Two Notes on Isoscelians. By R. TUCKER.

[Read Feb. 12th, 1891.]

(i.) With the in-centre, O , of ABC as centre, describe a circle to cut BC in D, D' ; CA in E, E' ; AB in F, F' . Then the chords DD', EE', FF' are equidistant from O , and are therefore equal to one another; also, AFE', BDF', CED' are isosceles triangles.

The lines FE', DF', ED' are *direct* Isoscelians.*

The lines DE', EF', FD' , which form the triangle $\alpha\beta\gamma$, are evidently parallel to the sides of the triangle $\delta\epsilon\zeta$ ($\delta\epsilon\zeta$ being the in-circle).

Hence the set $A'B'C'$ (got by producing DF', FE', ED') and the set $\alpha\beta\gamma$ are inversely similar.

The lines $Aa, B\beta, C\gamma$ meet in

$$aa(s-a) = b\beta(s-b) = c\gamma(s-c),$$

i.e., in the "Gergonne" point (G) of the triangle ABC . But $A\delta, B\epsilon, C\zeta$ meet in G ; hence the set $A'B'C'$, $\alpha\beta\gamma$ are in perspective with ABC , and G is their centre of perspective.

Now

$$\angle aDc = 90^\circ - C/2 = \angle \gamma,$$

therefore $\beta\gamma$ is an antiparallel to a in the triangle aDD' , and $A\delta$ is a median of this triangle; hence it is a Symmedian line of the triangle $\alpha\beta\gamma$, i.e., G is the Symmedian point of the sets $A'B'C'$, $\alpha\beta\gamma$; and therefore all the circles $DD'EE'FF'$ are Tucker circles of the two sets.

(ii.) If F, F' are the *permanent centres of similitude* of the immovable triangles $abc, a'b'c'$ of the triangle ABC (the nomenclature and the figure are adopted from *Proc.*, Vol. XVI., p. 185), and if we put

$$AcF = BaF' = ObF = \phi \text{ (say),}$$

$$Bc'F' = Ca'F' = Ab'F' = \phi';$$

then

$$\left. \begin{aligned} FA &= 2\rho \sin \phi \\ F'A &= 2\rho' \sin \phi' \end{aligned} \right\} \text{ (say),}$$

and

$$bc = 2\rho \sin A,$$

$$b'c' = 2\rho' \sin A.$$

* The writer proposes to call the Isoscelians previously discussed by him *Oblique Isoscelians*.

Now, if (α, β, γ) , $(\alpha', \beta', \gamma')$ are the trilinear coordinates of F, F' , we have

$$\gamma = AF \sin FAc, \quad \beta' = AF' \sin F'Ab,$$

therefore
$$\frac{AF'}{AF} = \frac{\beta'}{\gamma} = \frac{\beta\beta'}{\beta\gamma} \propto \alpha$$

(because F, F' are isogonal conjugates).

Hence
$$\alpha \propto \frac{AF'}{AF} \propto \frac{\rho'}{\rho} \propto \frac{b'c'}{bc};$$

that is, if the sides of abc , $a'b'c'$ are respectively p, q, r, p', q', r' ,

we have
$$\alpha : \beta : \gamma = \frac{p'}{p} : \frac{q'}{q} : \frac{r'}{r},$$

and
$$\alpha' : \beta' : \gamma' = \frac{p}{p'} : \frac{q}{q'} : \frac{r}{r'}.$$

The following is the extract, from Mr. Heal's letter, which was communicated to the Society by Prof. Cayley:—

“About a year ago I wrote you giving the equation of the Bitangential of the Quintic in the form

$$\begin{aligned} & [297HJ(n, H, \phi) - 456HJ'(n, H, \phi) - 20J(n, H, \theta)]^2 \\ & = 3025(4\theta - 9H\phi)^3. \end{aligned}$$

I showed how to reduce this equation by the well-known formula for the square of the Jacobian, and thus found the equation to be divisible by H , and all the terms except

$$\left[\left(\frac{\theta}{H} \right) - 2\theta\phi \right] H$$

were shown to be divisible by H^2 .

“I have recently found the following equation, which shows that this expression is also divisible by H^2 . The equation in question is

$$8 \left[\left(\frac{\theta}{H} \right) - 2\theta\phi \right] = \left[9 \left(\frac{\phi'}{H} \right) - 19\psi \right] H \dots\dots\dots (A),$$

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where ψ is

$$\begin{vmatrix} a, & h, & g, & H_1, & \delta_x \\ h, & b, & f, & H_2, & \delta_y \\ g, & f, & c, & H_3, & \delta_z \\ H_1, & H_2, & H_3, & & \\ \delta_x, & \delta_y, & \delta_z, & & \end{vmatrix} H.$$

I have not been able to prove this equation, but I have no doubt of its correctness.

"It is easy to show that

$$\left(\frac{\theta}{H}\right) - 2\theta\phi = \left(\frac{\theta'}{H}\right) + 2H\psi,$$

and I have tried to make the proof of equation (A) depend upon the formula

$$\square = 8(8\theta - 9H\phi)$$

(see your paper on the bitangential in *Phil. Trans.*).

"Accepting equation (A) as correct, the equation of the bitangential, cleared of extraneous factors, is

$$\begin{aligned} & 32400 \left(\frac{\theta}{\phi}\right) + 376200\psi\theta - 1674675\theta\phi^2 - 1163160\theta \left(\frac{\phi'}{H}\right) \\ & - 1259280\phi \left(\frac{\theta}{H}\right) + 1389960\theta \left(\frac{\phi}{H}\right) \\ & + \left[2205225\phi^3 - 6014250\phi \left(\frac{\phi}{H}\right) + 20516544\phi \left(\frac{\phi'}{H}\right) \right. \\ & \quad \left. - 962280 \left(\frac{\theta}{\phi}\right) + 1480680 \left(\frac{\theta}{\phi'}\right) \right] H \\ & + \left[7144929 \left(\frac{\phi}{\phi}\right) - 21939984 \left(\frac{\phi}{\phi'}\right) + 16701816 \left(\frac{\phi'}{\phi'}\right) \right] H^2 = 0. \end{aligned}$$

"A remarkable property of the function ψ is that, regarding it as a quadric function in $\delta_x, \delta_y, \delta_z$, the discriminant vanishes, or it breaks up into two linear factors. I am now working to prove equation (A), which will complete the proof of the above equation of the bitangential."