

563. Note on the Integration of the Difference between Two Fagnano Arcs of an Ellipse

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If then $\frac{u_{n+1}}{u_n} = 1 - \frac{1}{n} - \frac{u}{n^2}$, where $|u|$ is always $< C$, we have only to take for k a number $> C$ to ensure that ultimately, *even if u is positive*, $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$. But Σv_n is divergent; $\therefore \Sigma u_n$ is divergent. With this addition the rules cover all cases that ordinarily arise.

Two more points.

(1) It is not always remembered that (keeping still to series of positive terms) if $\frac{u_n}{v_n}$ is always finite, $< C$, say, and Σv_n is convergent, Σu_n is also convergent; for

$$|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < C(v_{n+1} + v_{n+2} + \dots + v_{n+p}) < C\epsilon < \epsilon',$$

where ϵ has the usual meaning. In this way we may often substitute for a series a much simpler series before applying the test involving $\frac{u_{n+1}}{u_n}$; or convergence may be obvious for the simplified series without any such test.

(2) In the comparison with the series $1^{-\beta} + 2^{-\beta} + 3^{-\beta} + \dots (\beta + ^\circ)$, where the ratio of the $(n+1)^{\text{th}}$ term to the $n^{\text{th}} = \left(1 + \frac{1}{n}\right)^{-\beta}$, it is unnecessary to use the binomial theorem for a fractional or negative index, *which is undesirable in a fundamental test which is required for the full discussion of the binomial theorem*, if (as Chrystal) we use the important and easily proved inequality $\frac{a^p - 1}{p} > \frac{a^q - 1}{q}$, if $p > q$ (a being any positive number). For it follows that

$$\frac{\left(1 + \frac{1}{n}\right)^{-\beta} - 1}{-\beta} < \frac{\left(1 + \frac{1}{n}\right)^{+1} - 1}{+1} \quad (\beta \text{ being positive}), \text{ i.e. } < \frac{1}{n};$$

and therefore

$$\left(1 + \frac{1}{n}\right)^{-\beta} > 1 - \frac{\beta}{n} > 1 - \frac{B}{n} - \frac{u}{n^2}$$

ultimately, if $B > \beta$ and u finite; so that the latter ratio $<$ that in the series $1^{-\beta} + 2^{-\beta} + 3^{-\beta} + \dots$, which is convergent if $\beta > 1$; which proves that Σu_n is convergent if $B > 1$, β being taken as above.

If in any case the terms are not all positive, the corresponding tests become tests for the *absolute* convergence of the series in question.

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PERCY J. HEAWOOD.

563. [L¹. 9 b.] *Note on the integration of the difference between two Fagnano arcs of an ellipse.* See *Elliptic Trammels and Fagnano Points*, by Prof. P. J. Harding. *Mathematical Gazette*, vol. vi. pp. 68-78; 117-124.

The object of the following paper is to give a simple proof of the property of the Fagnano integrals.

In the ordinary figure for the description of elliptic arcs by trammels $HK(a+b)$ and $hk(a-b)$, let I, i be the instantaneous centres; $\phi_1 = HCI = hCi$ the eccentric angle of P and CF the perp'n. on the normal IPi through P ; ψ_1 the angle Ii makes with the major axis; ϕ_2 the angle HIi .

Since

$$\cot \phi_2 = \tan \psi_1 = \frac{b}{a} \tan \phi_1,$$

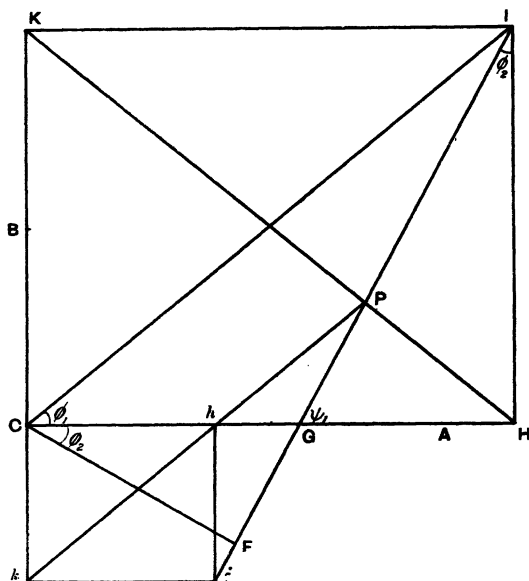
$$\therefore \tan \phi_1 \tan \phi_2 = \frac{a}{b}; \dots\dots\dots(i)$$

$$\therefore d\phi_1 \sin 2\phi_2 + d\phi_2 \sin 2\phi_1 = 0. \dots\dots\dots(ii)$$

Also, since HK , CI are equally inclined to CA , if an infinitesimal arc ds_1 is described by P through the motion of the trammel,

$$ds_1 = \pm IP d\phi_1, \dots\dots\dots(iii)$$

according as the arc is measured from A or from B .



Again, since H, I, C, F are concyclic,

$$\angle FCH = \angle HIG = \phi_2;$$

$$\therefore \angle ICF = \phi_1 + \phi_2.$$

$$\frac{a}{\cos \phi_1 \cos \phi_2} = \frac{b}{\sin \phi_1 \sin \phi_2} = \frac{CI}{\cos(\phi_1 - \phi_2)} = \frac{Ci}{\cos(\phi_1 + \phi_2)}$$

$$= \frac{Ii}{\sin 2\phi_1} \text{ from } \triangle ICI;$$

$$\therefore IP = \frac{a+b}{2 \cos(\phi_1 - \phi_2)} \sin 2\phi_1. \dots\dots\dots(iv)$$

Hence if a second figure were drawn with eccentric angle ϕ_2 , the angles ϕ_1 and ϕ_2 having been interchanged and accented letters used in it, if s_2 denote the arc BP' ,

$$ds_2 = -I'P'd\phi_2 = -\frac{a+b}{2 \cos(\phi_2 - \phi_1)} \sin 2\phi_2;$$

$$\therefore ds_1 - ds_2 = \frac{a+b}{2 \cos(\phi_1 - \phi_2)} (\sin 2\phi_1 d\phi_1 + \sin 2\phi_2 d\phi_2)$$

$$= \frac{a+b}{2 \cos(\phi_1 - \phi_2)} (\sin 2\phi_1 + \sin 2\phi_2)(d\phi_1 + d\phi_2), \text{ by (ii),}$$

$$= (a+b) \sin(\phi_1 + \phi_2) d(\phi_1 + \phi_2)$$

$$= -d(CF);$$

$$\therefore s_2 - s_1 = CF,$$

no constant being required, since s_1 , s_2 and CF vanish together.

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