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XXII. *The Equilibrium of Revolving Liquid under Capillary Force.* By Lord RAYLEIGH, O.M., F.R.S.*

THE problem of a mass of homogeneous incompressible fluid revolving with uniform angular velocity (ω) and held together by capillary tension (T) is suggested by well known experiments of Plateau. If there is no rotation, the mass assumes a spherical form. Under the influence of rotation the sphere flattens at the poles, and the oblateness increases with the angular velocity. At higher rotations Plateau's experiments suggest that an annular form may be one of equilibrium. The earlier forms, where the liquid still meets the axis of rotation, have been considered in some detail by Beer†, but little attention seems to have been given to the equilibrium in the form of a ring. A general treatment of this case involves difficulties, but if we assume that the ring is *thin*, viz. that the diameter of the section is small compared with the diameter of the circular axis, we may prove that the form of the section is approximately circular and investigate the small departures from that figure. It is assumed that in the cases considered the surface is one of revolution about the axis of rotation.

* Communicated by the Author.

† Pogg. *Ann.* vol. xevi. p. 210 (1855); compare Poincaré's *Capillarité*, 1895.

Fig. 1 represents a section by a plane through the axis Oy , O being the point where the axis meets the equatorial plane. One of the principal curvatures of the surface at P

Fig. 1.

is that of the meridional curve, the radius of the other principal curvature is PQ —the normal as terminated on the axis. The pressure due to the curvature is thus

$$T\left(\frac{1}{\rho} + \frac{1}{PQ}\right),$$

and the equation of equilibrium may be written

$$\frac{1}{\rho} + \frac{1}{PQ} = \frac{\sigma\omega^2 x^2}{2T} + \frac{p_0}{T}, \quad \dots \quad (1)$$

where p_0 is the pressure at points lying upon the axis, and σ is the density of the fluid.

The curvatures may most simply be expressed by means of s , the length of the arc of the curve measured say from A . Thus

$$\frac{1}{PQ} = \frac{1}{x} \frac{dy}{ds}, \quad \frac{1}{\rho} = \frac{d^2y/ds^2}{dx/ds}, \quad \dots \quad (2)$$

so that (1) becomes

$$\frac{dy}{ds} \frac{dx}{ds} + x \frac{d^2y}{ds^2} = \frac{\sigma\omega^2 x^3}{2T} \frac{dx}{ds} + \frac{p_0 x}{T} \frac{dx}{ds},$$

or on integration

$$x \frac{dy}{ds} = \frac{\sigma\omega^2 x^4}{8T} + \frac{p_0 x^2}{2T} + \text{const.} \quad \dots \quad (3)$$

Thus dy/ds is a function of x of known form, say X , and we get for y in terms of x

$$y = \pm \int \frac{X dx}{\sqrt{(1-X^2)}}, \quad \dots \quad (4)$$

as given by Beer.

If, as in fig. 1, the curve meets the axis, (3) must be satisfied by $x=0$, $dy/ds=0$. The constant accordingly disappears, and we have the much simplified form

$$\frac{dy}{ds} = \frac{\sigma \omega^2 x^3}{8T} + \frac{p_0 x}{2T} \quad \dots \quad (5)$$

At the point A on the equator $dy/ds=1$. If $OA=a$,

$$1 = \frac{\sigma \omega^2 a^3}{8T} + \frac{p_0 a}{2T};$$

whence eliminating p_0 and writing

$$\Omega = \frac{\sigma \omega^2 a^3}{8T}, \quad \dots \quad (6)$$

we get

$$\frac{dy}{ds} = \Omega \frac{x^3}{a^3} + (1-\Omega) \frac{x}{a} \quad \dots \quad (7)$$

In terms of y and x from (7)

$$\pm \frac{dy}{dx} = \frac{x \left(\Omega \frac{x^2}{a^2} + 1 - \Omega \right)}{\sqrt{\left\{ a^2 - x^2 \left(\Omega \frac{x^2}{a^2} + 1 - \Omega \right)^2 \right\}}}, \quad \dots \quad (8)$$

or if we write

$$x^2/a^2 = 1-z, \quad \dots \quad (9)$$

$$\begin{aligned} -\frac{2dy}{adz} &= \frac{1-\Omega z}{\sqrt{z} \cdot \sqrt{\{1+2(1-z)\Omega - z(1-z)\Omega^2\}}} \\ &= (1-\Omega + \tfrac{3}{2}\Omega^2)z^{-\frac{1}{2}} - \tfrac{3}{2}\Omega^2 z^{\frac{1}{2}}, \quad \dots \quad (10) \end{aligned}$$

when we neglect higher powers of Ω than Ω^2 . Reverting to x , we find for the integral of (10)

$$\pm \frac{y}{a} = (1-\Omega + \tfrac{3}{2}\Omega^2) \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{\Omega^2}{2} \left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}}, \quad (11)$$

no constant being added since $y=0$ when $x=a$.

If we stop at Ω , we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-\Omega)^2} = 1 \quad \dots \quad (12)$$

representing an ellipse whose minor axis OB is $a(1-\Omega)$.

When Ω^2 is retained,

$$OB = (1-\Omega + \Omega^2) a \quad \dots \quad (13)$$

The approximation in powers of Ω could of course be continued if desired.

So long as $\Omega < 1$, p_0 is positive and the (equal) curvatures at B are convex. When $\Omega = 1$, $p_0 = 0$ and the surface at B is flat. In this case (8) gives

$$\pm \frac{dy}{dx} = \frac{x^3}{\sqrt{a^6 - x^6}}, \quad \dots \quad (14)$$

or if we set $x = a \sin^{\frac{1}{3}} \phi$,

$$\pm \frac{dy}{d\phi} = \frac{a}{3} \sin^{\frac{1}{3}} \phi. \quad \dots \quad (15)$$

Here $x = a$ corresponds to $\phi = \frac{1}{2}\pi$, and $x = 0$ corresponds to $\phi = 0$. Hence

$$OB = \frac{a}{3} \int_0^{\frac{1}{2}\pi} \sin^{\frac{1}{3}} \phi \, d\phi. \quad \dots \quad (16)$$

The integrals in (16) may be expressed in terms of gamma functions and we get

$$OB = a \sqrt{\pi} \cdot \Gamma\left(\frac{2}{3}\right) \div \Gamma\left(\frac{1}{6}\right) = \cdot 4312a. \quad \dots \quad (17)$$

When $\Omega > 1$, the curvature at B is concave and p_0 is negative, as is quite permissible.

In order to trace the various curves we may calculate by quadratures from (4) the position of a sufficient number of points. This, as I understand, was the procedure adopted by Beer. An alternative method is to trace the curves by direct use of the radius of curvature at the point arrived at. Starting from (7) we find

$$\frac{d^2y}{ds^2} = \left(\Omega \frac{3x^2}{a^3} + \frac{1-\Omega}{a} \right) \frac{dx}{ds},$$

and thence

$$\frac{a}{\rho} = a \frac{d^2y/ds^2}{dx/ds} = \Omega \frac{3x^2}{a^2} + 1 - \Omega. \quad \dots \quad (18)$$

From (18) we see at once that $\Omega = 0$ makes $\rho = a$ throughout, and that when $\Omega = 1$, $x = 0$ makes $\rho = \infty$.

In tracing a curve we start from the point A in a known direction and with $\rho = a/(2\Omega + 1)$, and at every point arrived at we know with what curvature to proceed. If, as has been assumed, the curve meets the axis, it must do so at right angles, and a solution is then obtained.

The method is readily applied to the case $\Omega = 1$ with the advantage that we know where the curve should meet the axis of y . From (18) with $\Omega = 1$ and $a = 5$,

$$\frac{1}{\rho} = \frac{24x^2}{1000}. \quad \dots \quad (19)$$

Starting from $x = 5$ we draw small portions of the curve corresponding to decrements of x equal to $\cdot 2$, thus arriving

in succession at the points for which $x=4.8, 4.6, 4.4$, &c. For these portions we employ the *mean* curvatures corresponding to $x=4.9, 4.7$, &c. calculated from (19). It is convenient to use squared paper and fair results may be obtained with the ordinary ruler and compasses. There is no need actually to draw the normals. But for such work the procedure recommended by Boys* offers great advantages. The ruler and compasses are replaced by a straight scale divided upon a strip of semi-transparent celluloid. At one point on the scale a fine pencil point protrudes through a small hole and describes the diminutive circular arc. Another point of the scale at the required distance occupies the centre of the circle and is held temporarily at rest with the aid of a small brass tripod standing on sharp needle points. After each step the celluloid is held firmly to the paper and the tripod is moved to the point of the scale required to give the next value of the curvature. The ordinates of the curve so drawn are given in the second and fifth columns of the annexed table. It will be seen that from $x=0$ to $x=2$ the curve is very flat.

$\pm x.$	$\pm y.$	$\pm y'.$	$\pm x.$	$\pm y.$	$\pm y'.$
0.0	2.16	0.00	2.6	2.06	0.75
0.2	2.16	0.01	2.8	2.03	0.83
0.4	2.16	0.03	3.0	1.99	0.90
0.6	2.16	0.06	3.2	1.95	0.95
0.8	2.16	0.10	3.4	1.89	0.99
1.0	2.15	0.14	3.6	1.81	1.01
1.2	2.15	0.20	3.8	1.72	1.02
1.4	2.15	0.27	4.0	1.61	1.00
1.6	2.15	0.34	4.2	1.49	0.98
1.8	2.14	0.42	4.4	1.32	0.89
2.0	2.12	0.50	4.6	1.11	0.78
2.2	2.11	0.58	4.8	0.80	0.67
2.4	2.09	0.65	4.9	0.59	0.41
			5.0	0.00	0.00

Another case of special interest is the last figure reaching the axis of symmetry at all, which occurs at the point $x=0$. We do not know beforehand to what value of Ω this corresponds, and curves must be drawn tentatively. It appears that $\Omega=2.4$ approximately, and the values of y obtained from this curve are given in columns 3 and 6 of the table. There is a little difficulty in drawing the curve through the point of zero curvature. I found it best to begin at both ends ($x=0, y=0$) and ($x=5, y=0$) with an assumed value of Ω and examine whether the two parts could be made to fit.

* Phil. Mag. vol. xxxvi. p. 75 (1893). I am much indebted to Mr. Boys for the loan of suitable instruments. The use is easy after a little practice.

When $\Omega > 2.4$ and the curve does not meet the axis at all, the constant in (3) must be retained, and the difficulty is much increased. If we suppose that $dy/ds = +1$ when $x = a_2$ and $dy/ds = -1$ when $x = a_1$, we can determine p_0 as well as the constant of integration, and (3) becomes

$$x \frac{dy}{ds} = \frac{\sigma \omega^2}{8T} (x^2 - a_1^2)(x^2 - a_2^2) + \frac{x^2 - a_1 a_2}{a_2 - a_1}. \quad (20)$$

We may imagine a curve to be traced by means of this equation. We start from the point A where $y=0, x=a_2$ and in the direction perpendicular to OA, and (as before) we are told in what direction to proceed at any point reached. When $x=a_1$, the tangent must again be parallel to the axis, but there is nothing to ensure that this occurs when $y=0$. To secure this end and so obtain an annular form of equilibrium, $\sigma \omega^2/T$ must be chosen suitably, but there is no means apparent of doing this beforehand. The process of curve tracing can only be tentative.

If we form the expression for the curvature as before, we obtain

$$\frac{1}{\rho} = \frac{\sigma \omega^2}{8T} \left(3x^2 - a_1^2 - a_2^2 - \frac{a_1^2 a_2^2}{x^2} \right) + \frac{1}{a_2 - a_1} + \frac{a_1 a_2}{x^2(a_2 - a_1)} \quad (21)$$

by means of which the curves may be traced tentatively.

If we retain the normal PQ, as we may conveniently do in using Boys' method, we have the simpler expression

$$\frac{1}{\rho} + \frac{1}{PQ} = \frac{\sigma \omega^2}{4T} (2x^2 - a_1^2 - a_2^2) + \frac{2}{a_2 - a_1}. \quad (22)$$

When the radius CP of the section is very small in
Fig. 2.

comparison with the radius of the ring OC, the conditions are approximately satisfied by a circular form. We write $CP = r$,

OC= a , PCX= θ . Then, r being supposed constant, the principal radii of curvature are r and $a \sec \theta + r$, so that the equation of equilibrium is

$$\frac{p_0}{T} = \frac{1}{r} + \frac{\cos \theta}{a + r \cos \theta} - \frac{\omega^2}{2T} (a + r \cos \theta)^2, \quad (23)$$

in which p_0 should be constant as θ varies. In this

$$\frac{\cos \theta}{a + r \cos \theta} = \frac{1}{a} \left\{ -\frac{r}{2a} + \left(1 + \frac{3r^2}{4a^2}\right) \cos \theta - \frac{r}{2a} \cos 2\theta + \frac{r^2}{4a^2} \cos 3\theta \right\},$$

$$\left(1 + \frac{r}{a} \cos \theta\right)^2 = 1 - \frac{r^2}{2a^2} + \frac{2r}{a} \cos \theta + \frac{r^2}{2a^2} \cos 2\theta.$$

Thus approximately

$$\begin{aligned} \frac{ap_0}{T} = & \frac{a}{r} - \frac{r}{2a} - \frac{\omega^2 a^3}{2T} \left(1 + \frac{r^2}{2a^2}\right) + \cos \theta \left\{ 1 + \frac{3r^2}{4a^2} - \frac{\omega^2 a^3}{2T} \frac{2r}{a} \right\} \\ & + \cos 2\theta \left\{ -\frac{r}{2a} - \frac{\omega^2 a}{2T} \frac{r^2}{2a^2} \right\} + \cos 3\theta \cdot \frac{r^2}{4a^2}. \end{aligned} \quad (24)$$

The term in $\cos \theta$ will vanish if we take ω so that

$$\frac{\omega^2 a^2}{T} = \frac{1}{r} \left(1 - \frac{3r^2}{a^2}\right). \quad (25)$$

The coefficient of $\cos 2\theta$ then becomes

$$-\frac{3r}{4a} + \text{cubes of } \frac{r}{a}. \quad (26)$$

If we are content to neglect r/a in comparison with unity, the condition of equilibrium is satisfied by the circular form; otherwise there is an inequality of pressure of this order in the term proportional to $\cos 2\theta$. From (25) it is seen that if a and T be given, the necessary angular velocity increases as the radius of the section decreases.

In order to secure a better fulfilment of the pressure equation it is necessary to suppose r variable, and this of course complicates the expressions for the curvatures. For that in the meridional plane we have

$$\frac{1}{\rho} = \frac{r^2 - r \frac{d^2 r}{d\theta^2} + 2 \left(\frac{dr}{d\theta}\right)^2}{\left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{3}{2}}},$$

or with sufficient approximation

$$\frac{1}{\rho} = \frac{1}{r} \left\{ 1 - \frac{1}{r} \frac{d^2 r}{d\theta^2} + \frac{1}{2r^2} \left(\frac{dr}{d\theta}\right)^2 \right\}. \quad (27)$$

For the curvature in the perpendicular plane we have to

Fig. 3.

substitute PQ' , measured along the normal, for PQ , whose expression remains as before. Now

$$\frac{PQ}{PQ'} = \frac{\sin Q'}{\sin Q} = \cos QPQ' - \tan \theta \sin QPQ'$$

in which

$$\cos QPQ' = \frac{CN}{r} = \left\{ 1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right\}^{-\frac{1}{2}} = 1 - \frac{1}{2r^2} \left(\frac{dr}{d\theta} \right)^2$$

approximately,

$$\sin QPQ' = -\frac{1}{r} \frac{dr}{d\theta} \left\{ 1 - \frac{1}{2r^2} \left(\frac{dr}{d\theta} \right)^2 \right\}.$$

Thus

$$\begin{aligned} \frac{1}{PQ'} &= \frac{\cos \theta}{a + r \cos \theta} \left\{ 1 - \frac{1}{2r^2} \left(\frac{dr}{d\theta} \right)^2 \right\} \\ &+ \frac{\sin \theta}{a + r \cos \theta} \frac{1}{r} \frac{dr}{d\theta} \left\{ 1 - \frac{1}{2r^2} \left(\frac{dr}{d\theta} \right)^2 \right\}. \end{aligned} \quad (28)$$

It will be found that it is unnecessary to retain $(dr/d\theta)^2$, and thus the pressure equation becomes

$$\begin{aligned} \frac{ap_0}{T} &= \frac{a}{r} \left\{ 1 - \frac{1}{r} \frac{d^2 r}{d\theta^2} \right\} + \frac{a \cos \theta}{a + r \cos \theta} \\ &+ \frac{a \sin \theta}{a + r \cos \theta} \frac{1}{r} \frac{dr}{d\theta} - \frac{\omega^2 a^3}{2T} \left(1 + \frac{r \cos \theta}{a} \right)^2. \end{aligned} \quad (29)$$

It is proposed to satisfy this equation so far as terms of the order r^2/a^2 inclusive.

As a function of θ , r may be taken to be

$$r = r_0 + \delta r = r_0 + r_1 \cos \theta + r_2 \cos 2\theta + \dots, \quad (30)$$

where r_1 , r_2 , &c. are constants small relatively to r_0 . It

will appear that to our order of approximation $(\delta r/r_0)^2$ may be neglected and that it is unnecessary to include the r 's beyond r_3 inclusive. We have

$$\begin{aligned} \frac{a}{r} &= \frac{a}{r_0} \left\{ 1 - \frac{\delta r}{r_0} + \left(\frac{\delta r}{r_0} \right)^2 \right\}, \\ \frac{a \cos \theta}{a + r \cos \theta} &= -\frac{r_0}{2a} - \frac{r_2}{4a} + \cos \theta \left\{ 1 + \frac{3r_0^2}{4a^2} - \frac{3r_1}{4a} - \frac{r_3}{4a} \right\} \\ &\quad - \cos 2\theta \left\{ \frac{r_0}{2a} + \frac{r_2}{2a} \right\} + \cos 3\theta \left\{ \frac{r_0^2}{4a^2} - \frac{r_1}{4a} - \frac{r_3}{2a} \right\}, \\ \left(1 + \frac{r \cos \theta}{a} \right)^2 &= 1 + \frac{r_1}{a^2} + \frac{r_0^2}{2a^2} + \frac{r_0 r_2}{2a^2} \\ &\quad + \cos \theta \left\{ \frac{2r_0}{a} + \frac{r_2}{a} + \frac{3r_0 r_1}{2a^2} + \frac{r_0 r_3}{2a^2} \right\} \\ &\quad + \cos 2\theta \left\{ \frac{r_1}{a} + \frac{r_3}{a} + \frac{r_0^2}{2a^2} + \frac{r_0 r_2}{a^2} \right\} \\ &\quad + \cos 3\theta \left\{ \frac{r_2}{a} + \frac{r_0 r_3}{a^2} + \frac{r_0 r_1}{2a^2} \right\}, \\ -\frac{a}{r^2} \frac{d^2 r}{d\theta^2} &= \frac{a}{r_0^2} \left\{ r_1 \cos \theta + 4r_2 \cos 2\theta + 9r_3 \cos 3\theta \right\}, \\ \frac{a \sin \theta}{a + r \cos \theta} \frac{1}{r} \frac{dr}{d\theta} &= -\frac{r_1}{2r_0} + \frac{r_2}{2a} + \cos \theta \left\{ \frac{r_1}{4a} - \frac{r_2}{r_0} + \frac{3r_3}{4a} \right\} \\ &\quad + \cos 2\theta \left\{ \frac{r_1}{2r_0} - \frac{3r_3}{2r_0} \right\} + \cos 3\theta \left\{ \frac{r_2}{r_0} - \frac{r_1}{4a} \right\}. \end{aligned}$$

Thus altogether for the coefficient of $\cos \theta$ on the right of (29) we get

$$1 + \frac{3r_0^2}{4a^2} - \frac{r_1}{2a} - \frac{r_2}{r_0} - \frac{\omega^2 a^3}{2\Gamma} \left\{ \frac{2r_0}{a} + \frac{r_2}{a} \right\}.$$

This will be made to vanish if we take ω such that

$$\frac{\omega^2 a^2 r_0}{\Gamma} = 1 + \frac{3r_0^2}{4a^2} - \frac{r_1}{2a} - \frac{3r_2}{2r_0}. \quad \dots \quad (31)$$

The coefficient of $\cos 2\theta$ is

$$\frac{3ar_2}{r_0^2} - \frac{r_0}{2a} + \frac{r_1}{2r_0} - \frac{3r_3}{2r_0} - \frac{\omega^2 a^3}{2\Gamma} \left\{ \frac{r_1}{a} + \frac{r_3}{a} + \frac{r_0^2}{2a^2} \right\},$$

or when we introduce the value of ω from (31)

$$\frac{3ar_2}{r_0^2} - \frac{3r_0}{4a} - \frac{2r_3}{r_0}. \quad \dots \quad (32)$$

The coefficient of $\cos 3\theta$ is in like manner

$$\frac{8ar_3}{r_0^3} + \frac{r_0^2}{4a^2} + \frac{r_2}{2r_0} \dots \dots \dots (33)$$

These coefficients are annulled and ap_0/T is rendered constant so far as the second order of r_0/a inclusive, when we take r_4, r_5 , &c. equal to zero and

$$r_2/r_0 = r_0^2/4a^2, \quad r_3/r_0 = -3r_0^3/64a^3. \quad \dots \dots (34)$$

We may also suppose that $r_1 = 0$.

The solution of the problem is accordingly that

$$r = r_0 \left\{ 1 + \frac{r_0^2}{4a^2} \cos 2\theta - \frac{3r_0^3}{64a^3} \cos 3\theta \right\} \dots \dots (35)$$

gives the figure of equilibrium, provided ω be such that

$$\frac{\omega^2 a^2 r_0}{T} = 1 + \frac{3r_0^2}{8a^2} \dots \dots \dots (36)$$

The form of a thin ring of equilibrium is thus determined; but it seems probable that the equilibrium would be unstable for disturbances involving a departure from symmetry round the axis of revolution.

XXIII. *The Positive Ionization from Heated Platinum.*

By CHARLES SHEARD, *Ph.D.*, *Ohio State University* *.

I. *Introduction.*

THIS paper will present the results of some investigations made as to the dependence of the positive ionization from heated platinum wires upon (1) temporary heatings of the wire when earthed and at temperatures higher than a chosen lower temperature at which the positive thermionic current was subsequently measured; (2) temporary heatings of the wire under negative potential and at such temperatures as to give small negative currents, the positive emission being investigated at an arbitrary, chosen temperature; and (3) heating the wire in a Bunsen flame and in carbon dioxide, and also re-washing in nitric acid. Some experiments carried out some time ago with a form of apparatus somewhat different from that used in this work and exhausted to a pressure of about 0.0001 mm., showed that the positive currents at any given temperature could be enhanced, by either heating the wire at a higher temperature, the wire being connected to earth, or by discharging

* Communicated by the Author.