A simple Proof of the Existence of Irreducible Invariants of Degrees 20 and 30 for the Binary Seventhic.

By

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"Made whole with very easy arguments".

(King John, act 1, sc. 1).

The present paper contains an a priori solution of the question, whether it is possible or not for the Binary Seventhic to have an irreducible invariant whose degree is any multiple of 10. As an immediate consequence of the existence of such invariants, the representative form of the Generating Function for the covariants of the Seventhic can be expressed in many ways as a fraction with a finite numerator. The simplest expression for this Function is given at the end of the paper.

The degree of any invariant of the Seventhic must (as is well known) be an even number, which cannot be 2, 6, or 10.

Let $2m$ be the degree (and $7m$ the weight) of an invariant which does not vanish when the Seventhic has the special form

$$(a, 0, 0, 0, 0, f, 0, 0)(x, y)^7.$$

Since this invariant (if it exists) must contain a term $a^2 f^\mu$, we have

$$\lambda + \mu = 2m \quad \text{and} \quad 5\mu = 7m,$$

which can only be satisfied by writing

$$m = 5n, \quad \lambda = 3n, \quad \mu = 7n.$$

Hence the only invariants which do not vanish are of degree $10n$ and contain a term $a^{3n} f^{7n}$.

But, since no invariant of degree 10 exists, $n > 1$, so that the only invariants which do not vanish (if any such exist) are of the degrees

$$20, \quad 30, \quad 40, \quad 50, \quad \dots$$

and contain the terms $a^6 f^{14}, a^9 f^{21}, a^{12} f^{28}, a^{15} f^{35}, \dots$

Let us call them $I_{20}, I_{30}, I_{40}, I_{50}, \dots$

then, since it is obvious that every *reducible* invariant, whose degree is either 20 or 30, vanishes for the special form assigned to the Seventhic*), it is only necessary, in order to completely establish the irreducibility of I_{20} and I_{30} to prove that neither of them vanishes when the Seventhic has this special form.

This is effected by identifying I_{20} and I_{30} with two of the invariants (of degrees 10 and 15, respectively, in its coefficients) of the well known sextic covariant whose leading term is

$$(ae - 4bd + 3c^2)x^6.$$

The complete value of this covariant is

$$\begin{aligned} & (ax^3 + 3bx^2y + 3cxy^2 + dy^3)(ex^3 + 3fx^2y + 3gxy^2 + hy^3) \\ & - 4(bx^3 + 3cx^2y + 3dxy^2 + ey^3)(dx^3 + 3ex^2y + 3fxy^2 + gy^3) \\ & + 3(cx^3 + 3dx^2y + 3exy^2 + fy^3)^2, \end{aligned}$$

which, for the special Seventhic under consideration, reduces to

$$3(afx^5y + f^2y^6).$$

All the roots of this special sextic are unequal, and therefore its discriminant (which is of degree 10 in its coefficients) does not vanish. This discriminant then is the irreducible I_{20} we have been in search of; and a direct calculation of its value will show that it reduces to a definite numerical multiple of a^6f^{14} .

It can also be shown in many ways that the invariant of degree 15 in the coefficients of the above sextic covariant does not vanish for the special form in which

$$b = c = d = e = g = h = 0.$$

Perhaps the easiest way is to take the invariantive syzygy for the sextic, which von Gall (Math. Ann. Bd. XXXV, S. 80) writes as follows

$$2R^2 - \begin{vmatrix} A_{ll} & A_{lm} & D \\ A_{lm} & D & A_{mn} \\ D & A_{mn} & A_{nn} \end{vmatrix} = 0.$$

All the A 's vanish for our special form, but D becomes a definite numerical multiple of the discriminant I_{20} which does not vanish. Consequently R (the 15th invariant in question) does not vanish, and may therefore be identified either with our irreducible I_{30} or, if we

*) This argument does not apply to the degrees 40, 50, or any higher multiple of 10; for, among the invariants of degrees 40 and 50, we have the *reducible* invariants I_{20}^2 and $I_{20}I_{30}$, which *do not vanish*; and so for all higher multiples of 10.

please, with a numerical multiple of I_{30} so chosen that v. Gall's syzygy reduces to $I_{30}^2 = I_{20}^3$ for the special form under consideration*).

The method we have employed only shows that the Binary Seventhic has *at least* one irreducible invariant of degree 20, and *at least* one of degree 30, but gives no further information as to the actual number of such invariants**). What we have proved is, notwithstanding, sufficient to enable us to express the representative Generating Function in many ways as a fraction with a finite numerator, though it will not help us to see the exact number of ways in which this is possible.

We shall now show how the simplest *representative* form of the Generating Function may be obtained.

When expressed in its simplest form the Generating Function for the covariants of the Binary Seventhic is an ordinary algebraic fraction whose denominator is

$$(1 - a^4) (1 - a^6) (1 - a^8) (1 - a^{10}) (1 - a^{12}) (1 - ax) (1 - ax^3) \\ (1 - ax^5) (1 - ax^7),$$

*) I take this opportunity of congratulating v. Gall on the extreme elegance of the forms which he has given to his Sextic Syzygies, and of expressing the hope that he will calculate afresh the omitted Syzygies of the 9th and lower degrees, so as to have the complete set in their simplest form and in a uniform notation. The results in the numerical table at the end of his paper are in exact accordance with those obtained by me shortly after publishing the paper (in the American Journal of Mathematics, Vol. VII) to which he refers for the syzygies in question. But though all the syzygies were in my possession, they were so inelegantly expressed that I refrained from publishing them, hoping at some time or other to find leisure to put them into a better shape. I have never done so; and the syzygies, entered in my note-book rather more than 4 years ago, still remain there "with all their imperfections on their head".

As an example, contrast v. Gall's expression for the invariantive syzygy, given in the text, with the following expression for the same syzygy copied from my note-book

$$Z^2 + \alpha W + \beta(4B^2I^2 + 8BIP + 48I^3 + 3P^2) \\ + 2\gamma(2B^3I^2 - 8BI^3 - BP^2 + 6IW) = 0,$$

where

$$\alpha = 4W^2 - B^2PW - 3BP^3 + 4I(B^3IP + 176B^2I^3 + 204BI^2P - 36BIW \\ + 576I^4 + 36IP^2 - 18PW),$$

$$\beta = BW(3P + 4BI) + P(3P + 4BI)^2 + 4(B^2I + BP + 12I^2) \\ \cdot (3IP + 4BI^2 - W),$$

$$\gamma = (BP + 48I^2)(B^2I + BP + 12I^2) + 4I(3P + 4BI)^2 - W(3P + 4BI),$$

and B, I, P, W, Z are used to denote the same invariants (of degrees 2, 4, 6, 10, 15, respectively) as in Cayley's "Tables for the Binary Sextic" (American Journal, Vol. IV, pp. 379—384).

**) It has however been proved by v. Gall (Math. Ann. Bd. XXXI, 1887) that there are *two* irreducible invariants of degree 20 and one of degree 30. See his „Formensystem der binären Form 7^{ter} Ordnung."

and whose numerator is a *finite* series of terms of the form $\pm Na^\lambda x^\mu$, where N, λ, μ are positive integers*).

Here each of the factors $(1 - a^4), (1 - a^8), (1 - a^{12}), (1 - ax^7)$ represents an irreducible invariant or covariant (viz. $1 - ax^7$ represents the Seventhic, and $1 - a^4, 1 - a^8, 1 - a^{12}$ represent irreducible invariants of the degrees 4, 8, 12), but the remaining factors $(1 - a^6), (1 - a^{10}), (1 - ax), (1 - ax^3), (1 - ax^5)$ do not correspond to any invariants or covariants; for the Seventhic has no invariant whose degree is either 6 or 10, and no covariant of any of the degree-orders $(1 \cdot 1), (1 \cdot 3),$ or $(1 \cdot 5)$.

Now any representative form of the Generating Function may be obtained by transforming it into an equivalent fraction in which *every* factor of the denominator corresponds to an irreducible invariant or covariant. This may be done in many ways; for example, since the Seventhic has irreducible covariants whose degree-orders are $(2 \cdot 2), (3 \cdot 3), (4 \cdot 4), (5 \cdot 5), (6 \cdot 6), (7 \cdot 7)$ we may multiply both numerator and denominator by $1 + ax + a^2x^2 + \dots + a^{x-1}x^{x-1}$, and thus replace the factor $1 - ax$, which occurs in the denominator, by the representative factor $1 - a^xx^x$, where x has any of the values 2, 3, 4, 5, 6, 7: and in like manner the factor $1 - a^{10}$ may be replaced by $1 - a^{20}$ or by $1 - a^{30}$, for either of these will correspond to an irreducible invariant.

Professor Sylvester's representative form is found by multiplying both the numerator and denominator of the above fraction (his "reduced form") by

$$(1 + a^6)(1 + a^{10} + a^{20} + a^{30} + \dots \text{ad. inf.})(1 + ax)(1 + ax^3)(1 + ax^5),$$

so that the denominator becomes

$$(1 - a^4)(1 - a^8)(1 - a^{12})^2(1 - a^2x^2)(1 - a^2x^6)(1 - a^2x^{10})(1 - ax^7),$$

and the numerator is a finite function of a, x multiplied by the infinite series $1 + a^{10} + a^{20} + a^{30} + \dots$.

But for the existence of an irreducible invariant whose degree is a multiple of 10, this would be the simplest possible expression for the representative form of the Generating Function: as it is, the existence of irreducible invariants of degrees 20 and 30 enables us to find two simpler expressions for it, in each of which the numerator is finite. We have, in fact, only to multiply both numerator and denominator of Sylvester's representative form either by $1 - a^{20}$ or by

*) For its actual value see either of the following papers in the American Journal: Cayley, "Calculation of the Minimum N. G. J. of the Binary Seventhic". (Vol. II, pp. 71-84). — Sylvester, assisted by F. Franklin: "Tables of the Generating Functions and Groundforms for the Binary Quantics of the first Ten Orders". (Vol. II, pp. 223-251).

$1 - a^{30}$, and then the infinite series which occurs as a factor of the numerator is replaced either by $1 + a^{10}$ or by $1 + a^{10} + a^{20}$.

Multiplying then by $1 - a^{20}$ we obtain the simplest possible expression for the representative generating function, which is a fraction with the denominator

$$(1 - a^4) (1 - a^8) (1 - a^{12})^2 (1 - a^{20}) (1 - a^2 x^2) (1 - a^2 x^6) \\ (1 - a^2 x^{10}) (1 - a x^7),$$

its numerator being

$$\begin{aligned} & 1 + a^3(x^3 + x^5 + x^7 + x^9 + x^{11} + x^{15}) \\ & + a^4(2x^4 + x^6 + 2x^8 + x^{10} + x^{14}) \\ & + a^5(x + 2x^3 + 2x^5 + 2x^7 + 2x^9 - x^{17} - x^{21}) \\ & + a^6(3x^2 + 2x^4 + 3x^6 + 3x^8 + 2x^{12} - x^{14} - x^{16}) \\ & + a^7(3x + 2x^3 + 4x^5 + 4x^7 + x^{11} - 2x^{15} - x^{19} + x^{23}) \\ & + a^8(2 + 3x^2 + 4x^4 + 6x^6 + x^8 + 3x^{10} - x^{12} - 2x^{14} - x^{18}) \\ & + a^9(3x + 5x^3 + 7x^5 + x^7 + 4x^9 + 2x^{13} - x^{15} - 2x^{17} + x^{21}) \\ & + a^{10}(5x^2 + 8x^4 + 6x^6 + 4x^8 + x^{10} - 4x^{12} - 3x^{16} - x^{18}) \\ & + a^{11}(5x + 8x^3 + 8x^5 + 8x^7 + 4x^9 - 4x^{11} - x^{13} - 5x^{15} - x^{17}) \\ & + a^{12}(4 + 9x^2 + 9x^4 + 12x^6 + 4x^8 - x^{10} - 3x^{12} - 5x^{14} - 6x^{16} \\ & \quad - x^{20} + x^{22}) \\ & + a^{13}(9x + 9x^3 + 12x^5 + 6x^7 - x^9 - 3x^{11} - 8x^{13} - 9x^{15} - 3x^{17} \\ & \quad - x^{19} + x^{21}) \\ & + a^{14}(4 + 9x^2 + 13x^4 + 11x^6 - x^8 - 3x^{10} - 9x^{12} - 10x^{14} - 7x^{16} \\ & \quad - 2x^{18} + 3x^{22}) \\ & + a^{15}(9x + 12x^3 + 16x^5 + 3x^7 + 2x^9 - 10x^{11} - 11x^{13} - 8x^{15} \\ & \quad - 3x^{17} + 3x^{21} + 2x^{23}) \\ & + a^{16}(5 + 14x^2 + 15x^4 + 12x^6 + x^8 - 5x^{10} - 16x^{12} - 9x^{14} \\ & \quad - 9x^{16} - x^{18} + 3x^{20} + 3x^{22}) \\ & + a^{17}(12x + 15x^3 + 16x^5 + 6x^7 - 3x^9 - 17x^{11} - 13x^{13} - 15x^{15} \\ & \quad - 5x^{17} + 2x^{19} + 3x^{21}) \\ & + a^{18}(9 + 14x^2 + 15x^4 + 14x^6 - 3x^8 - 13x^{10} - 20x^{12} - 15x^{14} \\ & \quad - 15x^{16} + 2x^{18} + 2x^{20} + 5x^{22}) \\ & + a^{19}(15x + 16x^3 + 18x^5 - 8x^9 - 18x^{11} - 20x^{13} - 19x^{15} - 3x^{17} \\ & \quad + 3x^{19} + 5x^{21} + 4x^{23}) \\ & + a^{20}(6 + 14x^2 + 18x^4 + 12x^6 - 10x^8 - 16x^{10} - 25x^{12} - 19x^{14} \\ & \quad - 12x^{16} + 2x^{18} + 5x^{20} + 9x^{22}) \\ & + a^{21}(14x + 17x^3 + 19x^5 - x^7 - 8x^9 - 27x^{11} - 25x^{13} - 16x^{15} \\ & \quad - 2x^{17} + 4x^{19} + 8x^{21} + 4x^{23}) \\ & + a^{22}(9 + 17x^2 + 19x^4 + 11x^6 - 8x^8 - 18x^{10} - 31x^{12} - 17x^{14} \\ & \quad - 15x^{16} + 6x^{18} + 9x^{20} + 9x^{22}) \end{aligned}$$

$$\begin{aligned}
& + a^{23}(17x + 18x^3 + 17x^5 - 4x^7 - 14x^9 - 32x^{11} - 25x^{13} - 22x^{15} \\
& \quad - 4x^{17} + 9x^{19} + 9x^{21} + 5x^{23}) \\
& + a^{24}(8 + 17x^2 + 15x^4 + 9x^6 - 14x^8 - 28x^{10} - 32x^{12} - 23x^{14} \\
& \quad - 16x^{16} + 9x^{18} + 9x^{20} + 12x^{22}) \\
& + a^{25}(17x + 15x^3 + 17x^5 - 8x^7 - 19x^9 - 31x^{11} - 28x^{13} - 22x^{15} \\
& \quad + 4x^{17} + 10x^{19} + 13x^{21} + 9x^{23}) \\
& + a^{26}(9 + 15x^2 + 16x^4 + 8x^6 - 20x^8 - 23x^{10} - 36x^{12} - 20x^{14} \\
& \quad - 9x^{16} + 10x^{18} + 14x^{20} + 15x^{22}) \\
& + a^{27}(14x + 15x^3 + 15x^5 - 13x^7 - 16x^9 - 37x^{11} - 29x^{13} - 17x^{15} \\
& \quad + 3x^{17} + 14x^{19} + 14x^{21} + 6x^{23}) \\
& + a^{28}(6 + 14x^2 + 14x^4 + 3x^6 - 17x^8 - 29x^{10} - 37x^{12} - 16x^{14} \\
& \quad - 13x^{16} + 15x^{18} + 15x^{20} + 14x^{22}) \\
& + a^{29}(15x + 14x^3 + 10x^5 - 9x^7 - 20x^9 - 36x^{11} - 23x^{13} - 20x^{15} \\
& \quad + 8x^{17} + 16x^{19} + 15x^{21} + 9x^{23}) \\
& + a^{30}(9 + 13x^2 + 10x^4 + 4x^6 - 22x^8 - 28x^{10} - 31x^{12} - 19x^{14} \\
& \quad - 8x^{16} + 17x^{18} + 15x^{20} + 17x^{22}) \\
& + a^{31}(12x + 9x^3 + 9x^5 - 16x^7 - 23x^9 - 32x^{11} - 28x^{13} - 14x^{15} \\
& \quad + 9x^{17} + 15x^{19} + 17x^{21} + 8x^{23}) \\
& + a^{32}(5 + 9x^2 + 9x^4 - 4x^6 - 22x^8 - 25x^{10} - 32x^{12} - 14x^{14} \\
& \quad - 4x^{16} + 17x^{18} + 18x^{20} + 17x^{22}) \\
& + a^{33}(9x + 9x^4 + 6x^5 - 15x^7 - 17x^9 - 31x^{11} - 18x^{13} - 8x^{15} \\
& \quad + 11x^{17} + 19x^{19} + 17x^{21} + 9x^{23}) \\
& + a^{34}(4 + 8x^2 + 4x^4 - 2x^6 - 16x^8 - 25x^{10} - 27x^{12} - 8x^{14} \\
& \quad - x^{16} + 19x^{18} + 17x^{20} + 14x^{22}) \\
& + a^{35}(9x + 5x^3 + 2x^5 - 12x^7 - 19x^9 - 25x^{11} - 16x^{13} - 10x^{15} \\
& \quad + 12x^{17} + 18x^{19} + 14x^{21} + 6x^{23}) \\
& + a^{36}(4 + 5x^2 + 3x^4 - 3x^6 - 19x^8 - 20x^{10} - 18x^{12} - 8x^{14} \\
& \quad + 18x^{18} + 16x^{20} + 15x^{22}) \\
& + a^{37}(5x + 2x^3 + 2x^5 - 15x^7 - 15x^9 - 20x^{11} - 13x^{13} - 3x^{15} \\
& \quad + 14x^{17} + 15x^{19} + 14x^{21} + 9x^{23}) \\
& + a^{38}(3x^2 + 2x^4 - 5x^6 - 15x^8 - 13x^{10} - 17x^{12} - 3x^{14} + 6x^{16} \\
& \quad + 16x^{18} + 15x^{20} + 12x^{22}) \\
& + a^{39}(3x + 3x^3 - x^5 - 9x^7 - 9x^9 - 16x^{11} - 5x^{13} + x^{15} + 12x^{17} \\
& \quad + 15x^{19} + 14x^{21} + 5x^{23}) \\
& + a^{40}(2 + 3x^2 - 3x^6 - 8x^8 - 11x^{10} - 10x^{12} + 2x^{14} + 3x^{16} \\
& \quad + 16x^{18} + 12x^{20} + 9x^{22}) \\
& + a^{41}(3x - 2x^5 - 7x^7 - 10x^9 - 9x^{11} - 3x^{13} - x^{15} + 11x^{17} \\
& \quad + 13x^{19} + 9x^{21} + 4x^{23}) \\
& + a^{42}(x^2 - x^4 - 3x^6 - 9x^8 - 8x^{10} - 3x^{12} - x^{14} + 6x^{16} + 12x^{18} \\
& \quad + 9x^{20} + 9x^{22})
\end{aligned}$$

$$\begin{aligned}
& + a^{43}(x - x^3 - 6x^7 - 5x^9 - 3x^{11} - x^{13} + 4x^{15} + 12x^{17} + 9x^{19} \\
& \qquad \qquad \qquad + 9x^{21} + 4x^{23}) \\
& + a^{44}(-x^6 - 5x^8 - x^{10} - 4x^{12} + 4x^{14} + 8x^{16} + 8x^{18} + 8x^{20} + 5x^{22}) \\
& + a^{45}(-x^5 - 3x^7 - 4x^{11} + x^{13} + 4x^{15} + 6x^{17} + 8x^{19} + 5x^{21}) \\
& + a^{46}(x^3 - 2x^6 - x^8 - 2x^{10} + 4x^{14} + x^{16} + 7x^{18} + 5x^{20} + 3x^{22}) \\
& + a^{47}(-x^5 - 2x^9 - x^{11} + 3x^{13} + x^{15} + 6x^{17} + 4x^{19} + 3x^{21} + 2x^{23}) \\
& + a^{48}(1 - x^4 - 2x^8 + x^{12} + 4x^{16} + 4x^{18} + 2x^{20} + 3x^{22}) \\
& + a^{49}(-x^7 - x^9 + 2x^{11} + 3x^{15} + 3x^{17} + 2x^{19} + 3x^{21}) \\
& + a^{50}(-x^2 - x^6 + 2x^{14} + 2x^{16} + 2x^{18} + 2x^{20} + x^{22}) \\
& + a^{51}(x^9 + x^{13} + 2x^{15} + x^{17} + 2x^{19}) \\
& + a^{52}(x^8 + x^{12} + x^{14} + x^{16} + x^{18} + x^{20}) \\
& + a^{55}x^{23}.
\end{aligned}$$

It is not worth while to write down the somewhat less simple form of the Generating Function with the denominator $(1-a^4)(1-a^8)(1-a^{12})^2(1-a^{30})(1-a^2x^2)(1-a^2x^6)(1-a^2x^{10})(1-ax^7)$ which only differs slightly from the above: viz. its denominator contains the factor $1-a^{30}$ instead of $1-a^{20}$, and its numerator extends to $a^{65}x^{23}$ instead of to $a^{55}x^{23}$.