ART. LIV.—*Notes on the Electromagnetic Theory of Light;* by J. WILLARD GIBBS. NO. II.—*On Double Refraction in perfectly transparent Media which exhibit the Phenomena of Circular Polarization.*

1. IN the April number of this Journal,* the velocity of propagation of a system of plane waves of light, regarded as oscillating electrical fluxes, was discussed with such a degree of approximation as would account for the dispersion of colors and give Fresnel's laws of double refraction. It is the object of this paper to supplement that discussion by carrying the approximation so much further as is necessary in order to embrace the phenomena of circularly polarizing media.

2. If we imagine all the velocities in any progressive system of plane waves to be reversed at a given instant without affecting the displacements, and the system of wave-motion thus obtained to be superposed upon the original system, we obtain a system of stationary waves having the same wave-length and period of oscillation as the original progressive system. If we then reduce the magnitude of the displacements in the uniform ratio of two to one, they will be identical, at an instant of maximum displacement, with those of the original system at the same instant.

Following the same method as in the paper cited, let us especially consider the system of stationary waves, and divide the whole displacement into the *regular* part, represented by ξ, η, ζ, and the *irregular* part, represented by ξ', η', ζ', in accordance with the definitions of § 2 of that paper.

3. The *regular* part of the displacement is subject to the equations of wave-motion, which may be written (in the most general case of plane stationary waves)

$$\left.\begin{aligned}\xi&=\left(\alpha_1\cos 2\pi\frac{u}{l}+\alpha_2\sin 2\pi\frac{u}{l}\right)\cos 2\pi\frac{t}{p},\\ \eta&=\left(\beta_1\cos 2\pi\frac{u}{l}+\beta_2\sin 2\pi\frac{u}{l}\right)\cos 2\pi\frac{t}{p},\\ \zeta&=\left(\gamma_1\cos 2\pi\frac{u}{l}+\gamma_2\sin 2\pi\frac{u}{l}\right)\cos 2\pi\frac{t}{p},\end{aligned}\right\}\qquad(1)$$

where l denotes the wave-length, p the period of oscillation, u the distance of the point considered from the wave-plane passing through the origin, α_1, β_1, γ_1 the amplitudes of the displacements ξ, η, ζ in the wave-plane passing through the origin, and α_2, β_2, γ_2 their amplitudes in a wave-plane one-quarter of a

* See page 262 of this volume.

wave-length distant and on the side toward which u increases. If we also write L, M, N for the direction-cosines of the wave-normal drawn in the direction in which u increases, we shall have the following necessary relations:

$$L^2+M^2+N^2=1, \tag{2}$$

$$u=Lx+My+Nz, \tag{3}$$

$$L\alpha_1+M\beta_1+N\gamma_1=0, \quad L\alpha_2+M\beta_2+N\gamma_2=0. \tag{4}$$

4. That the *irregular* part of the displacement (ξ', η', ζ') at any given point is a simple harmonic function of the time, having the same period and phase as the regular part of the displacement (ξ, η, ζ), may be proved by the single principle of superposition of motions, and is therefore to be regarded as exact in a discussion of this kind. But the further conclusion of the preceding paper (§ 4), "that the values of ξ', η', ζ' at any given point in the medium are capable of expression as linear functions of ξ, η, ζ in a manner which shall be independent of the time and of the orientation of the wave-planes and the distance of a nodal plane from the point considered, so long as the period of oscillation remains the same," is evidently only approximative, although a very close approximation. A very much closer approximation may be obtained, if we regard ξ', η', ζ', at any given point of the medium and for light of a given period, as linear functions of ξ, η, ζ and the nine differential coëfficients

$$\frac{d\xi}{dx}, \quad \frac{d\eta}{dx}, \quad \frac{d\zeta}{dx}, \quad \frac{d\xi}{dy}, \quad \text{etc.}$$

We shall write ξ, η, ζ *and diff. coëff.* to denote these twelve quantities.

From this it follows immediately that with the same degree of approximation $\dot\xi'$, $\dot\eta'$, $\dot\zeta'$ may be regarded, for a given point of the medium and light of a given period, as linear functions of $\dot\xi$, $\dot\eta$, $\dot\zeta$ and the differential coëfficients of $\dot\xi$, $\dot\eta$, $\dot\zeta$ with respect to the coördinates. For these twelve quantities we shall write $\dot\xi$, $\dot\eta$, $\dot\zeta$ *and diff. coëff.*

5. Let us now proceed to equate the statical energy of the medium at an instant of no velocity with its kinetic energy at an instant of no displacement. It will be convenient to estimate each of these quantities for a unit of volume.

6. The statical energy of an infinitesimal element of volume may be represented by $\sigma\, dv$, where σ is a quadratic function of the components of displacement $\xi+\xi'$, $\eta+\eta'$, $\zeta+\zeta'$. Since for

that element of volume ξ', η', ζ' may be regarded as linear functions of ξ, η, ζ *and diff. coëff.*, we may regard σ as a quadratic function of ξ, η, ζ *and diff. coëff.*, or as a linear function of the seventy-eight squares and products of these quantities. But the seventy-eight coëfficients by which this function is expressed will vary with the position of the element of volume with respect to the surrounding molecules.

In estimating the statical energy or any considerable space by the integral

$$\int \sigma dv,$$

it will be allowable to substitute for the seventy-eight coëfficients contained implicitly in σ their average values throughout the medium. That is, if we write s for a quadratic function of ξ, η, ζ, *and diff. coëff.* in which the seventy-eight coëfficients are the space-averages of those in σ, the statical energy of any considerable space may be estimated by the integral

$$\int s\, dv.$$

(This will appear most distinctly if we suppose the integration to be first effected for a thin slice of the medium bounded by two wave-planes.) The seventy-eight coëfficients of this function s are determined solely by the nature of the medium and the period of oscillation.

We may divide s into three parts, of which the first ($s_{/}$) contains the squares and products of ξ, η, ζ, the second ($s_{//}$) contains the products of ξ, η, ζ with the differential coëfficients, and the third ($s_{///}$) contains the squares and products of the differential coëfficients. It is evident that the average statical energy of the whole medium per unit of volume is the space-average of s, and that it will consist of three parts, which are the space-averages of $s_{/}$, $s_{//}$ and $s_{///}$ respectively. These parts we may call $S_{/}$, $S_{//}$, and $S_{///}$. Only the first of these was considered in the preceding paper.

Now the considerations which justify us in neglecting, for an approximate estimate, the terms of s which contain the differential coëfficients of ξ, η, ζ with respect to the coördinates, will apply with especial force to the terms which contain the squares and products of these differential coëfficients. Therefore, to carry the approximation one step beyond that of the preceding paper, it will only be necessary to take account of $s_{/}$ and $s_{//}$ and of $S_{/}$ and $S_{//}$.

7. We may set

$$s_{/} = A\xi^2 + B\eta^2 + C\zeta^2 + E\eta\zeta + F\zeta\xi + G\xi\eta, \qquad (5)$$

where, for a given medium and light of a given period, A, B, C, E, F, G are constant.

Since the average values of

$$\sin^2 2\pi\frac{u}{l}, \quad \cos^2 2\pi\frac{u}{l}, \quad \sin 2\pi\frac{u}{l}\cos 2\pi\frac{u}{l}$$

are respectively $\frac{1}{2}$, $\frac{1}{2}$, and 0, and since at the time to be considered

$$\cos^2 2\pi\frac{t}{p}=1,$$

it will appear from inspection of equations (1) that

$$\begin{aligned}S_{\prime}=&\tfrac{1}{2}(A\alpha_1^2+B\beta_1^2+C\gamma_1^2+E\beta_1\gamma_1+F\gamma_1\alpha_1+G\alpha_1\beta_1)\\&+\tfrac{1}{2}(A\alpha_2^2+B\beta_2^2+C\gamma_2^2+E\beta_2\gamma_2+F\gamma_2\alpha_2+G\alpha_2\beta_2). \qquad (6)\end{aligned}$$

This is the first part of the statical energy of the whole medium per unit of volume.

8. The second part of the statical energy of the whole medium per unit of volume ($S_{\prime\prime}$) is the space-average of $s_{\prime\prime}$, which is a linear function of the twenty-seven products of ξ, η, ζ with their differential coëfficients with respect to the coördinates. Now since

$$\xi\frac{d\xi}{dx}=\tfrac{1}{2}\frac{d(\xi^2)}{dx}, \quad \eta\frac{d\eta}{dx}=\tfrac{1}{2}\frac{d(\eta^2)}{dx}, \quad \text{etc.,}$$

the space-average of such products will be zero, and they will contribute nothing to the value of $S_{\prime\prime}$. There will be nine of these products, in which the same component of displacement appears twice. The remaining eighteen products may be divided into pairs according to the letters which they contain, as

$$\eta\frac{d\zeta}{dx} \quad \text{and} \quad \zeta\frac{d\eta}{dx}.$$

A linear function of the eighteen products may also be regarded as a linear function of the sums and differences of the products in such pairs. But since

$$\eta\frac{d\zeta}{dx}+\zeta\frac{d\eta}{dx}=\frac{d(\eta\zeta)}{dx},$$

the terms of $s_{\prime\prime}$ containing such sums will contribute nothing to the value of $S_{\prime\prime}$. We have left a linear function of the nine differences

$$\eta\frac{d\zeta}{dx}-\zeta\frac{d\eta}{dx}, \quad \zeta\frac{d\xi}{dx}-\xi\frac{d\zeta}{dx}, \quad \xi\frac{d\eta}{dx}-\eta\frac{d\xi}{dx}, \quad \text{etc.,}$$

(the unwritten expressions being obtained by substituting in the denominators dy and dz for dx,) which constitutes the part of $s_{,,}$ that we have to consider. $S_{,,}$ is therefore a linear function of the space-averages of these nine quantities. But by (3)

$$\eta\frac{d\zeta}{dx}-\zeta\frac{d\eta}{dx}=\mathrm{L}\left(\eta\frac{d\zeta}{du}-\zeta\frac{d\eta}{du}\right),$$

and the space-average of this, at a moment of maximum displacement, is by (1)

$$\frac{2\pi\mathrm{L}}{l}(\beta_1\gamma_2-\gamma_1\beta_2).$$

By such reductions it appears that $l\mathrm{S}_{,,}$ is a linear function of the nine products of L, M, N with

$$\beta_1\gamma_2-\gamma_1\beta_2,\quad \gamma_1\alpha_2-\alpha_1\gamma_2,\quad \alpha_1\beta_2-\beta_1\alpha_2.$$

Now if we set

$$\Theta=\mathrm{L}\,(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}\,(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}\,(\alpha_1\beta_2-\beta_1\alpha_2), \qquad (7)$$

we have by (4) and (2)

$$\mathrm{L}\Theta=\beta_1\gamma_2-\gamma_1\beta_2,\quad \mathrm{M}\Theta=\gamma_1\alpha_2-\alpha_1\gamma_2,\quad \mathrm{N}\Theta=\alpha_1\beta_2-\beta_1\alpha_2. \quad (8)$$

Therefore $l\mathrm{S}_{,,}$ is a linear function of the nine products of L, M, N with $\mathrm{L}\Theta$, $\mathrm{M}\Theta$, $\mathrm{N}\Theta$. That is, $l\mathrm{S}_{,,}$ is the product of Θ and a quadratic function of L, M and N. We may therefore write

$$\mathrm{S}_{,,}=\frac{\Phi}{l}\Theta=\frac{\Phi}{l}[\mathrm{L}\,(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}\,(\alpha_1\beta_2-\beta_1\alpha_2)], (9)$$

where Φ is a quadratic function of L, M and N, dependent, however, on the nature of the medium and the period of oscillation.

9. It will be useful to consider more closely the geometrical significance of the quantity Θ. For this purpose it will be convenient to have a definite understanding with respect to the relative position of the coördinate axes.

We shall suppose that the axes of X, Y, and Z are related in the same way as lines drawn to the right, forward and upward, so that a rotation from X to Y appears clock-wise to one looking in the direction of Z.

Now if from any same point, as the origin of coördinates, we lay off lines representing in direction and magnitude the displacements in all the different wave-planes, we obtain an ellipse, which we may call the *displacement-ellipse.** Of this, one radius vector (ρ_1) will have the components α_1, β_1, γ_1, and

* This ellipse, which represents the simultaneous displacements in different parts of the field, will also represent the successive displacements at any same point in the corresponding system of progressive waves.

another (ρ_2) the components α_2, β_2, γ_2. These will belong to conjugate diameters, each being parallel to the tangent at the extremity of the other. The area of the ellipse will therefore be equal to the parallelogram of which ρ_1, and ρ_2 are two sides, multiplied by π. Now it is evident that $\beta_1\gamma_2-\gamma_1\beta_2$, $\gamma_1\alpha_2-\alpha_1\gamma_2$, $\alpha_2\beta_1-\beta_1\alpha_2$ are numerically equal to the projections of this parallelogram on the planes of the coördinate axes, and are each positive or negative according as a revolution from ρ_1 to ρ_2 appears clock-wise or counter-clock-wise to one looking in the direction of the proper coördinate axis. Hence, θ will be numerically equal to the parallelogram, that is, to the area of the displacement-ellipse divided by π, and will be positive or negative according as a revolution from ρ_1 to ρ_2 appears clock-wise or counter-clock-wise to one looking in the direction of the wave-normal. Since ρ_1 and ρ_2 are determined by displacements in planes one-quarter of a wave-length distant from each other, and the plane to which the latter relates lies on the side toward which the wave-normal is drawn, it follows that θ is positive or negative according as the combination of displacements has the character of a right-handed or a left-handed screw.

10. The kinetic energy of the medium, which is to be estimated for an instant of no displacement, may be shown as in § 7 of the former paper (page 266 of this volume) to consist of two parts, of which one relates to the regular flux $(\dot\xi, \dot\eta, \dot\zeta)$, and the other to the irregular flux $(\dot\xi', \dot\eta', \dot\zeta')$. The first, in the notation of that paper, is represented by

$$\tfrac{1}{2}\int(\dot\xi \text{ Pot } \dot\xi+\dot\eta \text{ Pot } \dot\eta+\dot\zeta \text{ Pot } \dot\zeta)\, dv,$$

which reduces to

$$\frac{l^2}{2\pi}\int(\dot\xi^2+\dot\eta^2+\dot\zeta^2)\, dv.$$

By substitution of the values given by equations (1), we obtain for the kinetic energy due to the regular flux in a unit of volume

$$\mathrm{T}=\frac{\pi l^2}{p^2}(\alpha_1^{\,2}+\beta_1^{\,2}+\gamma_1^{\,2}+\alpha_2^{\,2}+\beta_2^{\,2}+\gamma_2^{\,2}). \tag{10}$$

11. The kinetic energy of the irregular part of the flux is represented by the volume-integral

$$\int\tfrac{1}{2}(\dot\xi' \text{ Pot } \dot\xi'+\dot\eta' \text{ Pot } \dot\eta'+\dot\zeta' \text{ Pot } \dot\zeta')\, dv.$$

Now, since $\dot\xi'$, $\dot\eta'$, $\dot\zeta'$ are everywhere linear functions of $\dot\xi$, $\dot\eta$, $\dot\zeta$ *and diff. coëff.* (see § 4), and since the integrations implied in the notation *Pot* may be confined to a sphere of which the radius is small in comparison with a wave-length,* and since within such a sphere $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ *and diff. coëff.* are sufficiently determined (in a linear form), by the values of the same twelve quantities at the center of the sphere, it follows that Pot $\dot{\xi}'$, Pot $\dot{\eta}'$, Pot $\dot{\zeta}$ must be linear functions of the values of $\dot{\xi}$, $\dot{\eta}'$, $\dot{\zeta}$ *and diff. coëff.* at the point for which the potential is sought. Hence,

$$\tfrac{1}{2}(\dot{\xi}' \text{ Pot } \dot{\xi}' + \dot{\eta}' \text{ Pot } \dot{\eta}' + \dot{\zeta}' \text{ Pot } \dot{\zeta}')$$

will be a quadratic function of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ *and diff. coëff.* But the seventy-eight coëfficients by which this function is expressed will vary with the position of the point considered with respect to the surrounding molecules.

Yet, as in the case of the statical energy, we may substitute the average values of these coëfficients for the coëfficients themselves in the integral by which we obtain the energy of any considerable space. The kinetic energy due to the irregular part of the flux is thus reduced to a quadratic function of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ *and diff. coëff.* which has constant coëfficients for a given medium and light of a given period.

The function may be divided into three parts, of which the first contains the squares and products of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, the second the products of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ with their differential coëfficients, and the third, which may be neglected, the squares and products of the differential coëfficients.

We may proceed with the reduction precisely as in the case of the statical energy, except that the differentiations with respect to the time will introduce the constant factor $\frac{4\pi^2}{p^2}$. This will give for the first part of the kinetic energy of the irregular flux per unit of volume

$$\begin{aligned} T'_{\prime} &= \frac{2\pi^2}{p^2}(A'\alpha_1^2 + B'\beta_1^2 + C'\gamma_1^2 + E'\beta_1\gamma_1 + F'\gamma_1\alpha_1 + G'\alpha_1\beta_1) \\ &+ \frac{2\pi^2}{p^2}(A'\alpha_2^2 + B'\beta_2^2 + C'\gamma_2^2 + E'\beta_2\gamma_2 + F'\gamma_2\alpha_2 + G'\alpha_2\beta_2), \end{aligned} \qquad (11)$$

and for the second part of the same

$$\begin{aligned} T'_{\prime\prime} &= \frac{4\pi^2\Phi'}{p^2 l}\Theta \\ &= \frac{4\pi^2\Phi'}{p^2 l}[L(\beta_1\gamma_2 - \gamma_1\beta_2) + M(\gamma_1\alpha_2 - \alpha_1\gamma_2) + N(\alpha_1\beta_2 - \beta_1\alpha_2)], \end{aligned} \qquad (12)$$

* See § 9 of the former paper, on page 268 of this volume.

where A', B', C', E', F', G' are constant, and Φ' a quadratic function of L, M, and N, for a given medium and light of a given period.

12. Equating the statical and kinetic energies, we have

$$S_{,}+S_{,,}=T+T'_{,}+T'_{,,},$$

that is, by equations (6), (9), (10), (11), and (12),

$$\begin{aligned}
&\tfrac{1}{2}(A\alpha_1^2+B\beta_1^2+C\gamma_1^2+E\beta_1\gamma_1+F\gamma_1\alpha_1+G\alpha_1\beta_1)\\
&+\tfrac{1}{2}(A\alpha_2^2+B\beta_2^2+C\gamma_2^2+E\beta_2\gamma_2+F\gamma_2\alpha_2+G\alpha_2\beta_2)\\
&+\frac{\Phi}{l}[L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2)\\
&=\frac{\pi l^2}{p^2}(\alpha_1^2+\beta_1^2+\gamma_1^2+\alpha_2^2+\beta_2^2+\gamma_2^2)\\
&+\frac{2\pi^2}{p^2}(A'\alpha_1^2+B'\beta_1^2+C'\gamma_1^2+E'\beta_1\gamma_1+F'\gamma_1\alpha_1+G'\alpha_1\beta_1)\\
&+\frac{2\pi^2}{p^2}(A'\alpha_2^2+B'\beta_2^2+C'\gamma_2^2+E'\beta_2\gamma_2+F'\gamma_2\alpha_2+G'\alpha_2\beta_2)\\
&+\frac{4\pi^2\Phi'}{p^2 l}[L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2). \qquad (13)
\end{aligned}$$

If we set

$$a=\frac{A}{2\pi}-\frac{2\pi A'}{p^2}, \quad b=\frac{B}{2\pi}-\frac{2\pi B'}{p^2}, \quad \text{etc.,} \qquad (14)$$

and

$$\varphi=\frac{\Phi}{2\pi p}-\frac{2\pi\Phi'}{p^3}, \qquad (15)$$

the equation reduces to

$$\begin{aligned}
&a\alpha_1^2+b\beta_1^2+c\gamma_1^2+e\beta_1\gamma_1+f\gamma_1\alpha_1+g\alpha_1\beta_1\\
&+a\alpha_2^2+b\beta_2^2+c\gamma_2^2+e\beta_2\gamma_2+f\gamma_2\alpha_2+g\alpha_2\beta_2\\
&+\frac{2p\varphi}{l}[L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2)]\\
&=\frac{l^2}{p^2}(\alpha_1^2+\beta_1^2+\gamma_1^2+\alpha_2^2+\beta_2^2+\gamma_2^2), \qquad (16)
\end{aligned}$$

where a, b, c, e, f, g are constant, and φ a quadratic function of L, M, N, for a given medium and light of a given period.

13. Now this equation, which expresses a relation between the constants of the equations of wave-motion (1), will apply, with those equations, not only to such vibrations as actually take place, but also to such as we may imagine to take place under the influence of constraints determining the type of vibration. The free or unconstrained vibrations, with which alone we are concerned, are characterized by this, that infinitesimal variations (by constraint) of the type of vibration,

that is, of the ratios of the quantities α_1, β_1, γ_1, α_2, β_2, γ_2, will not affect the period by any quantity of the same order of magnitude.* These variations must however be consistent with equations (4), which require that

$$\mathrm{L}d\alpha_1+\mathrm{M}d\beta_1+\mathrm{N}d\gamma_1=0, \quad \mathrm{L}d\alpha_2+\mathrm{M}d\beta_2+\mathrm{N}d\gamma_2=0. \qquad (17)$$

Hence, to obtain the conditions which characterize free vibration, we may differentiate equation (16) with respect to α_1, β_1, γ_1, α_2, β_2, γ_2, regarding all other letters as constant, and give to $d\alpha_1$, $d\beta_1$, $d\gamma_1$, $d\alpha_2$, $d\beta_2$, $d\gamma_2$, such values as are consistent with equations (17). Now $d\alpha_1$, $d\beta_1$, $d\gamma_1$, are independent of $d\alpha_2$, $d\beta_2$, $d\gamma_2$, and for either three variations, values proportional either to α_1, β_1, γ_1, or to α_2, β_2, γ_2, are possible. If, then, we differentiate equation (16) with respect to α_1, β_1, γ_1, and substitute first α_1, β_1, γ_1, and then α_2, β_2, γ_2, for $d\alpha_1$, $d\beta_1$, $d\gamma_1$, and also differentiate with respect to α_2, β_2, γ_2, with similar substitutions, we shall obtain all the independent equations which this principle will yield.

If we differentiate with respect to α_1, β_1, γ_1, and write α_1, β_1, γ_1 for $d\alpha_1$, $d\beta_1$, $d\gamma_1$, we obtain

$$\begin{aligned}&a\alpha_1^2+b\beta_1^2+c\gamma_1^2+e\beta_1\gamma_1+f\gamma_1\alpha_1+g\alpha_1\beta_1\\&+\frac{p\varphi}{l}[\mathrm{L}(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}(\alpha_1\beta_2-\beta_1\alpha_2)]\\&=\frac{l^2}{p^2}(\alpha_1^2+\beta_1^2+\gamma_1^2).\end{aligned} \qquad (18)$$

If we differentiate with respect to α_1, β_1, γ_1, and write α_2, β_2, γ_2 for $d\alpha_1$, $d\beta_1$, $d\gamma_1$, we obtain

$$\begin{aligned}2a\alpha_1\alpha_2+2b\beta_1\beta_2+2c\gamma_1\gamma_2+e(\beta_1\gamma_2+\gamma_1\beta_2)+f(\gamma_1\alpha_2+\alpha_1\gamma_2)&\\+g(\alpha_1\beta_2+\beta_1\alpha_2)=\frac{2l^2}{p^2}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2).&\end{aligned} \qquad (19)$$

If we differentiate with respect to α_2, β_2, γ_2, and write α_2, β_2, γ_2 for $d\alpha_2$, $d\beta_2$, $d\gamma_2$, we obtain

$$\begin{aligned}&a\alpha_2^2+b\beta_2^2+c\gamma_2^2+e\beta_2\gamma_2+f\gamma_2\alpha_2+g\alpha_2\beta_2\\&+\frac{p\varphi}{l}[\mathrm{L}(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}(\alpha_1\beta_2-\beta_1\alpha_2)]\\&=\frac{l^2}{p^2}(\alpha_2^2+\beta_2^2+\gamma_2^2).\end{aligned} \qquad (20)$$

The equation derived by differentiating with respect to α_2, β_2, γ_2, and writing α_1, β_1, γ_1 for $d\alpha_2$, $d\beta_2$, $d\gamma_2$, is identical with (19). We should also observe that equations (18) and (20) by addi-

* Compare § 11 of the former paper, page 270 of this volume.

tion give equation (16), which therefore will not need to be considered in addition to the last three equations.

14. The geometrical signification of our equations may now be simplified by a suitable choice of the position of the origin of coördinates, which is as yet wholly arbitrary.

We shall hereafter suppose that the origin is placed in a plane of maximum or minimum displacement,* if such there are. In the case of circular polarization, in which the displacements are everywhere equal, its position is immaterial. The lines ρ_1 and ρ_2, of which $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are respectively the components, will now be the semi-axes of the displacement-ellipse, and therefore at right angles. (See § 9.) The case of circular polarization will not constitute any exception. Hence,

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0. \tag{21}$$

and by § 9,

$$\Theta=\mathrm{L}(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}(\alpha_1\beta_2-\beta_1\alpha_2)=\pm\rho_1\rho_2, \tag{22}$$

where we are to read + or − in the last member according as the system of displacements has the character of a right-handed or a left-handed screw.

15. Equation (19) is now reduced to the form

$$\begin{aligned}2a\alpha_1\alpha_2+2b\beta_1\beta_2+2c\gamma_1\gamma_2+e(\beta_1\gamma_2+\gamma_1\beta_2)&\\+f(\gamma_1\alpha_2+\alpha_1\gamma_2)+g(\alpha_1\beta_2+\beta_1\alpha_2)&=0,\end{aligned} \tag{23}$$

which has a very simple geometrical signification. If we consider the ellipsoid

$$ax^2+by^2+cz^2+eyz+fzx+gxy, \tag{24}$$

and especially its central section by a plane parallel to the planes of the wave-system which we are considering, it will easily appear that the equation

$$\begin{aligned}2ax_1x_2+2by_1y_2+2cz_1z_2+e(y_1z_2+z_1y_2)&\\+f(z_1x_2+x_1z_2)+g(x_1y_2+y_1x_2)&=0\end{aligned}$$

will hold of any two points x_1, y_1, z_1 and x_2, y_2, z_2 which belong to conjugate diameters of this central section. Therefore equation (23) expresses that the displacements $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are parallel to conjugate diameters of the central section of the ellipsoid (24) by a wave-plane. But since the displacements $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are also at right angles to each other, it follows that they are parallel to the axes of the central section of the ellipsoid (24) by a wave-plane. That is:—

* The reader will perceive that an earlier limitation of the position of the origin by a supposition of this nature, involving a limitation of the values of $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2$, would have been embarrassing in the operations of the last paragraph.

The axes of the displacement-ellipse coincide in direction with those of a central section of the ellipsoid (24) by a wave-plane.

16. If we write U_1, U_2 for the reciprocals of the semi-axès of the central section of the ellipsoid (24) by a wave-plane, U_1 being the reciprocal of the one to which the displacement α_1, β_1, γ_1 is parallel, we have

$$a\alpha_1^2+b\beta_1^2+c\gamma_1^2+e\beta_1\gamma_1+f\gamma_1\alpha_1+g\alpha_1\beta_1=U_1^2(\alpha_1^2+\beta_1^2+\gamma_1^2), \quad (25)$$

as is at once evident, if we substitute the coördinates of an extremity of the axis, for the proportional quantities α_1, β_1, γ_1. So also

$$a\alpha_2^2+b\beta_2^2+c\gamma_2^2+e\beta_2\gamma_2+f\gamma_2\alpha_2+g\alpha_2\beta_2=U_2^2(\alpha_2^2+\beta_2^2+\gamma_2^2). \quad (26)$$

If we write V for the velocity of propagation of the system of progressive waves corresponding to the system of stationary waves which we have been considering, we shall have

$$V=\frac{l}{p}. \quad (27)$$

By equations (22), (25), and (26), equations (18) and (20) are reduced to the form

$$U_1^2\rho_1^2\pm\frac{\varphi}{V}\rho_1\rho_2=V^2\rho_1^2, \quad U_2^2\rho_2^2\pm\frac{\varphi}{V}\rho_1\rho_2=V^2\rho_2^2, \quad (28)$$

where we are to read + or − according as the disturbance has the character of a right-handed or a left-handed screw. In a progressive system of waves, when the combination of displacements has the character of a right-handed screw, the rotations will be such as appear clock-wise to the observer, who looks in the direction opposite to that of the propagation of light. We shall call such a ray *right-handed.*

We may here observe that in case $\varphi=0$ the solution of these equations is very simple. We have necessarily either $\rho_2=0$ and $V^2=U_1^2$, or $\rho_1=0$ and $V^2=U_2^2$. In this case, the light is linearly polarized, and the directions of oscillation and the velocities of propagation are given by Fresnel's law. Experiment has shown that this is the usual case. We wish, however, to investigate the case in which φ does not vanish. Since the term containing φ arises from the consideration of those quantities which it was allowable to neglect in the first approximation, we may assume that φ is always very small in comparison with V^3, U_1^3, or U_2^3.

17. Equations (28) may be written

$$V^2-U_1^2=\pm\frac{\varphi}{V}\frac{\rho_2}{\rho_1}, \quad V^2-U_2^2=\pm\frac{\varphi}{V}\frac{\rho_1}{\rho_2}. \quad (29)$$

By multiplication we obtain

$$V^2(V^2-U_1^2)(V^2-U_2^2)=\varphi^2. \tag{30}$$

Since φ is a very small quantity, it is evident from inspection of this equation that it will admit three values of V^2, of which one will be a very little greater than the greater of the two quantities U_1^2 and U_2^2, another will be a very little less than the less of the same two quantities, and the third will be a very small quantity. It is evident that the values of V^2 with which we have to do are those which differ but little from U_1^2 and U_2^2.*

For the numerical computation of V, when U_1, U_2, and φ are known numerically, we may divide the equation by V^2, and then solve it as if the second member were known. This will give

$$V^2=\frac{U_1^2+U_2^2}{2}\pm\sqrt{\frac{\varphi^2}{V^2}+\frac{(U_1^2-U_2^2)^2}{4}}. \tag{31}$$

By substituting U_1U_2 for V^2 in the second member, we may obtain a close approximation to the two values of V^2. Each of the values obtained may be improved by substitution of that value for V^2 in the second member of the equation.

For either value of V^2, we may easily find the ratio of ρ_1 to ρ_2, that is, the ratio of the axes of the displacement-ellipse, from one of equations (29), or from the equation

$$\frac{\rho_2^2}{\rho_1^2}=\frac{V^2-U_1^2}{V^2-U_2^2} \tag{32}$$

obtained by combining the two.

In equations (29), we are to read + or − in the second members, according as the ray is right-handed or left-handed. (See § 16.) It follows that if the value of φ is positive, the greater velocity will belong to a right-handed ray, and the smaller to a left-handed, but if the value of φ is negative, the opposite is the case. Except when $\varphi=0$, and the polarization is linear, there will be one right-handed and one left-handed ray for any given wave-normal and period.

18. When $U_1=U_2$, equations (29) give

$$\rho_1=\rho_2,\quad V^2=U^2\pm\frac{\varphi}{V},$$

* We should not attribute any physical significance to the third value of V^2. For this value would imply a wave-length very small in comparison with the length of ordinary waves of light, and with respect to which our fundamental assumption that the wave-length is very great in comparison with the distances of contiguous molecules would be entirely false. Our analysis, therefore, furnishes no reason for supposing that any such velocities are possible for the propagation of electrical disturbances.

where U represents the common value of U_1 and U_2. The polarization is therefore circular. The converse is also evident from equations (29), viz: that a ray can be circularly polarized only when the direction of its wave-normal is such that $U_1 = U_2$. Such a direction, which is determined by a circular section of the ellipsoid (24) precisely as an optic axis of a crystal which conforms to Fresnel's law of double refraction, may be called an optic axis, although its physical properties are not the same as in the more ordinary case.* If we write V_R and V_L, respectively, for the wave-velocities of the right-handed and left-handed rays, we have

$$V_R^2 = U^2 + \frac{\varphi}{V_R}, \quad V_L^2 = U^2 - \frac{\varphi}{V_L}; \tag{33}$$

whence

$$V_R^2 - V_L^2 = \varphi\left(\frac{1}{V_R} + \frac{1}{V_L}\right) = \varphi\frac{V_R + V_L}{V_R V_L},$$

and

$$V_R - V_L = \frac{\varphi}{V_R V_L}. \tag{34}$$

The phenomenon best observed with respect to an optic axis is the rotation of the plane of linearly polarized light. If we denote by θ the amount of this rotation per unit of the distance traversed by the wave-plane, regarding it as positive when it appears clock-wise to the observer, who looks in the direction opposite to that of the propagation of the light,† we have

$$\theta = \frac{\pi}{p}\left(\frac{1}{V_L} - \frac{1}{V_R}\right), \tag{35}$$

By the preceding equation, this reduces to

$$\theta = \frac{\pi\varphi}{p V_R^2 V_L^2}. \tag{36}$$

* Our experimental knowledge of circularly or elliptically polarizing media is confined to such as are optically either isotropic or uniaxial. The general theory of such media, embracing the case of two optic axes, has however been discussed by Professor von Lang. (*Theorie der Circularpolarization*, Sitz.-Ber. Wiener Akad., vol. lxxv, p. 719.) The general results of the present paper, although derived from physical hypotheses of an entirely different nature, are quite similar to those of the memoir cited. They would become identical, the writer believes, by the substitution of a constant for $\frac{p\phi}{l}$ or $\frac{\phi}{V}$ in the equations of this paper. [See especially equations (18), (20), (28).]

That a complete discussion of the subject on any theory must include the case of biaxial media having the property of circular or elliptical polarization, is evident from the consideration that it must at least be possible to produce examples of such media artificially. An isotropic or uniaxial crystal may be made biaxial by pressure. If it has the property of circular and elliptic polarization, that property cannot be wholly destroyed by the application of small pressures.

† When the rotation of the plane of polarization appears clock-wise to the observer, it has the character of a *left-handed* screw. But the circularly polarized ray to which V_R relates, the rotation of which also appears clock-wise to the observer, has the character of a *right-handed* screw.

Without any appreciable error, we may substitute U^4 for $V_R^2\,V_L^2$, which will give*

$$\theta = \frac{\pi\varphi}{pU^4}. \tag{37}$$

19. Since these equations involve unknown functions of the period, they will not serve for an exact determination of the relation between θ and the period. For a rough approximation, however, we may assume that the manner in which the general displacement in any small part of the medium distributes itself among the molecules and intermolecular spaces is independent of the period, being determined entirely by the values of ξ, η, ζ, and their differential coëfficients with respect to the coördinates.† For a fixed direction of the wave-normal, Φ and Φ' will then be constant. Now equations (15) and (36) give

$$\theta = \frac{\Phi}{2p^2 V_R^2 V_L^2} - \frac{2\pi^2\Phi'}{p^4 V_R^2 V_L^2} \tag{38}$$

To express this result in terms of the quantities directly observed, we may use the equations

$$p = \frac{\lambda}{k}, \quad V_R = \frac{k}{n_R}, \quad V_L = \frac{k}{n_L}, \quad U = \frac{k}{n},$$

where k denotes the velocity of light *in vacuo*, λ the wave-length *in vacuo* of the light employed, n_R, n_L the absolute indices of refraction of the two rays, and n the index for the optic axis as derived from the ellipsoid (24) by Fresnel's law. We thus obtain

$$\theta = \frac{\Phi n_R^2 n_L^2}{2k^2\lambda^2} - \frac{2\pi^2\Phi' n_R^2 n_L^2}{\lambda^4} \tag{39}$$

In the case of uniaxial crystals, the direction of the optic axis is fixed. We may therefore write

$$\theta = n_R^2 n_L^2 \left(\frac{K}{\lambda^2} + \frac{K'}{\lambda^4}\right), \tag{40}$$

regarding K and K′ as constants. If we had used equation (37), we should have had the factor n^4 instead of $n_R^2 n_L^2$.

* The degree of accuracy of this substitution may be shown as follows. By (33)

$$V_R(V_R^2 - U^2) = V_L(U^2 - V_L^2),$$

whence

$$V_R^3 + V_L^3 = (V_R + V_L)U^2,$$
$$V_R^2 - V_R V_L + V_L^2 = U^2,$$
$$V_R V_L = U^2 - (V_R - V_L)^2.$$

† Compare § 12 of the former paper, on page 270 of this volume.

Since this factor varies but slowly with λ, it may be neglected, if its omission is compensated in the values of K and K′. The formula being only approximative, such a simplification will not necessarily render it less accurate.

20. But without any such assumption as that contained in the last paragraph, we may easily obtain formulæ for the experimental determination of Φ and Φ' for the optic axis of an uniaxial crystal. Considerations analogous to those of § 13 of the former paper (page 271 of this volume), show that in differentiating equation (39) we may regard Φ and Φ' as constant, although they may actually vary with λ. This equation may be written

$$\frac{\theta\lambda^2}{n^4}=\frac{\Phi}{2k^2}-\frac{2\pi^2\Phi'}{\lambda^2} \qquad (41)$$

Therefore,

$$\frac{d\left(\frac{\theta\lambda^2}{n^4}\right)}{d\left(\frac{1}{\lambda^2}\right)}=-2\pi^2\Phi'. \qquad (42)$$

When Φ' has been determined by this equation, Φ may be found from the preceding.

21. If we wish to represent φ geometrically, like U_1 and U_2, we may construct the surfaces

$$\text{A}x^2+\text{B}y^2+\text{C}z^2+\text{E}yz+\text{F}zx+\text{G}xy=\pm 1, \qquad (43)$$

the coëfficients A, B, etc., being the same by which φ is expressed in terms of L^2, M^2, etc. The numerical value of φ, for any direction of the wave-normal, will thus be represented by the square of the reciprocal of the radius vector of the surface drawn in the same direction. The positive or negative character of φ must be separately indicated. There are here two cases to be distinguished. If the sign of φ is the same in all directions, the surface will be an ellipse, and we have only to know whether all the values of φ are to be taken positively or all negatively. But if φ is positive for some directions and negative for others, the surface will consist of two conjugate hyperboloids, to one of which the positive, and to the other the negative values belong.

22. The manner in which the ellipsoid (24) may be partially determined by the relations of symmetry which the medium may possess, has been sufficiently discussed in the former paper.

With respect to the quantity φ, and the surfaces which determine it, the following principle is of fundamental importance. If one body is identical in its internal structure with the image by reflection of another, the values of φ in corresponding lines in the two bodies will be numerically equal but have opposite signs.*

It follows that if a body is identical in internal structure with its own image by reflection, the value of φ (if not zero for all directions) must be positive for some directions and negative for others. Moreover, the above described surface by which φ is represented must consist of two conjugate hyperboloids, of which one is identical in form with the image by reflection of the other. This requires that the hyperboloids shall be right cylinders with conjugate rectangular hyperbolas for bases. A crystal characterized by such properties will belong to the tetragonal system. Since $\varphi=0$ for the optic axis, it would be difficult to distinguish a case of this kind from an ordinary uniaxial crystal, unless the ellipsoid (24) should approach very closely to a sphere.†

It is only in the very limited case described in the last paragraph that a medium which is identical in its internal structure with its image by reflection can have the property of circular or elliptic polarization. To media which are unlike their images by reflection, and have the property of circular polarization, we may apply the following general principles.

If the medium has any axis of symmetry, the ellipsoid or hyperboloids which represent the values of φ will have an axis in the same direction. If the medium after a revolution of less than 180° about any axis is equivalent to the medium in its first position, the ellipsoid or hyperboloids will have an axis of revolution in that direction.

23. The laws of the propagation of light in plane waves, which have thus been derived from the single hypothesis that the disturbance by which light is transmitted consists of solenoidal electrical fluxes, and which apply to light of different colors and to the most general case of perfectly transparent and sensibly homogeneous media not subject to magnetic action,‡ are essentially those which are generally received as

* The necessity of the opposite signs will perhaps appear most readily from the consideration that the direction of rotation of the plane of polarization must be opposite in the two bodies.

† There is no difficulty in conceiving of the constitution of a body which would have the properties described above. Thus, we may imagine a body with molecules of a spiral form, of which one-half are right-handed and one-half left-handed, and we may suppose that the motion of electricity is opposed by a less resistance within them than without. If the axes of the right-handed molecules are parallel to the axis of X, and those of the left-handed molecules to the axis of Y, their effects would counterbalance one another when the wave-normal is parallel to the axis of Z. But when the wave-normal (of a beam of linearly polarized light). is parallel to the axis of X, the left-handed molecules would produce a left-handed (negative) rotation of the plane of polarization, the right-handed molecules having no effect; and when the wave-normal is parallel to the axis of Y, the reverse would be the case.

‡ The rotation of the plane of polarization which is produced by magnetic action has been discussed by Maxwell (*Treatise on Electricity and Magnetism*, vol. ii, Chap. XXI), and by Rowland (Amer. Journ. Math., vol. iii, p. 107).

embodying the results of experiment. In no particular, so far as the writer is aware, do they conflict with the results of experiment, or require the aid of auxiliary and forced hypotheses to bring them into harmony therewith.

In this respect, the electromagnetic theory of light stands in marked contrast with that theory in which the properties of an elastic solid are attributed to the ether,—a contrast which was very distinct in Maxwell's derivation of Fresnel's laws from electrical principles, but becomes more striking as we follow the subject farther into its details, and take account of the want of absolute homogeneity in the medium, so as to embrace the phenomena of the dispersion of colors and circular and elliptical polarization.