, might be defined by saying that its rule of contiguity is that each unit has two, and only two, units contiguous to it, but such a definition could not be extended to higher groups, nor does the full significance of such a definition appear except in connection with the conception of a Boundary, which I define in general terms as follows:

A *Boundary* is a sub-group contained in a wider group of units, which divides the whole into two distinct parts, in such a way that—

(i) it is impossible to "pass in review" from any unit in one of the parts to any unit in the other, without passing at least one unit of the boundary on the way, and

(ii) every unit in the boundary is contiguous to at least one unit in each of the two parts; so that it is not necessary to pass in review more than one of the units in the boundary.

A boundary is in general an ordered group, and is said to be of an *order* less by one than that of the group it divides. Two units respectively in the two parts into which a boundary divides a group are said to be separated by the *boundary*, which is said to be *between* the units. In general, therefore, the units contiguous to a given unit, since they form a boundary separating it from all other units, form an ordered group. In the case of a group of the first order, however, this is not the case, since the two units cannot by themselves be passed in review. We may, however, speak of them as forming a boundary of the order zero.

If we pass in review units of a group of order higher than the first, the units passed in review form a sub-group of the whole assemblage, which is of the first order as far as any part of it excepting the first and last units are concerned. If, however, the first and last units are the same, and no unit was passed in review twice, then the sub-group is a complete example of a group of the first order, in which any two units, not contiguous to one another, form a boundary. It is an *uniform* group, *i.e.* the rule of contiguity is the same at every unit in it. If the first and last units are not the same, the rule of contiguity is not the same for them, for each is contiguous to only one unity, instead of two, as in the case of the other units. Such a group may be called a *terminated* one, and the first and last units its *terminal* units.

Unless otherwise stated, when I speak of groups, I shall mean ordered, uniform, continuous, isomorphic group. I use the word *continuous* in its strict mathematical sense; by *isomorphic* I mean that it is such that the two parts into which any boundary divides the whole (except the limiting case of the boundary formed by the units contiguous to an unit, or to those of a sub-group of order less than that of a boundary group) are from the point of view of Order indistinguishable from one another. Terminated groups occur only as parts of complete groups, or in a special sense which does not really violate this definition.

(1) I may, therefore, lay it down as a general proposition that—*In any group of the first order a single unit cannot form a boundary, or be said to be between two others.*

I do not propose to go at length into the question of how a continuous group, even of the first order, can be catalogued, nor of the mathematical theory of continuity in connection with this problem, though these questions might well repay investigation at another time. It is sufficient for my present purpose to employ the twenty-six letters of the alphabet as a catalogue; but as the groups to be catalogued are uniform, and their order is defined solely by boundaries, we must be prepared to "pass the alphabet in review" indifferently forwards or backwards, and, moreover, we must pass on from *Z* to *A*, or from *A* to *Z*, without pause, so to speak. With these conventions the alphabet is a perfect example of an uniform, albeit discrete, group of units of thought, of the first order; though, not being "isomorphic," there would be difficulties in applying a similar catalogue without reservations to groups of higher orders. And in applying it even to groups of the first order, it must be borne in mind that two letters which are consecutive in the alphabet are not to be taken to represent contiguous units. Indeed, the great advantage of defining Order by means of Boundaries is that it is not necessary to talk about contiguous units, except very rarely, and then only in limiting cases of general propositions.

In naming the units of a given group of the first order from the catalogue, any name, selected absolutely at random, may be given to the first unit. Suppose we choose *K*. For the second unit of the group we may similarly choose at random any name, *except K*. So also for the third unit we may choose at random any name but those already chosen. For we know nothing about, and cannot possibly say anything about, the order of the three units of thought, except that the units are different from one another. It takes at least two units, even in a group of the first order, to form a boundary, and two more to be separated by that boundary, so that four is the smallest number of units of thought with respect to which it is ever possible to predicate any order relation whatever. It is, therefore, only when we come to choose the fourth and subsequent names that the order of the units imposes any restriction upon our freedom of choice. Thus, if the first three names chosen were

K , B , and P , and we find the fourth unit is separated from the first by the boundary formed by the second and third, we cannot choose, for example, G , nor M , for neither of these is separated from K by the boundary (BP). But we might call the fourth unit, say, T , or Y , or A . In the latter case, it is true, we might be bothered to find a name for a subsequent unit, separated from the first and third units by the second and fourth, but this difficulty, which is only due to the inadequacy of our discrete catalogue, can easily be removed, though it is not necessary to go into the matter here. I have already taken what may seem unnecessary trouble about a very simple matter, but it will be found to be of the utmost importance that the amount of freedom of choice remaining at each stage of the process of naming units from the catalogue should be exactly appreciated.

To compare the orders of units of thought in two different groups, we may pass them in review together, thinking of one unit from each group at the same time, and so *collating* the two units, and when all the units of both groups have been passed in review, the two groups may be said to have been collated, or a *collation* effected between them. This assumes that the passing in review of both groups begins and ends with one and the same pair of collated units, and that no unit in either group is passed more than once. This will always be assumed to be the case here, and it may be observed that this condition is automatically fulfilled if we effect the collation indirectly, by naming the units of both groups from the same catalogue, for by so doing each unit in the one group is collated with one, and only one, unit in the other, namely, the one which receives the same name. We may in this way effect collations of the units in one group with themselves by simply re-naming the group. But at the same time such a re-naming may equally well be regarded as collating together the two names successively given to each unit, and so may be regarded as a "collation in the catalogue." It is in this light that I prefer to regard such collations, which form the subject matter now to be investigated.

II. COLLATIONS IN THE CATALOGUE.

In order to facilitate the discussion I shall represent Collations by what I call *Schemes*. In these, the catalogue names of units of a group (for which I always use capital letters), or symbols (small letters, etc.) representing them, if the catalogue names are not known, are written in one line, and the new names into which they are changed by the collation immediately below them on the next line. Where the order of the units is known, *e.g.* if catalogue names are employed, they should be written down in order on the lines. But this is only for convenience; the arguments deduced by means of the schemes do not depend in any way upon the order in which the letters are written down. The only data expressed by the schemes are the collations which are represented by placing two letters in the same vertical column.

In order to distinguish between collations which are given by the data of the problem and those which are deduced from them in the course of the proof, I put letters added in the scheme in virtue of such deductions between brackets.

If the conditions of the problem would admit of the determination of any name, if it were required, it may be represented by a $+$. But if the conditions of the problem determine that any letter whatever might be written in a given place, I represent it by the "indeterminate symbol," thus... The significance of this will appear later.

When writing about the names in a scheme, a numerical suffix may be used to denote to which line of the scheme the name belongs.

Thus in the scheme—

$$\begin{array}{cccccc} A & B & K & (P) & X & (x) \} \\ B & x & L & + & (P) & \dots \end{array} \dots\dots\dots (a)$$

the catalogue names A_1 and B_2 are collated together, while B_1 is collated with some catalogue name which is represented by the small letter x_2 ; K_1 is collated with L_2 . These collations constitute the data of the problem, and from them we deduce that a particular letter, which it is not necessary to name, is collated with P_1 , while X_1 is collated with P_2 , and x_1 is collated indeterminately.

It must be remembered that schemes do not prove things by themselves; they are only graphic aids to the mental process of passing in review; they only help to make the proofs *anschaulich*. The essence of the proofs always lies in the inferences as to boundaries, i.e. whether one can, or can not, pass in review without passing certain units, or one of them, on the way. Now it is true that passing in review is a process in time, and as such depends on an asymmetrical relation; it has sense; and in passing in review a group of the first order, I shall assume that we pass always in the same sense till the whole group has been passed. It is, however, easy to define sense by means of boundaries alone, provided we are referred to not less than three given units in the group, without assuming any asymmetrical relation between them; and moreover, without discriminating between contiguous units, and so introducing the difficulties involved in the mathematical conception of continuity. This, therefore, I proceed to do.

When passing in review a group of the first order, the initial unit, that is the first one thought of, and the unit under review at any moment constitute a boundary dividing the whole group into two parts, the one containing all the units already passed in review, and the other all those which have yet to be passed. This condition ensures that the sense of the passing in review shall be the same throughout.

Again, if we pass the same group in review twice, or collate a group of the first order with itself, the question will arise whether the senses in which we passed were the same, or different. More particularly it will be seen that the nature of collations in the catalogue depends essentially on whether the names under review pass in the same or in opposite senses, a distinction which may be defined as follows:

When passing in review a collation in the catalogue, if *initially*, i.e. before either of the pair of collated names passes the boundary in the catalogue formed by the first pair of names collated together, the names in each pair already passed were separated by that boundary, then the names are said to be passed in the *same sense*, and the collation is called a *positive* one. If, on the contrary, they are not so separated, the names are passing in *opposite senses*, and the collation is a *negative* one.

This definition applies primarily to uniform (closed) groups of the first order. But it may be applied also to terminated groups, provided we begin passing in review always from one of the terminal units.

Now, collations may themselves be regarded as units of thought; and, as will shortly be seen, ordered groups of them may be formed, which may be passed in review. In doing this it will be found that the whole group of collations is separated into two parts, containing positive and negative collations respectively, by a boundary formed by two limiting cases of collation, the nature of which I proceed to indicate.

I have already laid it down that in passing in review we pass in the same sense till the whole group has been reviewed; the unit under review is, so to speak, never to turn back. But this need not prevent the unit from stopping still for a while in one group, while it goes on in

the collated group; or to put it another way, we may in special cases change the old name of one unit for more than one new name.

(2) Such a collation I will call a discontinuous one, and it may be shown that: *In a discontinuous collation, an unit which is collated with more than one unit must be collated with a continuous group of units, no two of which are separated by any two units with which the former unit is not collated.*

(3) Moreover, it may be shown that in discontinuous collations in the catalogue: *If any name within a group of units collated discontinuously with a given name (not a terminal name of the group) is also collated discontinuously, then the groups of names collated with single names must in both bases consist of the whole catalogue of names.*

As a matter of fact, this is the only sort of discontinuous collation with which we shall have to deal, and we may, therefore, say that besides positive and negative collations there are zero collations in the catalogue, in which one old and one new name (which may or may not be the same name) is collated in each case with all the names in the catalogue.

As an illustration, consider the specimen scheme given above. If we pass in review the names in the upper line as they are written, the initial collation is (A, B_2) , and we shall not reach any given collation before the old name reaches B_1 . Let us, therefore, try the other way round, passing from A_1 to Z_1 , X_1 , P_1 , K_1 , and so on. The first given collation which we reach is (KL) . Now, on the upper line, we have been passing names in the same part with respect to the boundary (AB) as K is, but on the lower line all we can say is that the units we have been passing have been in one of the two parts into which (BL) divides the catalogue, but we cannot say which until we know what catalogue name is represented by x ; for x is not a catalogue name. We cannot yet say whether the old and new names already passed are or are not separated by (AB) , or whether the collation is positive or negative.

(4) *We cannot determine the sense of a collation, unless the collations of at least three pairs of units are known.* But given this much, the sense of a collation is determined: for I may say at once that we shall not have anything to do with cases like that represented in the specimen scheme above, where part of the collation is continuous and part discontinuous. In all schemes in which the indeterminate symbol appears, zero collations are represented; and the rest of the line containing the symbol will therefore consist of repetitions of a single letter.

Let us now consider any four units in a group of the first order; say the names C , F , L , and Q ; and let us pass the catalogue in review, starting from C .

Suppose the first of the named units which we come to is Q . Then (CQ) forms a boundary, and with respect to it L and F are in the same part of the group, which is passed after Q .

But we might equally well have passed this part of the group first. And in that case either F or L must have been the first named unit to be reached. Our knowledge of the alphabet tells us that it would have been F .

Thus in neither case could we reach L without passing either Q or F first. That is, (FQ) is a boundary, which separates C and L . Similarly it may be shown that (CL) forms a boundary separating F and Q .

(5) It follows that: *Of the three pairs of pairs which can be formed of any four units in a group of the first order, one consists of two pairs which mutually separate one another, while the other two consist each of two pairs which do not separate one another.*

Let us next discuss a collation in the catalogue, and pass in review, starting with (A_1K_2) as initial collation. We may denote the units

under review by x and y . And first, let the collation be a negative one, so that initially x and y are not separated by (AK) . Nor does (Ax) initially separate (Ky) . Therefore, initially, (Ay) must separate (Kx) .

But, before x_1 reaches K_1 , or y_2 reaches A_2 (whichever event occurs first), (Ay) has ceased to separate (Kx) .

And, since neither has x been A_1 , nor y K_2 ; x must have crossed the boundary (Ay) at y ; and y crossed the boundary (Kx) at x .

That is to say, x and y must have stood for the same name, say C , which was therefore collated with itself.

Similar reasoning shows that after both x and y have passed K and A , they must come again to represent the same letter, say P this time, before they return respectively to A and K .

But after leaving C , x and y are separated by (CP) until they reach P together, and then are again separated by the same boundary until they reach C together.

(6) Therefore: *In any negative collation, two, and only two, units are collated each with itself; or, in the catalogue, two, and only two, names are left unchanged by a negative collation.*

If, however, the collation is a positive one, x and y are initially separated by (AK) , and remain so until one or other of them crosses that boundary. The collation being a continuous one, if they reach K_1 and A_2 at the same time, they will cross the boundary together, and so still be separated by (AK) after they have crossed it, till they reach A and K once more, at the same time.

But if x does not reach K at the same time as y reaches A , then after one of these events, and before the other, we may have (Ax) separating (Ky) , and (AK) no longer separating (xy) . While this state of affairs lasts, therefore, x and y may come to mean the same name, once or more than once.

(7) But: *If (Ay) ever separates (Kx) , at least two names must remain unchanged by the collation.* For y must have passed the boundary (Kx) at x , and will have to pass it back again before x and y have both passed the boundary (KA) .

(8) Thus: *In a positive collation, there may be no name unchanged, or there may be any number of such names.*

In the case of zero collations it is obvious that one, or two, units collated discontinuously will each be collated with itself, amongst the rest, and that no other units can be so collated.

In general, if A_1 is collated with, say, K_2 , there is no reason why K_1 should be collated with A_2 ; but such *reciprocal* collations are of peculiar significance in the theory of Order. We have, indeed, come across one example of this already. We found that if in a positive collation the units under review reached the initially collated units K_1 and A_2 at the same time, i.e. if A and K were reciprocally collated, they would, afterwards as before, be separated by (AK) .

(9) That is: *If in a positive collation any one pair of units is collated reciprocally, then no unit is collated with itself.* It should, however, be noted that a single unit, collated with itself, is reciprocally collated; and this may occur once or oftener in a positive collation.

If in any collation every pair of units is collated reciprocally, the collation as a whole may be called reciprocal. In the case of a negative reciprocal collation, two units will be each reciprocally collated with itself. I use the terms *conjugate pairs* and *self-conjugate units* to denote such reciprocal collations. In the case of zero collations, if, say, A_1 is collated discontinuously, and any other letter, say K_2 , is collated with it reciprocally, then K_1 is collated with A_2 . But in a zero collation there is only one name in the second group collated with every name

in the first; this one name must, therefore, be A ; and it follows, therefore, that the same name is collated discontinuously in both groups.

(10) Thus we conclude: *In a positive reciprocal collation every conjugate pair separates every other; in a negative reciprocal collation no conjugate pair separates any other; in a zero reciprocal collation, all conjugate pairs have a common unit.*

(11) And: *In a positive reciprocal collation no unit is self-conjugate; in a negative reciprocal collation two units are self-conjugate, and they form a self-conjugate boundary, which separates the two units of every self-conjugate pair; in a zero reciprocal collation it is the same unit in both groups which is collated discontinuously in both groups, and it alone is self-conjugate.*

E. T. DIXON.

(To be continued.)

PROOF BY RECURRENCE.

IN its most simple and general form the proof by recurrence of a formula containing constants and one variable consists in giving a number of successive integral values to the variable, and showing by trial that the formula is true for those values. The next step is to show that the *form* of the result remains the same when $n+1$ is substituted for n , or that each result depends upon the preceding one. If this second step is neglected, as it often used to be, the proof seems to be invalid.

This proof by mathematical or successive induction has been very generally accepted by mathematicians and neglected by logicians.

Todhunter's¹ account of the process and application to the binomial theorem may be accepted as authoritative, from his large acquaintance with previous writers and his special study of the theory of probability: "We prove that if a theorem is true in one case, whatever that case may be, it is true in another case which we may call the *next* case; we prove by trial that the theorem *is* true in a certain case; hence it is true in the next case, and hence in the next to that, and so on; hence it must be true in every case after that with which we began." "Hence if the laws hold when $n-1$ factors are multiplied together, they hold when n factors are multiplied together; but they have been proved to hold when four factors are multiplied together, therefore they hold when five factors are multiplied together, and so on; thus they hold universally."

The article "Induction" (Mathematics) in the *Penny Cyclopædia* by de Morgan has been so long out of print that a mere reference would be of little use: "The method of induction, in the sense in which the word is used in natural philosophy, is not known in pure mathematics. There certainly are instances in which a general proposition is proved by a collection of the demonstrations of different cases, which may remind the investigator of the inductive process, or the collection of the general from the particular. Such instances, however, must not be taken as permanent, for it usually happens that a general demonstration is discovered as soon as attention is turned to the subject.

"There is, however, one particular method of proceeding which is extremely common in mathematical reasoning, and to which we propose to give the name of *successive induction*. It has the main character of induction in physics, because it is really the collection of a general truth from a demonstration which implies the examination of every particular case; but it differs from the process of physics, inasmuch as each case depends upon the one which precedes. Substituting, however, demonstration for observation, the mathematical process bears an analogy to the experimental one,

¹ *Algebra*, 272, 288.