"On the Geometrical Representation of some familiar Cases of Reaction in Rigid Dynamics," by Prof. R. Townsend, F.R.S. ("Quarterly Journal of Mathematics," No. 51): from the author.

The Method of Reversion applied to the Transformation of Angles.

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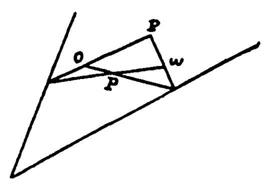
[*Abstract of Paper read May* 13*th*, 1875.]

The basis of this paper is a neglected work on Conic Sections (by G. Walker, F.R.S., Nottingham, 1794), which, for originality and thoroughness, is, in its own special department, unsurpassed.

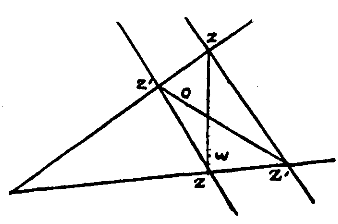
Walker establishes a connexion between a conic and a circle by means of a homographic transformation, which is a particular case of the following:—



Take fixed origins O, ω, and corresponding to each take a fixed straight line or axis, and let the law of correspondence between points, as P, p, be that PO, $p\omega$ intersect on the ω-axis, and Pω, pO on the O-axis. Call P, p REVERSE points. Then it is evident that each origin is the reverse of the other; that reverse curves are of the same degree; and, generally, that the relations of any point to either origin are of the same character as those of the reverse point to the reverse origin.

To determine the limits of deformation, take origins O, ω, and let it be required to determine the corresponding axes. Assume that a given point p shall be the reverse of a given point P. Then since PO, ωp meet on the ω-axis, and Pω, Op on the O-axis, one point on each axis is known. In like manner, by reversing a point Q into q, a second point on each axis is determined. Hence, in general, it is possible to reverse a quadrilateral OPQω into a quadrilateral ωpqO.

Construction for Reverse Lines.



It is evident that the reverse of a point z on the ω-axis is the point Z in which $z\omega$ meets the O-axis; and the reverse of a point Z′ on the O-axis is the point z' in which Z′O meets the ω-axis. Hence the straight line Zz' is the reverse of zZ′.

Taking the axes of reversion as axes of coordinates, and (h, k), (H, K) as the coordinates of the origins of reversion, it is easy to see that reverse lines, as zZ', Zz', may be represented by equations of the forms

$$\frac{x}{a} + \frac{y}{b} - 1 = 0 \quad \text{..............................(A)},$$

$$\frac{x}{H} + \frac{y}{k} - 1 = \frac{K}{H} \cdot \frac{x}{b} + \frac{h}{k} \cdot \frac{y}{a} \quad \text{.............. (B)}.$$

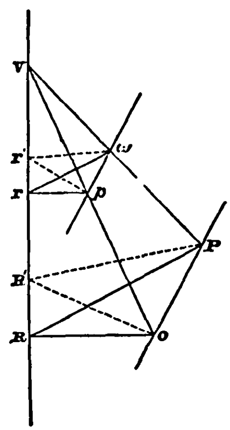
Let a, b vary proportionally. Then (A) represents a system of parallels, and (B) a system of straight lines intersecting on

$$\frac{x}{H} + \frac{y}{k} = 1 \quad \text{..............................(C)},$$

which is therefore reverse to the line at infinity. The line (C) might have been determined by removing Z' to infinity on the O-axis, and z to infinity on the ω-axis.

To proceed to the special case of reversion with which I am chiefly concerned. Let the ω-axis be at infinity, and let the O-axis be called the Base Line. We have then the construction for reverse points P, p :—

Through the origins of reversion O, ω *draw straight lines meeting, viz. in* V, *on the base line; and let* Vω, VO *be intersected in* P, p *by a pair of parallels through* O, ω.



Reverse Lines PR, pr meet the base line in points R, r, such that PR, OR are parallel respectively to ωr, pr.

The base line is the reverse of the line at infinity.

Hence the law of the

Reversion of Angles.

If the straight lines containing an angle P *meet the base line in* R, R′, *the lines containing the reverse angle* p *intercept on the base line a length* rr' *which subtends at* ω *an angle equal to* P.

Examples of the Reversion of Angles.

1. The Ortho-centre.—Let the sides of a triangle abc, the reverse of ABC, meet the base line in α, β, γ; draw through ω lines at right angles to $\omega\alpha$, $\omega\beta$, $\omega\gamma$ to meet the base line in a', b', c'. Then it is easily seen that aa', bb', cc' are reverse to the perpendiculars of the triangle ABC, and therefore *co-intersect* in a point θ, the reverse of the ortho-centre of ABC.

2. If the triangle *abc* envelopes a fixed conic which *touches the base line*, the point θ moves on a straight line, since the reverse triangle ABC envelopes a parabola, and its orthocentre therefore moves on a fixed straight line, viz. the directrix.

It is easy to deduce, that if a triangle *abc* envelopes a circle, and if the three parallel tangents meet a seventh tangent in a', b', c'; then aa', bb', cc' are parallel.

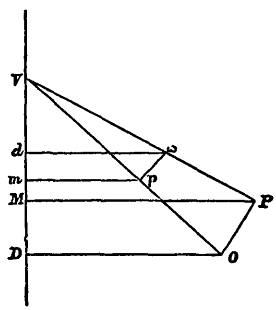
3. *Angles subtended at* O, ω *are equal each to each*, in consequence of the parallelism of OP, $p\omega$. To illustrate this special case, take the theorem that

A chord of a conic which subtends a right angle at a fixed point on the curve, passes through a fixed point on the normal;

which follows by reversion from the fact that the angle in a semicircle is a right angle, if this be first expressed in the form:

A chord of a circle which subtends a right angle at a fixed point on the circumference, passes through a fixed point on the normal.

WALKER'S CIRCLE.



Let fall perpendiculars ωd, OD, PM, pm upon the base line. Then, by parallels,

$$OP : \omega p = PV : \omega V = PM : \omega d.$$

Hence, if the locus of p be a circle about ω, the locus of P will be a conic having O for focus and the base line for directrix.

It is on this property that Walker's system of Geometrical Conics is based. See "Messenger of Mathematics," Vol. ii. p. 97 (1872).

It may be shewn that

$$PM . pm = OD . \omega d,$$

and hence that, if p (x, y) be referred to ωd and the base line as axes, and P (X, Y) be referred to OD and the base line as axes, the analytical reverse transformation will be

$$y = \frac{\omega d . OD}{Y}; \quad -x = \frac{\omega d . X}{Y}.$$

Compare Newton's "Principia," Lib. I., Lemma 22: "Figuras in alias ejusdem generis figuras mutare."

The method of *Reversion* is easily adapted to transformation in space.