A THREAD OF MATHEMATICAL HISTORY, AND SOME LESSONS.¹

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I.

Scientific work often waits long for its full fruitage. The seed of truth is frequently produced at a time when there is no soil ready to receive it; and consequently it must lie quiescent for an uncertain period until preparation has been made for its sowing. And even after it is put into the soil it may happen that centuries will elapse before it can germinate. But it retains its vitality until millenniums have come and passed. Ultimately it springs into growth, slow at first and more vigorous afterwards. Often it comes unexpectedly to florescence and bursts into magnificent flower. Fruit appears, and men begin with joy to reap the harvest of an ancient sowing. For year upon year and century upon century it not infrequently continues to yield its precious increase, and men rejoice in the ever renewed heritage of a remote past; and what they have received they hand on to the future in full vigor and undiminished usefulness. To have a part in creating a new heritage for those in the centuries to come is the most glorious labor which falls to the lot of man.

But to strive through a long life of severe scientific investigation without seeing the direct fruits of one's toil is difficult for many of us, and we need the encouragement which comes from seeing the magnificent usefulness of past research. Even the enthusiastic worker sometimes has a period of despondency when he can take no pleasure in his labors and when he turns away from them with a feeling of tire and disgust. Of Lagrange and his work on mechanics it is said that "for two years the printed volume of his *Mécanique*, which he had seen only in manuscript, lay unopened beside him."

That the teacher also suffers sometimes from the same spirit of dejection is known to most of us through our own experience.

It is thus clear that both the investigator and the teacher should be provided with all the encouragement which the past history of science affords. Considering what this is it is hard to see how they could ask for more. One purpose of this lecture is to illustrate the nature of this encouragement by means of a typical

¹An address delivered before the Indiana Association of Science and Mathematics Teachers at Indianapolis, Friday evening, March 7, 1918.

example taken from the history of mathematics and its applications to other sciences. We shall trace a certain thread of mathematical history from its origin in ancient times down to the most recent investigations. You are urged to remember that the example taken is merely one of a large number, any one of which would serve well the same purpose; it is not an isolated case, but one which has many instructive parallels.

But, first of all, before giving this account of certain applications of mathematical speculation, let me (as a mathematician) emphasize the fact that mathematics does not exist merely for these applications. In fact they are incidental to its main purpose. Mathematics exists primarily and essentially for itself. If we should attempt to cultivate it for its applications alone we would soon find our progress impeded by barriers which we could not remove. It must be cultivated for its own sake even though many of its results appear for a time entirely useless. How they often have unexpectedly a great value will be seen from the historical sketch which we are about to give.

Let us go back to ancient Greece where so many threads of history have their origin. Here we shall find the beginning of the theory of Conic Sections. One series of consequences growing out of this it is our purpose to treat.

Suppose that we have a right circular cone and cut its surface by means of a plane. If the plane passes through the vertex of the cone the section figure consists of two straight lines. We shall dismiss this case as trivial. When the cutting plane does not pass through the vertex several possibilities arise. In the first and simplest case the section figure is a circle. All the other cases we may look upon as generalizations of the circle. We have a closed figure, which resembles a circle elongated in one direction. We have next a figure with one branch going off to infinity. Finally we have a figure with two branches going off to infinity. These three "conic sections" are known as the ellipse, the parabola, and the hyperbola, respectively.

To the study of these three plane curves the ancient Greek mathematicians devoted much attention, and a large body of elegant theory was built up and rigorously established. The results had no apparent practical applications, and none were sought for them. They seemed to be remote from all experience—a creation of the mind, standing aloof from all useful service.

But this did not deter the investigators. They studied the

curves simply for the sake of their symmetry and elegance. Their object was the pure delight in the artistic beauty of the theory, made complete by the answer to every question which they were able to ask. Working thus and without knowing what they were really effecting, these Greek mathematicians wrote "the grammar of the language in which, it would be demonstrated in after ages, that the pages of the universe are written."

For many generations this body of Greek speculation was practically unaltered. For awhile it was almost forgotten, but it was preserved in the written records of Greek thought. When the revival of interest in ancient philosophy appeared, men were again delighted with the beauty of the theory of conic sections and it was passed along from teacher to pupil, unaltered for many generations and still without practical use or even the use which comes in application to other scientific theories. The era of scholasticism came and passed away. The spirit of the ages changed, and changed again; and yet this body of Greek thought remained unaltered and without application. It was still cultivated and preserved for its own sake.

Then came the dawn of the modern era, brought in by the writings of Bacon. Men still learned again what the Greeks knew of conic sections. A contemporary of Bacon's was at the same time versed in this ancient theory and interested in the study of astronomy. This man was the illustrious Johann Kepler. He was engaged in trying to determine the nature of the planetary orbits. He tried circles, but they would not fit into his data obtained by observation. He tried combinations of circular arcs, but this was no better. He cast about him for other curves of simple properties which might possibly agree with all observations. It occurred to him to try the ellipse, a curve whose properties were so well known to him through his acquaintance with Greek mathematics. To his delight he found that all his data fit perfectly into the scheme of an elliptic orbit. Nowhere was there greater deviation than might be accounted for by experimental error. His quest ended in success, because thousands of years before his time the Greek mathematicians had been interested in the theory of conic sections for its own sake.

But this was not all. Once the form of the planetary path was known to him, Kepler was able to take up successfully several fundamentally important questions which before had re-

mained inaccessible to treatment—in fact, they had hardly been framed by his predecessors. The same Greek tools which he had already employed again served his purpose; and the outcome of his investigations was the enunciation of those laws of astronomy which have since been associated with his name. Thus the foundation of modern astronomy was laid; and the methods by which it was accomplished depend essentially on the ancient theory of conic sections.

This, I believe, was the first important fruitage of that part of Greek mathematical speculation which we are considering; and, as the first, it is the beginning of a long line of magnificent discoveries which is not yet terminated. Let us follow the thread down to our own time, even though we shall have to watch closely to avoid losing it in the great body of our present day science.

A few years after the death of Bacon and of Kepler the great English physicist Sir Isaac Newton was born (1642). The great advances in astronomy due to the discoveries of Kepler naturally turned men's thought in that direction. In the early years of his life Newton meditated upon the elegant laws of Kepler. The question uppermost in his mind was this: What is the ultimate law of nature upon which the properties of the planetary orbits depend? Following out this line of investigation he was led to conceive the law of universal gravitation and to conjecture that the force of attraction between two bodies was directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

This law was simple enough. Could it be tested? Could it be applied in the derivation of the laws of planetary motion? Newton found that the laws of Kepler could be derived from his theory of gravitation. On the other hand it was also possible to derive his theory from Kepler's laws. The Greek theory of conic sections again played an important rôle in these discoveries. This was another fruitage of the ancient sowing.

It thus appears that speculations which were engaged in by the early philosophers solely from intellectual delight in them, became the corner-stone of the explanation of the Universe afforded by Newton's law of gravitation. Probably no discovery of man will ever surpass this in importance; for there is only one Universe in the large to explain, and Newton has explained this one.

The discoveries of Newton became the basis of the beautiful and extensive science of Celestial Mechanics. The general theory was first presented from a single point of view by Laplace in his *Mécanique Céleste*. As stated at the beginning of the first volume of this work, it was the author's purpose to "offer a complete solution of the great mechanical problem presented by the solar system, and to bring theory to coincide so closely with observation that empirical equations should no longer find a place in astronomical tables." His success in both respects brought him very close to a realization of his lofty ideal. He built his own researches and that of his contemporaries and predecessors into a magnificent explanation of the system of the universe. As so often happens, when a great work is finished, many years passed before essential additions were made to the theory; men were engaged in mastering it, in supplying its lacunae and developing to a further extent some conclusions contained in it. The late Henri Poincaré in his *Nouvelles Methodes* has in recent years made a notable and essentially new contribution to the theory. In all of this the theory of Newton is fundamental.

Let us return to the classic work of Laplace. The principal part of this, consisting of the first four volumes, was completed in the early years of the nineteenth century. This treatise holds a unique place in the history of science. It was the first instance in which a unified and logical theory (of such extent and importance) had been developed for a large class of observed phenomena. A very few fundamental principles lay at the basis of Laplace's entire work; and it was developed from these in its entirety solely by logical (that is, mathematical) processes. In such a body of truth there is something esthetically satisfying in a high degree. It could not fail to have a profound effect on the development of thought.

In fact, Celestial Mechanics became at once the model for the exact sciences. Men sought to give to other theories an equally beautiful and logically consistent form. The start from a few simple principles, easily enunciated and readily comprehended; the forward march of theory into new fields, comprising in the range of its explanation an ever-increasing portion of observed phenomena; and its ultimate comprehensiveness in this respect—these things gave it a hold on the imagination. It thus became a profound factor in the development of the whole of physical science in its mathematical form.

One has only to observe the remote origins of these theories to see what a profound rôle in scientific progress is played by the ancient Greek conic sections.

But there is still more to be said; it is becoming increasingly difficult to follow our thread, but still we shall be able to trace it further if we look carefully. We have just seen that the form of mathematical physics has been largely determined by that of Celestial Mechanics. Now experimental and theoretical physics have been profoundly influenced by mathematical speculation, both as to the direction of progress and as to the content of the discoveries.

In particular, Maxwell's theory of electricity and magnetism could not have been developed when it was and as it was without making essential use of mathematical physics. In fact, it is itself primarily a branch of mathematical physics.

Maxwell's theory, as is known to every student of science, has been a powerful element in leading to the recent advances in experimental physics. Some of the facts predicted in it waited twenty years for verification in the laboratory. Many conclusions reached in a speculative way were suggestive of new experiments; and these have been carried out—with essential addition to our knowledge of the nature and properties of electricity. It would be hard to overestimate the importance of Maxwell's work to the modern theory of electrons and that whole body of facts connected with them which give distinction and glory to the discoveries of the past few years.

Looking back now over our completed historical sketch we see that if all these developments had come without the ancient Greek theory of conic sections they must have come in an essentially different way. What are the consequences of this fact? That they actually came thus is probably sufficient evidence that this was the readiest way, and also the best for a mind constructed as ours is. From this we may conclude that if these discoveries had been made otherwise they would have been later in coming. But we have no assurance that they would ever have been made at all. When we notice to what extent it appears that discoveries are made by chance and how often an accidental juxtaposition of thoughts leads to the detection of a new principle, we feel that it would be hazardous to miss one link from the chain of scientific progress. We have not yet learned how to systematically explore new territories of thought. We can only

look over the field at random and hope to find here and there a pearl of great worth. On this account we cannot afford to lose anything which assists us to success. Every help, however remote, is essential; and we must hold it precious in our sight.

It would be foolish for us to conclude that modern physical science is a consequence of the old Greek mathematics. Many elements of diverse origin have entered into its structure—of philosophical speculation, of mathematical analysis, of experimental observation—and each of these has played an essential rôle. None can be omitted without marring or even destroying the whole.

Modern science is a web woven of threads from many sources, of which one is the Greek theory of conic sections. This thread is essential. Many others have a like place of importance, and all of these have the same claim to recognition. For the sake of definiteness we have put the emphasis upon one alone, because we have sought to trace out in one direction the value of a certain ancient speculation.

And we have indeed followed it out in only one direction. If we should seek to examine further the value of this same theory of the olden time, we should first have to observe that numerous threads of history spring from the same source. We have considered only one, and that perhaps not the most important. It would next be necessary to take up each of the others in their turn and find how they lead us into some important modern theory—whether of geometry or mathematical analysis or physical science or even philosophy—and see how they are thus woven into widely diverse parts of our present body of knowledge and opinion.

It is only when all of these have been considered in detail that we should be able to realize the magnificent consequences which owe to this ancient speculation an essential element in their development. Could those thinkers of the past have anticipated this glorious fruitage of their labors, with what a spirit of enthusiasm would they have performed their researches!

But the whole tale is not yet told. Further advances are yet to be essentially dependent upon this same fundamental basis. When one realizes the wide ramifications of the influence of research, once it is done, what greater incentive can one have to strive for the advancement of human knowledge—either directly by his own research or indirectly by his teaching others who shall

some day devote themselves to the conquest of the unknown!

The above discussion has an important corollary which we shall now state. It has to do with the problem of apparently useless research—especially in the realm of the mathematical sciences. There must be a large part of the science of every generation for which that generation knows no other use than the esthetic pleasure in its contemplation. What should be our attitude toward this part of the research of our contemporaries?

Let us recall that many generations came and went between the time of ancient Greek speculation and the gathering of the first fruitage in the researches of Kepler. But time has shown that the value to man of that early work was immeasurable; already we reap an abundant harvest. This is not an isolated case in the history of research. In fact, it is typical of a normal development. It is the way of science to burst into magnificent flower in unexpected places. We can never foresee its possibilities.

Consequently, we must treat every advance in permanent knowledge as a precious contribution to the wealth and worth of mankind, and must support it with our enthusiastic interest. America has not yet learned this lesson as she should and must learn it. The teachers of our country are to induct our people into a proper appreciation of the constructive thinking done among us.

The example which we have given is sufficient to show that future ages will remember gratefully the work which is accomplished under such encouragement.

It would doubtless be easy to find other illustrations to indicate the like value of researches in every field of scientific activity.

II.

An historical account of the past and distant is of no value to us unless we can bring it down into the here and now. Then let us ask what questions are suggested by, and what lessons are to be learned from, the history which we have reviewed together?

Consider the general subject which you are teaching today. What are the important threads of history with which the subject is shot through at the present time? What are those which came up from the past? and what are they which are just now making their appearance?

Of all these, which should be taught to a student at a given particular stage of his development—say at the stage at which you teach him?

These questions it is difficult to answer in the wisest way. The Greek theory of conic sections was studied and kept alive from generation to generation although the only reason was the esthetic delight in it. Can any one regret that it was so? In our day there is a great clamor for the practical—and the practical no doubt should be had. How much of the other should be retained? This generation is going to have to answer that question anew. Indeed, it is already thrust upon us. I am one who believes that we shall answer this question wisely only when we have studied it in the light of a profound knowledge of the forces which have heretofore determined human progress.

The glorious past must be our schoolmaster in the present—but it cannot be our taskmaster. We are living in a day of progress. I have no sort of notion that the great things were all done in the past; nor even that the most fundamental problems were all proposed. We shall ask far-reaching questions and shall also do notable things in our day and generation. The lesson of the past, however glorious, is not that we shall do the things of the past. It is rather that we shall do what seems to us good as the fathers did what seemed to them good.

We must solve anew for our generation many of the problems which the fathers solved for theirs. We have their example to inspire; but their results we can accept only as tentative. We must study out the questions anew for ourselves.

To take a special case, let us ask: What threads of mathematical history and mathematical knowledge are to be woven into the warp and woof of our young people's education? The scientific story which we have studied together shows the vast importance of our answering this question rightly. No less a matter is involved than the future progress of human thought. But it is obvious that the question is a difficult one. There are others similar to it of like difficulty and importance; every field of study affords one.

We have come upon a period of unrest. Nothing scarcely is accepted as established beyond question. The reasons for teaching every subject and every topic in each subject are undergoing critical scrutiny. This is evidenced by the large number of recent text-books and their varying content as well as by the frequent

and interested discussion among thoughtful teachers everywhere.

Many of our problems we have to solve again.

It is good for us. It is the only way to be progressive. No human being can ever do a thing so well that some other human being may not do it better. In the march of progress there is no room for the man who always wants to let well enough alone. That has been the attitude of monkeys; and it is one reason, I suppose, why they have never become men.

Nature hates a standpatter, and she never allows him the greatest success. We must always be taking hold of our questions—even the oldest ones—as if they have not been answered. It was refreshing to hear the illustrious Poincaré, a few years before his lamented death, ask in earnest whether the earth actually moves around the sun.

To settle the questions, old and new, which now confront us will require the combined wisdom of all of us.

At the present time it is hard to give a satisfactory reason why some subjects in our curricula should be there; it is harder still to see why others are excluded. Our generation is to witness a revision of courses of study which will make such a statement no longer possible. We want to gather all the threads of history out of the past, we want to find all those which are being spun today; then we must examine them, enquiring which of them can be used to best advantage in weaving the most beautiful and enduring fabric of modern civilization. These must be conserved, these must be wrought into human character through the processes of education.

For this to be done promptly and efficiently we need to bring out into the open light of free discussion our opinions, our practices and our arguments. It is a time for frankness; not merely the frankness which readily confesses all, but the frankness which finds opportunity to declare all. It is important that I know promptly and accurately what you think—what you think about your own work and what you think about mine. We need to see this whole matter from as many different angles as we have individual workers. We need also to see it from the point of view of intelligent men and women who are not professionally engaged in education. There is no other wisdom in a community so great as the combined wisdom of the whole community. It is this which we need in the solution of our present problems.

In order to get at our combined wisdom we shall need to

deliberate together. We require also to find some means of bringing our communities together for counsel. This latter in itself is a revolutionary proposal; but it is one which has been made by several persons who have approached our great modern problems from widely different points of view. We need to develop a public opinion in education which shall be virile and just. This opinion must be guided and developed by those who know the history of human thought and the needs of the present hour.

For the solution of our educational problems we also require suitable places of publication in which to record our more important ideas. America perhaps has fewer of these in proportion to the magnitude of her problems than any other of the greater countries. For those engaged in the teaching of mathematics there are two journals which stand out as especially valuable—SCHOOL SCIENCE AND MATHEMATICS and *The American Mathematical Monthly*. It is important that these journals receive the cordial and hearty support of teachers.

It is further necessary that the general discussion of our problems be examined by a large number of those engaged in educational work. The more thoughtful and valuable parts of this discussion are to be found regularly in the two journals mentioned. I want to urge teachers of mathematics to read both of these journals and all science teachers to read the former.

It is important not only that you shall know the opinions of others but also that you shall express your own—several of you no doubt in the permanent form of printed papers. It is thus that we shall disseminate most widely our varying opinions and contribute most to the early solution of our problems concerning the curriculum.

We are at the dawn of a new era in the teaching of mathematics, and probably of all the sciences. We must gather into the warp and woof of our courses other threads than those which have heretofore been woven in. The future is big with the possibilities which the present foretakens. I know that you also are inspired with the forelook of human progress. No other attitude is possible when we contemplate the magnificent achievement in the past through labors in every respect parallel to our own. It is beautiful to be comrades on such a road of progress; and I for one wish that our comradeship might grow more intimate.