SECT. II.—OTHER SELECTED PAPERS.

(Paper No. 4107.)

“Influence Diagrams for an Arch-Rib hinged at the Ends only.”

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THE arch in steel and in reinforced concrete, for both road- and railway-bridges in certain situations, and especially for moderately large spans, is not without favour with engineers at the present time. Three-hinged arches, two-hinged arches, and arches without hinges have been built. The three-hinged arch has a distinct advantage over the other two forms in that the stresses in the various parts of the arch are determinable easily and rigidly for any system of loading, but the possible lack of rigidity has prevented engineers from largely adopting it, and the two-hinged arch and the arch without hinges have been used to a greater extent. Before, however, the stresses in either of these two forms can be found, assumptions have to be made which depend upon the nature of the supports from which the arch springs. For two-hinged arches only those assumptions are necessary as appear quite justifiable.

It is not the aim of this Paper to discuss hingeless arches, but the Author is probably justified in saying that the stresses cannot be determined with the same confidence as in the two-hinged arch; and it is doubtful whether it is good engineering practice to adopt forms of structure in which the stresses are not determinable with some considerable degree of confidence. It is proposed therefore to deal with two-hinged arches only.

GENERAL EQUATIONS.

The determination of the shearing force and bending moment at any section of an arch-rib hinged at the ends only, especially when subjected to travelling loads, is not without difficulty and is somewhat tedious. The Author has shown in previous contributions to the Proceedings¹ that for single-span girders and for con-

¹ Minutes of Proceedings Inst. C.E., vol. clxi, p. 261, and vol. clxxxv, p. 277.

tinuous girders the method of influence diagrams makes it possible to determine the shearing and bending effects of travelling loads quickly and accurately. In the case of single-span girders it has been shown that the influence diagrams consist of straight lines; and similarly for two-span continuous girders it was shown that if a curved base is used for these diagrams the diagrams for any sections can be completed by straight lines. It is not difficult to show also that for three-hinged arch-ribs the influence diagrams for any sections consist of straight lines only. As for the continuous girder, so for the two-hinged arch-rib the influence diagrams when drawn on a particular curved base also consist of straight lines, and these diagrams can be used to determine shearing forces and bending moments, due to travelling or fixed loads, at the respective sections of an arch-rib.

Consider a curved beam AB , in Fig. 1, Plate 4.

Let it be assumed that a bending moment M is acting about the axis C of the section ae and that the portion AC of the beam is attached to the element $abce$. Due to the moment M a small element of length δs will turn through an angle $\delta\phi$, and due to the turning of this element the part of AC rigidly connected with it will turn about the axis C , and the point A will move a distance AF equal to $AC\delta\phi$, $\delta\phi$ being the small angle turned through by CA about the axis C . The angle $\delta\phi$ is very small and, therefore, the angle CAF is practically a right angle.

Let rectangular axis be taken through A , and let δx and δy be the co-ordinates of the point F to which A moves about the axis C ,

$$\text{then} \quad \frac{\delta y}{AF} = \frac{x}{AC}$$

$$\text{and} \quad \frac{\delta x}{AF} = \frac{y}{AC};$$

$$\text{therefore} \quad \delta y = x \delta\phi$$

$$\text{and} \quad \delta x = y \delta\phi.$$

At the section of any beam subjected to a bending moment M the angle turned through by an element of length at the section is

$$\delta\phi = \frac{M \delta s}{EI}.$$

E is the modulus of elasticity and I is the second moment (moment of inertia) of the section.

$$\text{Therefore} \quad \delta y = \frac{x M \delta s}{EI} \quad \text{and} \quad \delta x = \frac{y M \delta s}{EI}.$$

Integrating over the length A B the vertical movement of A relative to B is

$$\int \delta y = \delta_v = \int_A^B \frac{x M \delta s}{E I} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$\int \delta x = \delta_H = \int_A^B \frac{y M \delta s}{E I} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This result can be applied to determine the bending moment acting in an arch-rib hinged at the ends only, and to determine the influence diagram for any section of the rib.

Let A C B, Fig. 2, Plate 4, be an arch-rib hinged at the springings A and B only. When the rib is loaded deformation of the rib will take place, first due to the bending moments acting on the rib, secondly, due to the thrust in the arch-rib, and thirdly, due to shear; but the springings A and B can be made with practically any desirable degree of rigidity. If the assumption is made that they are perfectly rigid, then whatever deformation the rib undergoes the span cannot alter, and if it is assumed that the deformation due to the shear and thrust is negligible compared with that due to the bending moments, then equation (2) can be applied to this case, and integrating along the whole rib A B

$$\int_A^B \frac{M y \delta s}{E I} = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On the assumption thus made this result is true for any form of arch-rib.

If now the arch-rib is supposed to have a constant depth, and to be widened out from the centre towards the springings A and B in such a way that the width of the rib in plan at any point G, having co-ordinates y and z , is proportional to the secant of the angle of inclination of the tangent at G, then

$$ds = dz \sec \theta.$$

z now being the horizontal distance of the point G from A. Then if I_c is the moment of inertia of the section of the rib at the centre C the moment of inertia at the point G will be $I = I_c \sec \theta$.

Therefore

$$\int_A^B \frac{M y \delta s}{E I} = \int_A^B \frac{M y \delta z}{E I_c},$$

or, since I_c is a constant, the condition for no change of span is that

$$\int_A^B M y \delta z = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now let the centre line A C B of the arch-rib be assumed to be a parabola of span L and rise D.

The equation to the parabola is

$$\left(\frac{L}{2} - z\right)^2 = k(D - y).$$

At the point A $\left(\frac{L}{2}\right)^2 = kD,$

therefore $\left(\frac{L}{2} - z\right)^2 = \frac{L^2}{4D}(D - y),$

from which $y = \frac{4Dz(L - z)}{L^2} \dots \dots \dots (5)$

Substituting this value of y in equation (4) the equation for the parabolic rib becomes

$$\frac{4D}{L^2} \int_A^B (L - z) z dz = 0 \dots \dots \dots (6)$$

Now suppose any load W travelling over a platform resting on the arch-rib, and suppose the load to rest directly on the rib at any point F at a distance x from the centre C . Then the vertical reactions at A and B are

$$\frac{W}{L} \left(\frac{L}{2} - x\right) \text{ and } \frac{W}{L} \left(\frac{L}{2} + x\right) \text{ respectively.}$$

Throughout the rib the horizontal component of the thrust must be constant, therefore the horizontal reactions H at A and B are equal. For any section between A and F , and at a distance z from the vertical through A , the bending moment when the load W is at F will be—

$$Vz - Hy.$$

For a section between F and B the bending moment is—

$$Vz - Hy - W \left(z - \frac{L}{2} - x\right);$$

then substituting in equation (4)

$$\int_0^{\frac{L}{2}+x} (Vzy - Hy^2) dz + \int_{\frac{L}{2}+x}^L \left\{ Vzydz - Hy^2dz - yW \left(z - \frac{L}{2} - x\right) dz \right\} = 0$$

$$\text{or} \quad \int_0^L (V z y - H y^2) dz + \int_{\frac{L}{2}+x}^L y W \left(z - \frac{L}{2} - x \right) dz = 0,$$

substituting for y in terms of z

$$\begin{aligned} & \int_0^L \frac{4 V D z^2}{L^2} (L - z) dz - \int_0^L \frac{16 H D^2 z^2 (L - z)}{L^4} dz \\ & - \int_{\frac{L}{2}+x}^L \frac{4 D W z (L - z) \left(z - \frac{L}{2} - x \right) dz}{L^2} = 0, \end{aligned}$$

from which, taking W as unity,

$$\frac{V D L^2}{3} - \frac{8 H D^2 L}{15} = \frac{4 D}{L^2} \left(\frac{L^4}{64} - \frac{x L^3}{12} + \frac{x^2 L^2}{8} - \frac{x^4}{12} \right).$$

Substituting for V its value

$$\frac{\frac{L}{2} - x}{L},$$

$$\frac{8}{15} H D^2 L = \frac{4 D}{L^2} \left(\frac{x L^3}{12} + \frac{x^4}{12} - \frac{L^4}{64} - \frac{x^2 L^2}{8} \right) + \frac{D L}{3} \left(\frac{L}{2} - x \right),$$

$$\text{or} \quad \frac{8}{15} H D L^3 = \frac{5}{48} L^4 + \frac{1}{3} x^4 - \frac{1}{2} x^2 L^2,$$

$$\text{and} \quad H = \frac{\left(\frac{25}{4} L^2 - 5 x^2 \right) \left(\frac{L^2}{4} - x^2 \right)}{8 D L^3} \quad \dots \dots (7)$$

In determining this equation W was assumed to be fixed at some point to the right of the centre. If W is to the left of the centre, and x is defined as its distance from the centre, and the integration be made from B to A , an exactly similar expression is obtained for H , and therefore equation (7) is an equation to a curve which will give the value of H for any corresponding position of the unit load on the span. That is, this curve may be said to be the influence diagram for the horizontal thrust of the arch. On a base AB ,

Fig. 3, Plate 4, let such a curve be plotted, then for any position of unit load on the span

$$H = \text{ordinate of curve A D B,}$$

and for any load W

$$H = W \text{ (ordinate of curve under } W \text{);}$$

if there are any number of loads on the span

$$H = \Sigma W \text{ (ordinate of curve).}$$

The central ordinate of the curve is $\frac{25 L}{128 D}$

when $x = \frac{L}{4}$

$$\text{the ordinate} = \frac{285 L}{2048 D},$$

when $x = \frac{L}{2}$

the ordinate = 0.

Bending Moment at any Section of the Arch-Rib.—When unit load is to the right of the centre the bending moment at any section through G due to the unit load is

$$\begin{aligned} M_G &= V z - H y \\ &= \frac{1}{L} \left(\frac{L}{2} - x \right) z - y \text{ (ordinate of curve under the load).} \end{aligned}$$

The first part of the expression is an equation to a straight line, while the second is y times the ordinate of the H influence diagram.

Now set up at A an ordinate A E equal to $\frac{z}{y}$ and join B E. Then, as long as the load is to the right of the centre

$$H P = \frac{\frac{L}{2} - x}{L} \frac{z}{y}.$$

Therefore $\left(\frac{M_G}{y} \right) = H P - \text{ordinate P K}$

= ordinate between the line and the curve.

When the unit load is to the left of the centre

$$V = \frac{1}{L} \left(\frac{L}{2} + x \right)$$

and $\frac{Vz}{y} = \frac{1}{L} \left(\frac{L}{2} + x \right) \frac{z}{y}$

= ordinate of straight line B E.

So that, for any position of unit load between B and M on the span, $\frac{M_G}{y}$ = ordinate between the straight line B N and the curve A D B.

When any load is to the left of M and at a distance x from the centre the bending moment is

$$M_G = Vz - Hy - W \left(z - \frac{L}{2} + x \right)$$

$$\frac{M_G}{y} = \frac{Vz}{y} - \frac{Wy}{y} \left(z - \frac{L}{2} + x \right) - H.$$

Erect a perpendicular through M, Fig. 3, to meet the line BE in N, and join AN, then for any position of unit load W to the left of M, or G in Fig 2,

$$\frac{Vz}{y} - \frac{z - \frac{L}{2} + x}{y} = \text{the ordinates of the line AN,}$$

or $\frac{M_G}{y} = \text{the ordinate RT.}$

Then for any system of concentrated loads $\frac{M_G}{y} = \Sigma W$ (ordinates between the curved base A D B (which is constant for all points on the arch) and the influence line A N B), or A N B K D A is the influence diagram of bending moment for the section of the arch-rib through G.

It will be seen that the ordinates of the curve are undimensional and also those of the line. In setting out the number of units, the same scale must be used as for setting out the ordinates of the curve. For example, if $\frac{z}{y}$ is 1.75 and $\frac{L}{D}$ is 5, then A E is 1.75 on any unit and the central ordinate of the curve is 0.975 on the same scale.

All that need be done, therefore, to determine the bending moment at G for any system of loads is to multiply each load by the corresponding ordinate of the diagram and then multiply by the ordinate y in feet. For any uniformly distributed load $p q$ of w units per foot run

$$M_G = y w (r N s t u);$$

it should be again noted that since the ordinates are undimensional the area $r N s t u$ is really in square feet and equals—

(The area of $r N s t u$ in square inches) $\times$ (number of feet per inch in the scale of the span) $\times$ (number of units per inch to which the ordinates are set out).

For the case shown in Fig. 3, M_G will have its maximum positive value (counter-clockwise) due to a uniformly distributed load when the uniformly distributed load covers the portion B J of the span and its maximum negative value when the load covers A J. The bottom boom at the given section will be in compression in the first case and in tension in the second.

TWO-HINGED PARABOLIC ARCH-RIB HINGED AT THE ENDS ONLY.

Shear, and Diagonal Members: Booms.—To find the force in the diagonal of an arch for any loading, moments have to be taken about the points of intersection of the two booms which are cut by a section also cutting the diagonal member. This point, when the booms are parallel, is at infinity; if it is at a finite distance the moment can be found provided H and V are determinable. In the case under consideration the value of H has been determined on the assumption that the moment resisted by any normal section of the rib can be expressed as a continuous function, whereas the force in E J, Fig. 4, Plate 4, is found by taking moments about G, the force in F G by taking moments about E, and the force in E G by taking moments about some point Q, the intersection of E J and F G. The deflection of the point A cannot then be found by the integration of a continuous function. The only practical case consistent with the determination of H from equation (7) is to assume the booms are parallel, and that the dimensions of the bays are small compared with the span; in which case to find the force in both E J and F G moments can be taken about the point O on the centre line of the rib without considerable error.

Consider first the simple case when there are any number of loads, W_1, W_2 , etc., on the part of the arch to the right of the bay, and suppose H and V are the horizontal and vertical reactions

respectively at the springing A. Then the only forces acting on the part of the arch A J are H and V, and these can be replaced by forces H and V acting at O, together with a couple equal to $Vz - Hy$ acting about O.

Let the booms E J and F G be assumed parallel and inclined at an angle θ to the horizontal, and let the diagonal E G be inclined at any angle ϕ to the horizon.

Then, resolving H and V parallel to and perpendicular to F G, there is a direct thrust in a direction parallel to F G equal to $H \cos \theta + V \sin \theta$, and a component perpendicular to F G equal to $V \cos \theta - H \sin \theta$. The component, $V \cos \theta - H \sin \theta$ perpendicular to F G, is balanced by the shear in a continuous web or by the component of the force in the diagonal perpendicular to F G.

The force F in the diagonal is—

$$F = \frac{V \cos \theta - H \sin \theta}{\sin (\theta + \phi)}.$$

The normal thrust on the booms due to the components of H and V parallel to them is $H \cos \theta + V \sin \theta$, and this causes a stress equal to $\frac{H \cos \theta + V \sin \theta}{\text{Area of booms}}$; the bending moment $Vz - Hy$ also causes stresses on the booms which can be found when the moment of inertia of the section is known.

The values of $\sin \theta$ and $\sin (\theta + \phi)$ can be easily found by measuring the angles from the drawing, or can be found from the equation—

$$\sin \theta = \frac{4 D (L - 2z)}{\sqrt{L^4 + 16 D^2 (L - 2z)^2}};$$

$$\text{or} \quad \cot \theta = \frac{L^2}{4 D (L - 2z)}.$$

Now assume there is a load W to the left of the bay at a distance x from the centre of the arch; then the forces acting at O are H and $V - W$, and the couple is $Vz - Hy - W(z - \frac{L}{2} + x)$.

The force in the diagonal is then—

$$\frac{-H(\sin \theta) + (V - W) \cos \theta}{\sin (\theta + \phi)} = \frac{\sin \theta [-H + (V - W) \cot \theta]}{\sin (\theta + \phi)}. \quad (8)$$

and the stress due to the direct thrust on the boom is—

$$\frac{H \cos \phi + (V - W) \sin \theta}{\text{Area of the booms}} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Influence Diagrams.—To obtain the shear or the force in any diagonal EG proceed as follows:—On the base AB, Fig. 5, draw the curve ADB from equation (7), p. 351. At A erect a perpendicular AK equal to $\cot \theta$, and join BK. Draw AS parallel to BK, meeting the vertical through M in S. Then ASRBD A is the influence diagram for shear or for the force in EG. The proof is not difficult. The ordinate of the curve ADB gives the value of H for unit load in any position; the ordinate of BK gives the value of $V \cot \theta$ for unit load in any position; and the ordinate of AS gives the value of $(V - W) \cot \theta$ for unit load when the load is in any position between A and M.

For any system of loads the shear $S = \sin \theta \Sigma W.NP$.

$$F = \frac{\sin \theta}{\sin (\theta + \phi)} \Sigma W.NP.$$

When NP is measured downwards $W.NP$ produces a compressive stress, and when measured upwards a tensile stress.

When θ is small, as will be the case near the centre of the arch, $H \sin \theta$ is negligible, $\cos \theta$ is unity, and the influence diagram for the diagonal is then exactly the same as for an ordinary parallel-boomed girder.¹ The meaning of this on the diagram is that $\cot \theta$ becomes infinite and the ordinate of the H curve is negligible. The diagram is then drawn at two parallel straight lines AS and BR, which meet on a vertical through the centre. The length SR is unity. For the case shown in Fig. 5, Plate 4, when the travelling load is on BE or AM only, the diagonal will be in compression, when on ME only it will be in tension.

For a diagonal near the abutment AR, BDA is approximately the influence diagram. In this case the variation of stress in the diagonal as a uniformly distributed load passes over the arch will be considerable.

Example of a Parabolic Arch-Rib Hinged at the Ends only.

Let $L = 500$ feet.

$D = 100$ „

Consider a point for which the abscissa z is 150 feet.

$$\begin{aligned} \text{Then } y &= \frac{4 \times 100 \times 150 \times 350}{500 \times 500} \\ &= 84 \text{ feet.} \end{aligned}$$

¹ See Minutes of Proceedings Inst. C.E., vol. clxi, p. 261.

$$\begin{aligned}\text{Cot } \theta &= \frac{L^2}{4 D (L - 2 z)} \\ &= \frac{500 \times 500}{4 \times 100 \times 200} \\ &= 3.125.\end{aligned}$$

The central ordinate for the influence diagram for **H** is—

$$\frac{25}{128} \cdot \frac{L}{D} = 0.975.$$

The ordinate at quarter span is—

$$\frac{285}{2,048} \times 5 = 0.695.$$

When z is 50—

$$\text{Cot } \theta = \frac{500 \times 500}{4 \times 100 \times 400} = 1.56.$$

The influence diagrams for the diagonals at $z = 150$ and $z = 50$ are shown in Fig. 6, Plate 4, as **ASRBDA**, and **AS'R'BDA**, respectively. For a uniformly distributed load of 1 ton per foot run the maximum tensile force in the diagonal when z is 150 is—

$$F = \frac{\sin \theta}{\sin (\theta + \phi)} \times \text{area } \mathbf{ERB}, (\theta = 25^\circ 36'),$$

and the maximum compression force is **ASE**, nearly.

The scales of the original diagram were 1 inch to 1 ton, and 1 centimetre equals 40 feet, or 1 square inch equals 101.6 tons per unit load per foot run. The tensile force for a uniformly distributed load of 1 ton per foot run equals—

$$\frac{\sin \theta}{\sin (\theta + \phi)} \times 127 \text{ tons.}$$

The maximum compression force is—

$$\frac{\sin \theta}{\sin (\theta + \phi)} \times 138 \text{ tons.}$$

The areas **ERB** and **ASE** were obtained by the planimeter.

Resultant Stress in the Booms at any Section of a Parabolic Arch-Rib.—The resultant stress p_0 in the booms of the arch-rib is that due to the normal thrust¹ in the section together with the moment resisted by the section, or

$$p_0 = \frac{N}{\text{area of section}} \pm \frac{M \cdot \frac{d}{2}}{I} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

d being the depth of the rib, and I the moment of inertia of the section.

Let $2A$ be the combined area of the booms.

Then the stress in the lower boom, Fig. 3, Plate 4, is

$$p = \frac{H \cos \theta + V \sin \theta}{2A} + \frac{M d}{2I} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and in the upper booms is

$$p_1 = \frac{H \cos \theta + V \sin \theta}{2A} - \frac{M d}{2I} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

When the load W is to the right of the section O which is being considered

$$p = \frac{H \cos \theta + V \sin \theta}{2A} + \frac{d(Hy - Vz)}{2I}$$

from which

$$\frac{2pAI}{A dy + I \cos \theta} = H - \frac{V(zdA - I \sin \theta)}{A dy + I \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and when the load is to the left of O and at a distance x from the centre

$$p = \frac{H \cos \theta + (V - W) \sin \theta}{2A} + \frac{d}{2I} \left\{ Hy - Vz + W \left(z - \frac{L}{2} + x \right) \right\}$$

and

$$\frac{2pAI}{A dy + I \cos \theta} = H - \frac{V(zdA - I \sin \theta)}{A dy + I \cos \theta} + \frac{W \left\{ dA \left(z - \frac{L}{2} + x \right) - I \sin \theta \right\}}{A dy + I \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

¹ Although the change of span due to normal thrust is negligible, it does not follow that stress due to it is negligible compared with the stress due to bending.

Similarly, when W is to the right of O

$$p_1 = \frac{H \cos \theta + V \sin \theta}{2 A} + \frac{d (H y - V z)}{2 I}$$

from which

$$\frac{2 p_1 I A}{A d y - I \cos \theta} = -H + \frac{V (z A d + I \sin \theta)}{A d y - I \cos \theta}.$$

When W is to the left of O

$$p_1 = \frac{H \cos \theta + V \sin \theta - W \sin \theta}{2 A} - \frac{d}{2 I} \left\{ H y - V z + W \left(z - \frac{L}{2} + x \right) \right\}$$

from which

$$\frac{2 p_1 A I}{A d y - I \cos \theta} = -H + \frac{V (z A d + I \sin \theta)}{A d y - I \cos \theta} - \frac{W \left\{ A d \left(z - \frac{L}{2} + x \right) + I \sin \theta \right\}}{A d y - I \cos \theta}$$

It will be seen in the sequence how an influence diagram can be drawn from which the value of the right-hand part of equations (13) and (14) can be directly determined for any system of loading. A simplification is, however, here introduced, and the influence diagram for the simplified case first drawn.

Let it be assumed that the areas of the booms are not only equal, but that also, as is generally assumed for ordinary plate girders, the flanges take the bending moment.

$$\text{Then} \quad I = \frac{A d^2}{2}.$$

Equation (13) then becomes

$$\frac{A d p}{y + \frac{d}{2} \cos \theta} = H - \frac{V \left(z - \frac{d}{2} \sin \theta \right)}{y + \frac{d}{2} \cos \theta} \quad \dots \quad (15)$$

and equation (14)

$$\frac{A d p}{y + \frac{d}{2} \cos \theta} = H - \frac{V \left(z - \frac{d}{2} \sin \theta \right)}{y + \frac{d}{2} \cos \theta} + \frac{W \left(z - \frac{L}{2} + x \right) - W \left(\frac{d}{2} \sin \theta \right)}{y + \frac{d}{2} \cos \theta} \quad \dots \quad (16)$$

The stress in the upper boom when W is to the right of O is

$$p_1 = \frac{H \cos \theta + V \sin \theta}{2 A} - \frac{(H y - V z) d}{2 I}$$

and when W is to the left of O is

$$p_1 = \frac{H \cos \theta + (V - W) \sin \theta}{2 A} - \frac{d \left\{ H y - V z + W \left(z - \frac{L}{2} + x \right) \right\}}{2 I}$$

Substituting for I its approximate value $\frac{A d^3}{2}$

$$\frac{A d p_1}{y - \frac{d}{2} \cos \theta} = -H + \frac{V \left(z + \frac{d}{2} \sin \theta \right)}{y - \frac{d}{2} \cos \theta} \quad \dots (17)$$

when W is to the right of O , and

$$\frac{A d p_1}{y - \frac{d}{2} \cos \theta} = H + \frac{V \left(z + \frac{d}{2} \sin \theta \right)}{y - \frac{d}{2} \cos \theta} - \frac{W \left\{ \left(z - \frac{L}{2} + x \right) + \frac{d}{2} \sin \theta \right\}}{y - \frac{d}{2} \cos \theta} \quad \dots (18)$$

when W is to the left of the section.

Influence Diagrams for the Stress in the Top and Bottom Booms of a Parabolic Arch-Rib.—Draw $A D B$, the influence diagram for H ,

Fig. 7, Plate 4, the ordinate at the centre being $\frac{25 L}{128 D}$.

At A set up the ordinate

$$A T = \frac{z - \frac{d}{2} \sin \theta}{y + \frac{d}{2} \cos \theta}$$

and

$$A K = \frac{z}{y + \frac{d}{2} \cos \theta}.$$

Join $B T$ and draw $K R$ parallel to it, to meet the vertical through O in R . Join $A R$. Then for any load W to the right of O ,

$$H = W.gf$$

$$V = \frac{W.Bg}{A B} = \frac{W.ge}{A T} \text{ or } W.ge = V.A T.$$

and therefore

$$\frac{A d p}{y + \frac{d}{2} \cos \theta} = \Sigma W.ef.$$

for any load to the left of O,

$$H = W.g_1f_1,$$

and

$$V.A.T = W.g.c.$$

From the figure it can be seen that

$$Cd = SR = \frac{\frac{d}{2} \sin \theta}{y + \frac{d}{2} \sin \theta}$$

and

$$\frac{e_1d}{A.K} = \frac{z - \frac{L}{2} + x}{z}$$

and therefore

$$e_1d = \frac{z - \frac{L}{2} + x}{y + \frac{d}{2} \cos \theta}.$$

Also

$$e_1f_1 = g_1c + cd - e_1d - g_1f_1;$$

then for any load to the left

$$\begin{aligned} Wc_1f_1 &= -(g_1f_1 - g_1c + e_1d - cd) \\ &= -(H - VAK + \frac{W(z - \frac{L}{2} + x)}{y + \frac{d}{2} \cos \theta} - \frac{W\frac{d}{2} \sin \theta}{y + \frac{d}{2} \cos \theta}) \end{aligned}$$

or, if e_1f_1 be called negative on the diagram,

$$Wc_1f_1 \text{ is equal to } \frac{Adp}{y + \frac{d}{2} \cos \theta}.$$

And since Ad and $y + \frac{d}{2} \cos \theta$ are constant the value of p can at once be found by summing up all such quantities as $W.fe$. The diagram $ARS ebfNA$ is then the influence diagram for the stress p in the lower flange. The curved base is constant for all points of the arch, and the other three lines AR , Rs and SB determine the diagram. Near the centre of the arch $\frac{d}{2} \sin \theta$ and $\frac{d}{2} \cos \theta$ will be

negligible compared with z and y , and the influence diagram is then as in Fig. 3. Similarly from equations (17) and (18) it can be seen that Fig. 8, Plate 4, is the influence diagram for the stress in the upper boom. There would be little error in any case in neglecting $\frac{d}{2} \sin \theta$ and $\frac{d}{2} \cos \theta$. If the original form of the equations for the stresses in booms, namely, equations (13) and (14), are retained the influence diagrams are similar to Figs. 7 and 8, except that in Fig. 7

$$A K \text{ becomes } \frac{z.d A}{A.d y + I \cos \theta}$$

$$\text{and } A T \quad ,, \quad \frac{z d A - I \sin \theta}{A d y + I \cos \theta};$$

and in Fig. 8

$$A K \text{ becomes } \frac{z d A}{A d y - I \cos \theta}$$

$$\text{and } A T \quad ,, \quad \frac{z d A + I \sin \theta}{A d y - I \cos \theta}'$$

while the stress in the lower flange is then

$$p = \frac{A y d + I \cos \theta}{2 A I} \Sigma W.ef. = k \Sigma W.ef.$$

where k is a constant for any given section, but varies with the section.

The stress in the upper flange is

$$p_1 = \frac{A y d - I \cos \theta}{2 A I} \Sigma W.ef.$$

In designing the method of procedure must be quite tentative. The best way to proceed probably when the span and rise are fixed is to assign values to d and A at the centre of the arch such that

$$\frac{A d p}{y} = \Sigma W.fe.$$

(Fig. 11), where p is an allowable stress. The $\Sigma W.ef$ will of course include distributed loads as well as concentrated loads. If

the latter are travelling loads it will not be difficult to see where they should be for maximum moment.

The assumption made in determining H was, it will be remembered, that

$$I = I_c \sec \theta.$$

If then $A_1 d_1^2 = A d^2 \sec \theta$, where d_1 and A_1 are the depth of the rib and the area of the booms respectively at any other section, and if then

$$\frac{A_1 d_1 p}{y + \frac{d}{2} \cos \theta} = \Sigma W.c.f,$$

or neglecting $\frac{d}{2} \cos \theta$

$$\frac{A_1 d_1 p}{y} = \Sigma W.f.c$$

gives reasonable values of p , the booms may be said to be satisfactory from the point of view of stress allowable.

THE BENDING-MOMENT DIAGRAM FOR AN ARCH-RIB WHEN ANY NUMBER OF LOADS ARE RESTING ON THE ARCH.

The influence diagram $A D B$ for H is not only useful as acting as the base for the influence diagrams of bending moment, shearing force, and so on, for any section of the arch, but by means of it the bending-moment diagram for any system of loads can be easily found.

This new method of obtaining the bending-moment diagram is very simple and involves a very small amount of labour.

Let $G C J$, Fig. 10, Plate 4, be a parabolic arch-rib hinged at the ends only, and let it be supposed that on the rib there are any number of loads, as shown on the figure.

Draw the curve $A D B$ on the span AB to represent the influence diagram for H (equation 7).

The arch in the figure has a span of 500 feet and a rise of 100 feet. The central ordinate for $A D B$ is then $\frac{25 \times 500}{128 \times 100} = 0.975$.

Scale off the ordinates $g k$ under each of the loads and tabulate. Also make $A T$ equal to unity on any scale and join BT . Under each load scale off the ordinates $g h$ and tabulate as in Table I (p. 364).

TABLE I.

Load.	Ordinate gk .	Product Load by gk .	Ordinate gh .	Product Load by gh .
10	0.375	3.75	0.87	8.70
15	0.75	11.25	0.72	10.80
15	0.96	14.40	0.555	8.33
20	0.93	18.60	0.400	8.00
10	0.675	6.75	0.240	2.40
10	0.275	2.75	0.080	0.80
Totals $H = 57.50$			Reaction V at $A = 39.03$	

Reaction at $B = W - 39.03 = 40.97$ tons.

Set out a line of loads LL_1 , making LQ equal to the reaction V , i.e. 39.03 tons. Draw the horizontal QP , making QP equal to the reaction H (57.5 tons). Join PL and PL_1 . Through J draw a line parallel to PL_1 to meet the load of 10 tons in 6, and through C a line parallel to PL , and complete the funicular polygon $C123456J$, by drawing lines parallel to the respective lines on the load diagram $PLQL_1$. Then $C123456J$ is the lineal arch for the system of loads, and the bending moment due to the system of loads shown, at any point in the arch, is the vertical distance between the parabolic arch and the lineal arch multiplied by the horizontal thrust H (57.5 tons).

It will be seen that for the system of loads assumed the bending moment at O is very small and cannot be determined with very great precision. By use of the influence diagram it can be found, however, with considerable accuracy. Set up at A the ordinate AK equal to $\frac{z}{y}$. Join BK ; then $AFBDEA$ is the influence diagram for the point O and the bending moment at O is found from Table II:—

TABLE II.

Load.	Ordinate ef .	Product $W.ef$.
10	0.125	1.25
15	0.4	6.00
15	0.04	0.60
20	-0.22	-4.4
10	-0.25	-2.5
10	-0.125	-1.25
Algebraical sum $\Sigma W.ef = -0.30$		

$$M_e = -0.30 y.$$

$$= -25.2 \text{ foot-tons.}$$

Now, as a further example, suppose the two loads, 15 tons and 20 tons only, are moving across the arch, then it is not difficult to see that the maximum positive moment at *M* will occur when the 20 tons is at *M* and the 15 tons is between *M* and *A*;

then $M_o = 84 (20 \cdot EF + 15 e_1 f_1) = 932$ foot-tons about.

The linear arch, when the two loads, 20 tons and 15 tons, are on alone, and 20 tons is at *M*, is *CabJ*. The value of *H* is now 22·3 tons and the bending moment can be found from it, and

$$= 22 \cdot 3 \times b_o = 22 \cdot 3 \times 42 \text{ foot-tons about.}$$

This simple example illustrates the importance and usefulness of the influence diagram for any section of such an arch-rib, for, if over the arch are to pass various systems of wheel loads, a few moments' consideration will enable one to see which of the systems will produce the greatest bending effect. Evidently a system of heavy wheel loads of over-all length less than *AR* is likely to produce a much greater bending effect, both positively and negatively, than a longer load.

INFLUENCE DIAGRAMS FOR A SEGMENTAL ARCH OF CONSTANT SECTION HINGED AT THE ENDS.

Let the arch consist of a circle of radius *r*, Fig. 11, Plate 4.

Let *O* be the centre of the circular arch, and let the angle subtended by *AB* be 2ϕ . Let *z* and *y* be the abscissa and ordinate respectively of a point *G* on the arch measured from axes through *A*, and let the angle *AOG* be α .

$$\text{Then} \quad y = r \cos (\phi - \alpha) - r \cos \phi.$$

$$z = r \sin \phi - r \sin (\phi - \alpha).$$

The general equation of condition that the span shall not alter for any loading of the arch then becomes—

$$\int_A^B \frac{M \cdot y \cdot ds}{EI} = \int_A^B \frac{M y \cdot r d\alpha}{EI} = 0, \text{ or } \int_A^B M y d\alpha = 0.$$

Let it be assumed that *I* is constant, and that there is a single load *W* on the arch at distance *x*, which equals $r \sin \beta$ from the centre.

Then if V is the vertical reaction at A , and H the horizontal reaction, the bending moment at G is—

$$Hy - Vz,$$

as long as G is to the left of W .

When G is to the right of W , i.e., at G_1 say, the bending moment at G —

$$= Hy - Vz + Wa.$$

Still calling the angle $\angle AOG_1$ α —

$$a = r \sin (\alpha - \phi) - r \sin \beta.$$

Then—

$$\begin{aligned} \int_A^B M \cdot y \, d\alpha &= \int_0^{2\phi} (Hy^2 - Vz y) \, d\alpha + \int_{\phi+\beta}^{2\phi} W \cdot a y \, d\alpha = 0. \\ \int_0^{2\phi} H \{ \cos (\phi - \alpha) - \cos \phi \}^2 \, d\alpha - V \{ [\cos (\phi - \alpha) - \cos \phi] \\ &\quad [\sin \phi - \sin (\phi - \alpha)] \} \, d\alpha \\ + \int_{\phi+\beta}^{2\phi} W \{ \sin (\alpha - \phi) - \sin \beta \} [\cos (\phi - \alpha) - \cos \phi] \, d\alpha &= 0. \end{aligned}$$

or—

$$\begin{aligned} &\int_0^{2\phi} H \{ \cos^2 (\phi - \alpha) - 2 \cos (\phi - \alpha) \cos \phi + \cos^2 \phi \} \, d\alpha \\ &- V \{ \cos (\phi - \alpha) \sin \phi - \cos \phi \sin \phi - \cos (\phi - \alpha) \sin (\phi - \alpha) \\ &\quad + \cos \phi \sin (\phi - \alpha) \} \, d\alpha \\ + \int_{\phi+\beta}^{2\phi} W \{ \sin (\alpha - \phi) \cos (\phi - \alpha) - \sin \beta \cos (\phi - \alpha) - \cos \phi \sin (\alpha - \phi) \\ &\quad + \cos \phi \sin \beta \} \, d\alpha = 0. \end{aligned}$$

From which

$$\begin{aligned} &H (\phi - 3 \sin \phi \cos \phi + 2 \phi \cos^2 \phi) - V (2 \sin^2 \phi - \phi \sin 2 \phi) \\ + W \left(\frac{1}{2} + \frac{\cos^2 \phi}{2} - \cos (\phi - \beta) + \frac{\sin^2 \beta}{2} + (\phi - \beta) \cos \phi \sin \beta \right) &= 0 \end{aligned}$$

or

$$\begin{aligned} H &= 2 V (\sin^2 \phi - \phi \sin \phi \cos \phi) - W \left\{ \frac{1}{2} + \frac{\cos^2 \phi}{2} - \cos (\phi - \beta) \right. \\ &\quad \left. + \frac{\sin^2 \beta}{2} + (\phi - \beta) \cos \phi \sin \beta \right\} \\ &\quad \hline &\phi - 3 \sin \phi \cos \phi + 2 \phi \cos^2 \phi \end{aligned}$$

Now
$$V = \frac{W}{L} \left(\frac{L}{2} - r \sin \beta \right)$$

which, when W is unity,

$$= \frac{1}{2} - \frac{r}{L} \sin \beta.$$

From Fig. 11 it can be seen that

$$\frac{L}{2r} = \sin \phi,$$

therefore

$$2V = 1 - \frac{\sin \beta}{\sin \phi}.$$

When W is unity, and

$$H = (\sin \phi - \sin \beta) (\sin \phi - \phi \cos \phi) - \left\{ \frac{1}{2} + \frac{\cos^2 \phi}{2} - \cos(\phi - \beta) \right. \\ \left. + \frac{\sin^2 \beta}{2} + (\phi - \beta) \cos \phi \sin \beta \right\} \\ \hline \phi - 3 \sin \phi \cos \phi + 2 \phi \cos^2 \phi$$

When $\phi = \beta$, $H = 0$.

It will be seen that this equation for a given value of ϕ varies with β only. It should be noted that in determining H , W in the Figure is shown to the right of the centre. It is not necessary to consider any other case in detail, for since H_A and H_B are equal the curve must be symmetrical about the centre of the span. In Fig. 12, Plate 4, the curve $A D B$ has been plotted by giving values to β . It is only necessary to calculate a few points.

Influence Diagram for Bending Moments at any section of the Segmental Arch.—Consider the bending moment at N , Fig. 12.

When the unit load W is to the right of N —

$$M = H y - \frac{1}{L} \left(\frac{L}{2} - x \right) z;$$

or

$$\frac{M}{y} = H - \frac{1}{L} \left(\frac{L}{2} - x \right) \frac{z}{y}.$$

Set up $A K$ equal to $\frac{z}{y}$ and join $B K$, meeting the vertical through N in d . Then—

$$E F = \frac{1}{L} \left(\frac{L}{2} - x \right),$$

and

$$M = y (E H - E F) = y F J.$$

When W is between the centre and N—

$$V = \frac{1}{L} \left(\frac{L}{2} + x \right),$$

and

$$\frac{M}{y} = -F_1 J_1.$$

Now join A d.

When W is to the left of N—

$$M = H y - \frac{1}{L} \left(\frac{L}{2} + x \right) z + \left(z - \frac{L}{2} + x \right);$$

or

$$\begin{aligned} \frac{M}{y} &= H - E_2 e + \frac{\left(z - \frac{L}{2} + x \right) z}{z} \frac{1}{y} \\ &= E_2 J - E_2 e + F_2 e \\ &= -F_2 J. \end{aligned}$$

If, therefore, F J, when measured upward, is said to be positive, and negative when measured downward—

$$M = y \sum W.F J$$

for any system of loads; and A D B d A is the influence diagram of bending moment for the section of the arch.

It will be seen that a bending moment due to H is always making the arch more convex.

For a point N to the right of the centre of the arch it will clearly be convenient to set up $\frac{z}{y}$ at B. The bending moment due to H will still be making the arch more convex, and may still conveniently be considered positive.

Lattice Segmental Arch-Rib: Influence Diagram.—For the diagonal member or for shear it can be shown, as on p. 355, that the force F in the diagonal E G, Fig 13, Plate 4, when the booms are parallel and W is to the right of N, is—

$$F = \frac{V \cos \theta - H \sin \theta}{\sin (\theta + \gamma)};$$

or

$$\frac{F \cdot \sin (\theta + \gamma)}{\sin \theta} = V \cdot \cot \theta - H.$$

Set up at A, Fig. 14, Plate 4, $AK = \cot \theta$, which equals $\frac{OQ}{NQ}$,
 Fig. 13. Join BK. Through N raise a perpendicular to meet BK in d . Through A draw af parallel to Bd to meet dN in f .
 Then, when W is to the right of N ,

$$EF = V \cot \theta,$$

and when W is to the left of N ,

$$E_1F_1 = (V - W) \cot \theta,$$

and

$$J_1F_1 = (V - W) \cot \theta - H.$$

If ordinates JF upwards are called positive and JF downwards negative, then

$$F \frac{\sin(\theta + \gamma)}{\sin \theta} = \Sigma W \cdot JF.$$

$$F = \frac{\sin \theta}{\sin(\theta + \gamma)} \Sigma W \cdot JF.$$

If, instead of the force F in the diagonal, the shearing force S perpendicular to the direction of the centre line of the arch is required,

$$S = V \cos \theta - H \sin \theta.$$

When W is to the right of N , and

$$S = (V - W) \cos \theta - H \sin \theta.$$

When W is to the left of N , so that

$$S = \sin \theta \Sigma W \cdot JF.$$

Positive values of JF correspond to a tension in the diagonal FG or to the shear that is tending to raise the left of the arch relative to the right.

The influence diagram for the stresses in the booms of the rib, taking into account thrust and bending moment, can be drawn in the way described for parabolic arches on p. 360.

The Paper is accompanied by three sheets of diagrams, from which Plate 4 has been prepared.

INFLUENCE DIAGRAMS FOR TWO-HINGED ARCHES.

Fig. 1.

Fig. 2.

PARABOLIC ARCH-RIB
OF CONSTANT DEPTH AND
VARIABLE WIDTH.

Fig. 3.

$$M_G = \sum W.H.K$$

Fig. 4.

Fig. 5.

INFLUENCE DIAGRAM FOR
DIAGONAL MEMBER.
 $F = \frac{\sin \theta}{\sin (\theta + \phi)} \sum W.P.N$

Fig. 6.

PARABOLIC ARCH-RIB
INFLUENCE DIAGRAM FOR DIAGONAL MEMBER.

PLATE 4.
INFLUENCE DIAGRAMS
FOR TWO-HINGED ARCHES.

Fig. 7.

PARABOLIC ARCH-RIB
INFLUENCE DIAGRAM FOR THE
STRESS p IN THE LOWER ROOM.

$$p = \frac{y + \frac{1}{2} \cos \theta}{A.d} \sum W.e.f$$

Fig. 8.

PARABOLIC ARCH-RIB
INFLUENCE DIAGRAM FOR THE
STRESS IN THE UPPER ROOM.
 $p_1 = \frac{y + \frac{1}{2} \cos \theta}{A.d} \sum W.e.f$

Fig. 9.

INFLUENCE DIAGRAM FOR CENTRE OF ARCH.

Fig. 10.

BENDING MOMENT AT O = $\sum W.e.f$
 $V = \sum W.g.h$ (AT UNIT)
 $H = \sum W.g.h$

Fig. 11.

Fig. 12.

Fig. 13.

Fig. 14.