characteristic sub-group with respect to which the entire group is isomorphic with a cyclical group whose order is a divisor of $p-1$.

Most of these results can be readily applied to the theory of abstract groups. If a group contains a maximal sub-group that does not include any self-conjugate sub-group of the entire group besides identity, and if the order of this maximal sub-group is the quotient obtained by dividing the order of the entire group by n , then it is possible to represent the entire group as a primitive substitution group of degree n .* Hence it follows from what precedes that the operators of order p which are contained in a group in which n may be equal to p or $p+2$ generate a simple group, and that such a group cannot have more than one composite factor of composition. Similar remarks can evidently be made in regard to most of the other results.

Finite Current-Sheets. By J. H. JEANS. Received May 30th, 1899. Communicated by G. T. WALKER, M.A., June 8th, 1899.

1. The theory of the currents induced in an infinite current sheet has been completely worked out by Maxwell (*Elect. and Mag.*, §§ 647-666). I have attempted to develop a similar theory for the case of a finite current-sheet.

As in the case of the infinite current-sheet, the mathematical discussion falls naturally into two parts: in the former it is necessary to determine the currents induced by an instantaneous change in the inducing field; in the latter to examine the dissipation of these currents under the action of resistance and self-induction.

In §§ 4 and 5 I have discussed these two questions for the particular case of a semi-infinite plane, but the method is quite general, as I have tried to show in § 8. § 7 contains the extension of former results to the two cases of an infinite strip with parallel edges, and an infinite plane from which such a strip has been removed. In § 6 Maxwell's theory for an infinite plane is deduced

* Dyck, *Mathematische Annalen*, Vol. xxii., 1883, p. 102.

from the results obtained for the semi-infinite plane bounded by a straight edge by supposing the straight edge to be at infinity.

§ 2 merely contains the transformation of the ordinary electrical equations into a form suitable for dealing with any finite current-sheet.

2. We start with a current-function Φ defined as by Maxwell (§ 648). A knowledge of Φ at every point will give us a complete knowledge of the currents in the sheet. The magnetic effect of the current-sheet at any external point is the same as that of a magnetic shell of strength $C + \Phi$, where C is a constant so chosen that $C + \Phi$ vanishes at the boundary.

In dealing with plane finite current-sheets we shall suppose Φ measured from a point on the boundary, so that $C = 0$.

We next introduce Maxwell's function P , defined for all points in space by the equation

$$P = \iint \frac{\Phi}{r} dx' dy', \quad (1)$$

where $dx' dy'$ is an element of the current-sheet at which the current-function is Φ , and r is the distance from this element to the point at which we are evaluating P .

P is the gravitational potential due to a distribution of density Φ over the surface of the sheet, and therefore the magnetic potential Ω due to the currents in the sheet is given by

$$\Omega = - \frac{dP}{dz}. \quad (2)$$

Also, since the value of P is symmetrical as regards the two surfaces of the sheet, we have at the positive surface

$$\Omega = - \frac{dP}{dz} = 2\pi\Phi,$$

and at the negative surface

$$\Omega = - \frac{dP}{dz} = - 2\pi\Phi.$$

If F , G , H are the components of the vector potential due to these currents, we may take

$$F = \frac{dP}{dy}, \quad G = - \frac{dP}{dx}, \quad H = 0,$$

for these satisfy the equations

$$\left. \begin{aligned} \frac{dH}{dy} - \frac{dG}{dz} &= -\frac{d\Omega}{dx} \\ \frac{dF}{dz} - \frac{dH}{dx} &= -\frac{d\Omega}{dy} \\ \frac{dG}{dx} - \frac{dF}{dy} &= -\frac{d\Omega}{dz} \end{aligned} \right\},$$

in virtue of equation (2).

Let F' , G' , H' be the components of the vector potential of the inducing field, and P' a quantity analogous to P , defined by

$$\Omega' = -\frac{dP'}{dz}, \quad (3)$$

where Ω' is the scalar potential of the inducing field.

P' is, so far, defined except for an arbitrary function of x and y , and this can be so chosen that P' shall be the potential of matter none of which is indefinitely near to the current-sheet. For a value of P which satisfies equation (3) is the potential of thin cylinders of matter starting from each pole of the external field, and going to infinity in the direction of the positive axis of z , the line-density of any cylinder being equal to the strength of the pole from which it started. Take as the additional arbitrary function of x and y the potential due to a series of cylinders coinciding with such of the above cylinders as pass through the current-sheet, and of equal but opposite strength. Then P' will satisfy the condition mentioned above.

Exactly as in the former case, we may take

$$F' = \frac{dP'}{dy}, \quad G' = -\frac{dP'}{dx}, \quad H' = 0.$$

It is now possible to write down the electrical equations for the current-sheet in the form

$$\left. \begin{aligned} X &= \sigma \frac{d\phi}{dy} = -\frac{d}{dt} (F + F') - \frac{d\psi}{dx} \\ Y &= -\sigma \frac{d\phi}{dx} = -\frac{d}{dt} (G + G') - \frac{d\psi}{dy} \end{aligned} \right\}. \quad (4)$$

Let us now introduce two new functions χ, χ' defined for all space by the equations

$$\left. \begin{aligned} \chi &= \frac{\sigma}{2\pi} \frac{dP}{dz} - \frac{d}{dt} (P + P') \\ \chi' &= -\frac{\sigma}{2\pi} \frac{dP}{dz} - \frac{d}{dt} (P + P') \end{aligned} \right\} \quad (5)$$

At the positive surface of the current-sheet

$$\begin{aligned} \frac{d\chi}{dy} &= -\sigma \frac{d\phi}{dy} - \frac{d}{dt} (F + F') \\ &= \frac{d\psi}{dx}, \text{ by equation (4),} \\ \frac{d\chi}{dx} &= -\sigma \frac{d\phi}{dx} + \frac{d}{dt} (G + G') \\ &= -\frac{d\psi}{dy}, \text{ by the same equation.} \end{aligned}$$

Hence, at all points on the positive side of the current-sheet

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} = 0,$$

and, similarly, at all points on the negative surface

$$\frac{d^2\chi'}{dx^2} + \frac{d^2\chi'}{dy^2} = 0.$$

But χ, χ' are potentials due to gravitating matter none of which lies just outside either surface of the sheet, and therefore at either surface

$$\frac{d^2\chi}{dx^2} + \frac{d^2\chi}{dy^2} + \frac{d^2\chi}{dz^2} = 0,$$

$$\frac{d^2\chi'}{dx^2} + \frac{d^2\chi'}{dy^2} + \frac{d^2\chi'}{dz^2} = 0.$$

The electrical equations can therefore be put into the form

$$\frac{d^2\chi}{dz^2} = 0$$

at the positive surface of the current-sheet, and

$$\frac{d^2\chi'}{dz^2} = 0$$

at the negative surface of the current-sheet.

The expression of the fact that Φ vanishes at all points on the boundary takes the simple form

$$\chi = \chi'$$

at the boundary.

3. Following the method adopted by Maxwell, we begin by considering the currents induced by a sudden change in the external field.

Writing R for $\frac{\sigma}{2\pi}$, we have seen that at the positive surface of the sheet

$$\frac{d}{dt} \left[\frac{d^2}{dz^2} (P + P') \right] = R \frac{d^3 P}{dz^3}, \quad (6a)$$

and at the negative surface

$$\frac{d}{dt} \left[\frac{d^2}{dz^2} (P + P') \right] = -R \frac{d^3 P}{dz^3}. \quad (6b)$$

Integrating either equation with respect to the time throughout the indefinitely short period during which the change is supposed to occur,

$$\begin{aligned} \left[\frac{d^2}{dz^2} (P + P') \right] &= \pm R \int_0^r \frac{d^3 P}{dz^3} dr \\ &= 0, \end{aligned}$$

if r is supposed to vanish in the limit, since $\frac{d^3 P}{dz^3}$ is finite.

Now $\left[\frac{d}{dz} (P + P') \right]$ or $-\left[\Omega + \Omega' \right]$ is the potential due to an unknown distribution on the current-sheet and a known distribution in external space. Denoting this potential by ϖ , the electrical equations express that $\frac{d\varpi}{dz}$ vanishes at both surfaces of the current-sheet.

Also the poles of ϖ are known in the space external to the current-sheet, being identical with those of $-\left[\Omega' \right]$, and ϖ vanishes at infinity.

Hence sufficient is known about ϖ to uniquely determine its value for all space. It is, in fact, the velocity potential in an incompressible fluid in which the current-sheet is a rigid boundary, and in which a system of sources and sinks exists such that the velocity potential due to it in a free unbounded fluid would be $-\left[\Omega' \right]$.

If the hydrodynamical problem of finding this velocity potential can be solved, we can find the currents induced by a sudden change

in the external magnetic field at the moment immediately after the change. If the current-sheet is perfectly conducting, the solution thus found will represent the currents at any time.

The next two sections deal with the case of a semi-infinite current-sheet bounded by a straight edge; the further discussion of the general case is postponed until § 8.

4. In dealing with a semi-infinite plane, let us take cylindrical coordinates ξ, r, ϕ . Let the axis of ξ coincide with the free edge of the current-sheet, the positive side of its surface coinciding with the sectorial plane $\phi = 0$, and the negative side with $\phi = 2\pi$.

To find the value of the potential ω , we shall use the method of multiform potentials introduced by Dr. Sommerfeld.*

Let us construct a Riemann's space of two windings, such that the branch-line coincides with the edge of our current-sheet, and let the position of a point in this space be determined by the cylindrical coordinates ξ, r, ϕ , so that the point returns to its original position if ϕ is increased by 4π , but not, of course, if it is increased by 2π . Moreover, let that sheet† of the space in which ϕ varies from 0 to 2π be identical with our original real space, in which ϕ varied from 0 to 2π .

It can be shown that u , a function of position in Riemann's space, is uniquely determined throughout any region of that space which does not include a branch-line, by the following conditions:—

- (i.) Its infinities in this region must be known.
- (ii.) It must be finite and continuous and satisfy $\nabla^2 u = 0$ at every point of this region, except at the infinities.
- (iii.) $\frac{du}{dn}$ must be given at every point of the boundary, and u must vanish at infinity (supposing the region to extend to infinity), to a degree at least equal to that of the inverse distance.

The proof depends on Sommerfeld's theorem‡ that Green's theorem can be applied without alteration to Riemann's space. Thus, if u_1, u_2

* "Über verzweigte Potentiale im Raum," *Proc. Lond. Math. Soc.*, Vol. xxviii., p. 395.

† *Exemplar*.

‡ *L.c. ante*, p. 403.

be two values of u satisfying the above conditions,

$$\begin{aligned} & \iiint \left\{ \left(\frac{d(u_1 - u_2)}{dx} \right)^2 + \left(\frac{d(u_1 - u_2)}{dy} \right)^2 + \left(\frac{d(u_1 - u_2)}{dz} \right)^2 \right\} dv \\ &= \iint (u_1 - u_2) \frac{d}{dn} (u_1 - u_2) dS - \iiint (u_1 - u_2) \nabla^2 (u_1 - u_2) dv \\ &= 0, \end{aligned}$$

for the surface integral vanishes by the third condition, and the volume integral by the second.

Thus $u_1 - u_2 = 0$ at every point, and therefore any solution is unique.

Let us, then, try to find a solution for that region of our Riemann's space which extends from $\phi = 0$ to $\phi = 2\pi$, such as shall satisfy the following special conditions:—

- (i.) Its infinities must be the same as those of $-[\Omega']$.
- (ii.) It must satisfy $\nabla^2 u = 0$ except at these infinities.
- (iii.) It must vanish at infinity and $\frac{du}{dn}$ must vanish over the boundaries $\phi = 0$ and $\phi = 2\pi$.

Then, by what we have already proved, u will at all points within this region be identical with the potential which has been denoted by ω .

It is easily seen that a value of u which satisfies the specified conditions may be obtained as follows.

In the section of the Riemann's space in which ϕ varies from -2π to 0 (or, what is the same thing, from 2π to 4π) place a magnetic system which shall be a positive image with respect to the plane $\phi = 0$ of the system whose potential is $-[\Omega']$, and calculate the potential due to the combined systems in the manner appropriate to a Riemann's space (Sommerfeld, *l.c. ante*, p. 405). This will have the required infinities between $\phi = 0$ and $\phi = 2\pi$, it will be a solution of Laplace's equation, and its normal differential coefficients are easily seen to vanish both at $\phi = 0$ and at $\phi = 2\pi$.

This, then, is the solution sought, and it may be immediately identified with ω .

In Sommerfeld's paper the potential function is specially calculated for the case of the space of two windings with which we are dealing. If ξ, η, ϕ be the coordinates of the point P at which the

potential is to be estimated, ξ', r', ϕ' those of a single pole of unit strength at Q , then the potential at P due to this pole is shown to be

$$\frac{1}{R} \frac{2}{\pi} \tan^{-1} \sqrt{\frac{\sigma+r}{\sigma-r}},$$

where $R^2 = PQ^2 = (\xi - \xi')^2 + r^2 + r'^2 - 2rr' \cos(\phi - \phi')$,

$$r = \cos \frac{\phi - \phi'}{2},$$

and $\sigma = \cos \frac{i\alpha}{2}$,

where $\cos i\alpha = \frac{(\xi - \xi')^2 + r^2 + r'^2}{2rr'}$.

In the above expression, the symbol $\tan^{-1}$ is to denote that particular value which lies between 0 and $\pi/2$, the radical being always taken positively. It will be noticed that α and therefore σ are the same for the image of a pole as for the pole itself. Thus, since $\frac{1}{R}$ can be written in the form $\frac{1}{2\sqrt{rr'}(\sigma^2 - r^2)}$, the combined potential of a pole of strength m and its image is

$$\frac{m}{\pi\sqrt{rr'}} \left\{ \frac{1}{\sqrt{\sigma^2 - r^2}} \tan^{-1} \sqrt{\frac{\sigma+r}{\sigma-r}} + \frac{1}{\sqrt{\sigma^2 - r'^2}} \tan^{-1} \sqrt{\frac{\sigma+r'}{\sigma-r'}} \right\},$$

where $r' = \cos \frac{\phi + \phi'}{2}$,

and the complete value of ω is obtained from this expression by a summation extending to all the poles of the original system.

The potential $-\Omega'$ is, however, given by

$$\begin{aligned} -[\Omega'] &= \sum \frac{m}{R} \\ &= \sum \frac{m}{2\sqrt{rr'}} \frac{1}{\sqrt{\sigma^2 - r^2}} \\ &= \sum \frac{m}{\pi\sqrt{rr'}} \frac{1}{\sqrt{\sigma^2 - r^2}} \left\{ \tan^{-1} \sqrt{\frac{\sigma+r}{\sigma-r}} + \tan^{-1} \sqrt{\frac{\sigma+r'}{\sigma-r'}} \right\}. \quad (7) \end{aligned}$$

Incidentally we have shown that the potential in ordinary space due to a pole at ξ, r, ϕ is equal to the value of the Riemann's potential in either region, due to a pole at ξ, r, ϕ and an equal pole at $\xi, r, \phi + 2\pi$.

Subtracting the above value for $-\Omega'$ from that already found for ω ,

$$-\Omega = \Sigma \frac{m}{\pi \sqrt{rr'}} \left\{ \frac{1}{\sqrt{\sigma^2 - r'^2}} \tan^{-1} \sqrt{\frac{\sigma+r'}{\sigma-r'}} - \frac{1}{\sqrt{\sigma^2 - r^2}} \tan^{-1} \sqrt{\frac{\sigma-r}{\sigma+r}} \right\}. \quad (8)$$

At the positive surface of the current-sheet $\phi = 0$; and therefore

$$r = r' = \cos \phi/2.$$

If Φ be the current function due to the currents arising from the instantaneous change under consideration, $2\pi\Phi$ is equal to the value of $[\Omega]$ at the positive surface of the current-sheet, and this

$$= \Sigma \frac{m}{\pi \sqrt{rr'} \sqrt{\sigma^2 - r'^2}} \left\{ \tan^{-1} \sqrt{\frac{\sigma-r}{\sigma+r}} - \tan^{-1} \sqrt{\frac{\sigma+r}{\sigma-r}} \right\};$$

therefore
$$\Phi = \Sigma \frac{m}{2\pi R} \left\{ 1 - \frac{4}{\pi} \tan^{-1} \sqrt{\frac{\sigma+r}{\sigma-r}} \right\}. \quad (9)$$

This expression gives Φ at every point of the current-sheet. The boundary condition that Φ shall vanish at the edge of the sheet has been implicitly involved throughout in the continuity of $[\Omega]$ at the edge of the sheet. It is, however, easy to verify that the above equation does give a zero value of Φ at the edge.

For at a point on the edge, $r = 0$, $\sigma = \infty$; and therefore

$$\tan^{-1} \sqrt{\frac{\sigma+r}{\sigma-r}},$$

in accordance with the restrictions already laid upon the meaning of the inverse symbol, must be taken equal to $\pi/4$, and this gives the expected result.

Before proceeding further, it will be convenient to express the results arrived at in Cartesian coordinates, and in a somewhat symbolical form.

Let x, y, z be the coordinates of any point in ordinary space, the axes of reference being so chosen that the current-sheet is that part of the plane of xy for which x is positive. This is identical with the first region of the Riemann's space in which ϕ varies from 0 to 2π . When the point is supposed to lie in the second region, for which $2\pi < \phi < 4\pi$, its coordinates will be denoted by x', y', z' . Here $x = x'$, $y = y'$, $z = z'$, if the sign ($=$) is regarded as simply asserting the algebraical equality of the magnitudes which it separates.

Any system of poles in ordinary space can be denoted by $f(x, y, z)$, and any system in the second region of the Riemann's space by $f(x', y', z')$. In dealing with our Riemann's space of two windings the positive image of $f(x, y, z)$ in the current-sheet ($\phi = 0$) will be $f(x', y', -z')$.

The Newtonian potential of the system $f(x, y, z)$ will be denoted by $\Pi\{f(x, y, z)\}$ and the Riemann's potential by $P\{f(x, y, z)\}$. In future the word potential will refer to Newtonian potentials in ordinary space, unless the contrary is stated.

As has already been noticed (p. 158), the following equation holds for any system, viz.,

$$\Pi\{f(x, y, z)\} = P\{f(x, y, z)\} + P\{f(x', y', z')\}.$$

Identifying $f(x, y, z)$ with the magnetic system whose potential is $-\Omega'$, we have

$$(i.) \quad -[\Omega'] = \Pi\{f(x, y, z)\}.$$

$$= P\{f(x, y, z)\} + P\{f(x', y', z')\},$$

$$(ii.) \quad -[\Omega + \Omega'] = P\{f(x, y, z)\} + P\{f(x', y', -z')\},$$

$$(iii.) \quad -[\Omega] = P\{f(x', y', -z')\} - P\{f(x', y', z')\}, \quad (8a)$$

$$(iv.) \quad \Phi = \frac{1}{2\pi} [P\{f(x', y', z')\} - P\{f(x', y', -z')\}], \quad (9a)$$

the last equation having reference only to points for which $\phi = 0$.

5. We have thus arrived at a complete knowledge of the currents induced by a sudden change in the external field; let us now examine how these currents become dissipated under the influence of their own induction and the resistance of the sheet, no external field being supposed to act.

It is impossible to follow Maxwell's method any further: his discussion of this case depends on the fact that

$$\frac{d}{dz} \Pi\{f(x_0, y_0, z_0)\} = \Pi\left\{\frac{d}{dz_0} f(x_0, y_0, z_0)\right\},$$

an equation which is not true if P be substituted for Π .

$\frac{d}{dz} f(x, y, z)$ will, of course, be the symbolical expression for the system obtained by changing each pole of $f(x, y, z)$ into a doublet of the same strength whose axis is parallel to the positive axis of z .

To avoid the confusion which might arise as to the meaning of $\frac{d}{dz}f(x, y, -z)$, let the system $\frac{d}{dz}f(x, y, z)$ be denoted by $f'(x, y, z)$.

Then, if V be the potential of $f(x, y, z)$, $\frac{dV}{dz}$ is the potential of $f'(x, y, z)$, and so on. The *positive* image of $f'(x, y, z)$ with respect to the plane $z = 0$ will be, in ordinary space, $-f'(x, y, -z)$.

In the case now under discussion the electrical equations (6a, 6b) take the form

$$\frac{d}{dt} \left[\frac{d\Omega}{dz} \right] = R \left[\frac{d^2\Omega}{dz^2} \right] \quad (10a)$$

at the positive surface,

$$\frac{d}{dt} \left[\frac{d\Omega}{dz} \right] = -R \left[\frac{d^2\Omega}{dz^2} \right] \quad (10b)$$

at the negative surface.

In the last section we evaluated ϖ or $-(\Omega + \Omega')$ at the moment immediately following a sudden change in the inducing field. We can, in a similar manner, find $\frac{d\varpi}{dz}$ or $-\left[\frac{d\Omega}{dz} + \frac{d\Omega'}{dz}\right]$, since the electrical equations express that this potential vanishes at the boundary.

The result* is easily seen to be

$$-\left[\frac{d\Omega}{dz} + \frac{d\Omega'}{dz}\right] = P \{f'(x, y, z)\} + P \{f'(x', y', -z')\},$$

and therefore, since

$$-\left[\frac{d\Omega'}{dz}\right] = P \{f'(x, y, z)\} + P \{f'(x', y', z')\},$$

we have, by subtraction,

$$\left[\frac{d\Omega}{dz}\right] = P \{f'(x', y', z')\} - P \{f'(x', y', -z')\}. \quad (11)$$

The solution of equations (10a) and (10b) in terms of initial conditions is unique. For, knowing $\left[\frac{d\Omega}{dz}\right]$ at any instant, we are

* This problem is discussed by Sommerfeld (*l.c. ante*, p. 414). His result [equation (3), p. 417] is stated inaccurately: the corrected form is given in the *Proc. Lond. Math. Soc.*, Vol. xxx., p. 163.

explicitly given the rate of increase of $\left[\frac{d\Omega}{dz}\right]$ all over the boundary, and therefore, since this rate of increase is itself a potential, at every point of space. Hence $\left[\frac{d\Omega}{dz}\right]$ is known at the next instant, and so on.

I shall now show that, corresponding to the initial condition expressed by equation (11), a solution which satisfies (10a) and (10b) is

$$\left[\frac{d\Omega}{dz}\right] = P\{f'(x', y', z' + Rt)\} - P\{f'(x', y', -z' - Rt)\}, \quad (12)$$

the equation being supposed to hold for all space external to the current-sheet.

For at the positive surface

$$\begin{aligned} \frac{d}{dt} \left[\frac{d\Omega}{dz}\right] &= P \left\{ \frac{d}{dt} f'(x', y', z' + Rt) \right\} - P \left\{ \frac{d}{dt} f'(x', y', -z' - Rt) \right\} \\ &= R [P\{f''(x', y', z' + Rt)\} + P\{f''(x', y', -z' - Rt)\}] \\ &= R [P\{f''(x', y', z' + Rt)\} + P\{f''(x, y, z + Rt)\}] \\ &= R\Pi\{f''(x, y, z + Rt)\} \\ &= R \frac{d}{dz} \{\Pi f'(x, y, z + Rt)\} \\ &= R \frac{d}{dz} [P\{f'(x, y, z + Rt)\} + P\{f'(x', y', z' + Rt)\}] \\ &= R \frac{d}{dz} [-P\{f'(x', y', -z' - Rt)\} + P\{f'(x', y', z' + Rt)\}] \\ &= R \frac{d}{dz} \left[\frac{d\Omega}{dz}\right] \\ &= R \left[\frac{d^2\Omega}{dz^2}\right]. \end{aligned}$$

Since the solution contained in equation (12) makes $\left[\frac{d\Omega}{dz}\right]$ symmetrical about the surface of the sheet, it follows at once that at the negative surface

$$\frac{d}{dt} \left[\frac{d\Omega}{dz}\right] = -R \left[\frac{d^2\Omega}{dz^2}\right].$$

A verification exactly similar to that following equation (9) will show that this solution makes Φ vanish at the boundary for all time. For we have already verified that (12) satisfies the boundary

conditions when $t = 0$, and the verification cannot depend on t , since z' is entirely arbitrary.

Thus equation (12) contains the solution, for all space and time, of the potential due to the currents induced by a single sudden change in the external field. We pass at once to the case of a continuously changing field by a method exactly analogous to Maxwell's "Trail of Images."* Clearly, if the pair of images (in the second sheet of the Riemann's space) of which the Riemann's potential represents $\left[\frac{d\Omega}{dz}\right]$ [equation (11)] be formed at every instant, and made to pass away each with velocity R , the one whose Cartesian coordinates are equal to those of the original system parallel to the positive axis of z , and the other parallel to the negative axis of z , then the Riemann's potential due to the two continuous trails of images so formed will at any time and at any point in real space be equal to $\frac{d\Omega}{dz}$, where Ω is the magnetic potential due to the currents in the sheet.

The same theory can be put in a more symbolical form in the following manner:—

Let t , τ both be used to express the time: they are to be independent variables as regards differentiation. The magnetic system at any instant will be a function (the word being used in the same extended sense as the symbol f) of the time as well as the coordinates, and may properly be denoted by $f(x, y, z, \tau)$.

The system previously denoted by $f(x, y, z)$ will now be

$$\frac{d}{d\tau} f(x, y, z, \tau) d\tau,$$

and after a time t its image will be

$$\frac{d}{d\tau} f(x, y, z + Rt, \tau) d\tau.$$

Equation (12) therefore becomes

$$\begin{aligned} \frac{d}{d\tau} \left(\frac{d\Omega}{dz} \right) d\tau = & \left[P \left\{ \frac{d}{d\tau} f(x', y', z' + Rt, \tau) \right\} \right. \\ & \left. - P \left\{ \frac{d}{d\tau} f(x', y', -z' - Rt, \tau) \right\} \right] d\tau, \end{aligned} \quad (13)$$

* "On the Induction of Electric Currents, &c.," *Proc. Roy. Soc.*, Vol. xx., p. 160, §§ 2-10.

where, if we are evaluating $\frac{d\Omega}{dz}$ at the origin of time, t must be put equal to $-\tau$ after differentiation.

Dividing by $\delta\tau$ and integrating with respect to τ from $-\infty$ to 0,

$$\frac{d\Omega}{dz} = \int_{-\infty}^0 \left[P \left\{ \frac{d}{d\tau} f'(x', y', z' + Rt, \tau) \right\} - P \left\{ \frac{d}{d\tau} f'(x', y', -z' - Rt, \tau) \right\} \right] \delta\tau. \quad (13)$$

Integrating with respect to z ,

$$\Phi = -\frac{1}{2\pi} \int_0^\infty dz \int_{-\infty}^0 d\tau \left[P \left\{ \frac{d}{d\tau} f'(x', y', z' + Rt, \tau) \right\} - P \left\{ \frac{d}{d\tau} f'(x', y', -z' - Rt, \tau) \right\} \right]. \quad (14)$$

The first step towards the evaluation of Φ will be the evaluation of $P \{f'(x, y, z)\}$, where the system $f(x, y, z)$ consists of a single pole, for the right-hand member of equation (14) can be found from this by summation and integration. In other words, we require the Riemann's potential due to a doublet.

That due to a single pole has been seen to be

$$\frac{1}{\pi \sqrt{rr'} \sqrt{\sigma^2 - \tau^2}} \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}}.$$

Denoting this by p , and the corresponding potential for a doublet by p' ,

$$\begin{aligned} p' &= \frac{dp}{d\xi} \quad (\text{where } \xi = r' \sin \phi') \\ &= \frac{dp}{r' d\phi'} \cos \phi' + \frac{dp}{dr'} \sin \phi' \\ &= \frac{\partial p}{\partial r} \frac{\cos \phi' \sin \frac{\phi - \phi'}{2}}{8r'\tau} - p \frac{\sin \phi'}{2r'} \\ &\quad - \frac{\partial p}{\partial \sigma} \sin \phi' \frac{r^2 - r'^2 + (\xi - \xi')^2}{8\sigma r r'^2}. \end{aligned}$$

After a certain amount of reduction this takes the form

$$\begin{aligned} & \frac{1}{\pi R r'} \sin \phi' \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} \\ & - \frac{1}{2\pi R^3} \sqrt{\frac{r}{r'}} \left\{ \frac{1}{2} \cos \phi' \tan \frac{\phi - \phi'}{2} + \sin \phi' \cos \frac{\phi - \phi'}{2} \left(\frac{r^2 - r'^2 + (\xi - \xi')^2}{(r + r')^2 + (\xi - \xi')^2} \right) \right\} \\ & + \frac{1}{\pi R^3} \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} \left\{ r \cos \phi' \sin \frac{\phi - \phi'}{2} + \sin \phi' \left(\frac{r^2 - r'^2 + (\xi - \xi')^2}{r'} \right) \right\}, \end{aligned}$$

where R is now the distance between the points ξ, r, ϕ and ξ', r', ϕ' .

Combining this with the corresponding expression for the potential due to the image of this doublet, we get for the combined potential

$$\begin{aligned} & \frac{\sin \phi'}{\pi r'} \left\{ \frac{1}{R} \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} + \frac{1}{R'} \tan^{-1} \sqrt{\frac{\sigma + \tau'}{\sigma - \tau'}} \right\} \\ & - \frac{\cos \phi'}{4\pi} \sqrt{\frac{r}{r'}} \left\{ \frac{1}{R^2} \tan \frac{\phi - \phi'}{2} - \frac{1}{R'^2} \tan \frac{\phi + \phi'}{2} \right\} \\ & - \frac{\sin \phi'}{2\pi} \sqrt{\frac{r}{r'}} \left(\frac{r^2 - r'^2 + (\xi - \xi')^2}{(r + r')^2 + (\xi - \xi')^2} \right) \left\{ \frac{1}{R^2} \cos \frac{\phi - \phi'}{2} + \frac{1}{R'^2} \cos \frac{\phi + \phi'}{2} \right\} \\ & + \frac{r \cos \phi'}{\pi} \left\{ \frac{1}{R^3} \sin \frac{\phi - \phi'}{2} \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} - \frac{1}{R'^3} \sin \frac{\phi + \phi'}{2} \tan^{-1} \sqrt{\frac{\sigma + \tau'}{\sigma - \tau'}} \right\} \\ & + \frac{\sin \phi'}{\pi r'} [r^2 - r'^2 + (\xi - \xi')^2] \left\{ \frac{1}{R^3} \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} + \frac{1}{R'^3} \tan^{-1} \sqrt{\frac{\sigma + \tau'}{\sigma - \tau'}} \right\}. \end{aligned}$$

The currents arising from the creation of a single pole at x, y, z can now be obtained in the following manner. Let the point ξ', r', ϕ' in the above expression be made identical with the point $x, y, z + vt$ (v is the velocity of the images, the R of § 4). Then the above expression is the value of $\frac{d\Omega}{dz}$ at the point ξ, r, ϕ . An integration with respect to z between the limits 0 and ∞ will give the value of $-2\pi\Phi$ for any values of x, y , and t .

Lastly, the above theory can be stated in a form which shall contain no reference to Riemann's space in the following manner.

Any continuous change of a magnetic system, whether arising from changes in the strength or position of the magnets, or both, can be represented by the continued creation or annihilation of poles, or, what is the same thing, by the creation of positive or negative poles.

Let each pole, immediately it is created, move parallel to the axis of z with constant velocity v . Then at any time we have a number of continuous series of poles. Substitute the cylindrical coordinates of any such pole in the expression found above, multiply by the strength of the pole, and sum the expression so found for every pole of the system.

The result gives the value of $\frac{d\Omega}{dz}$, and, as before, the current function at a point in the sheet is found by an integration with respect to z .

6. Maxwell's theory for the complete sheet may be deduced from equation (13) by supposing the edge of the semi-infinite sheet removed to infinity.

We are now dealing only with pairs of points for which

$$r = r' = \infty,$$

$$\phi - \phi' = 0 \quad \text{or} \quad 2\pi;$$

therefore $ia = 0, \quad \sigma = 1,$

and $\tau = 1 \quad \text{or} \quad -1,$

according to whether the continuous path, in the Riemann's space, which joins the two points is finite or infinite.

In the former case

$$P \{f(x, y, z)\} = \Sigma \frac{m}{R} = \Pi \{f(x, y, z)\}.$$

In the latter case $P \{f(x, y, z)\} = 0.$

Thus, due to any system on the negative side of the sheet, we have at any point on the positive side, by equation (13),

$$\frac{d\Omega}{dz} = \int_{-\infty}^0 \Pi \left\{ \frac{d}{d\tau} f'(x, y, z + R\tau, \tau) \right\} d\tau.$$

Integrating twice with respect to z ,

$$\int \Omega dz = \int_{-\infty}^0 \Pi \left\{ \frac{d}{d\tau} \left[f(x, y, z + R\tau, \tau) dz \right] \right\} d\tau,$$

and this is the same as the equation which Maxwell writes in the form

$$P = - \int_0^{\infty} \frac{dP'}{dt} d\tau$$

[*Elect. and Mag.*, § 663, equation (27)].

7. After the semi-infinite current-sheet, the case which comes next in order of simplicity is that in which the boundary consists of two parallel and infinite straight lines, giving

(i.) An infinite current-sheet from which an infinite strip has been removed.

(ii.) A conducting strip bounded by two parallel straight lines.

A complete investigation of the potential function for case (i.) is given by Sommerfeld (§ 5, p. 419), and there is no difficulty in extending it to case (ii.). I shall accordingly only quote results, slightly altering Sommerfeld's notation.

We again take cylindrical coordinates ξ, ρ, θ , where ρ, θ are connected with r, ϕ by the transformations

$$(i.) \quad \rho + i\theta = \log \left(\frac{re^{i\phi} - a}{re^{i\phi} + a} \right),$$

$$(ii.) \quad \rho + i\theta = \log \left(\frac{re^{i\phi} + a}{re^{i\phi} - a} \right),$$

the former referring to case (i.), the latter to case (ii.). The variable a is now to be given by

$$(i.) \quad \cos ia = \cos i(\rho - \rho') + \frac{1}{2}(\cos i\rho - \cos \theta)(\cos i\rho' - \cos \theta')(\xi - \xi')^2,$$

$$(ii.) \quad \cos ia = \cos i(\rho - \rho') + \frac{1}{2}(\cos i\rho + \cos \theta)(\cos i\rho' + \cos \theta')(\xi - \xi')^2.$$

σ, r have now the same expression in terms of a, ρ, θ as they formerly had in terms of a, r, ϕ , and Sommerfeld's result [given in equation (10), p. 424, for case (i.), and easily seen to be true also for case (ii.) when the quantities concerned have the meanings assigned to them above], states that in either case the potential function is

$$\frac{1}{R} \frac{2}{\pi} \tan^{-1} \sqrt{\frac{\sigma + r}{\sigma - r}}.$$

On account of the identity of form between this potential function and the one already dealt with, all the results of § 4 can be immediately applied to this case.

Corresponding to equation (7), we obviously have

$$-[\Omega'] = \Sigma \frac{m}{R} \frac{2}{\pi} \left\{ \tan^{-1} \sqrt{\frac{\sigma + r}{\sigma - r}} + \tan^{-1} \sqrt{\frac{\sigma - r}{\sigma + r}} \right\},$$

and equation (8) will apply to this case if put in the form

$$-[\Omega] = \Sigma \frac{2m}{\pi} \left\{ \frac{1}{R'} \tan^{-1} \sqrt{\frac{\sigma + r}{\sigma - r}} - \frac{1}{R} \tan^{-1} \sqrt{\frac{\sigma - r}{\sigma + r}} \right\}. \quad (15)$$

Equation (9) is true without change,

$$\Phi = \Sigma \frac{m}{2\pi R} \left\{ 1 - \frac{4}{\pi} \tan^{-1} \sqrt{\frac{\sigma + \tau}{\sigma - \tau}} \right\}. \quad (16)$$

It will be noticed that at the two edges of the sheet $\rho = \pm \infty$; therefore $\sigma = \infty$. The same verification as before now shows that Φ vanishes at both edges.

It will also be noticed that all the symbolical work of § 5 holds good without alteration for the cases now under discussion, and consequently the value of $\left[\frac{d\Omega}{dz} \right]$ found in equation (12) holds if we assign the altered meaning to the symbol P . The alteration is necessitated by the fact that the equations of transformation from R, σ, τ to x, y, z are no longer the same. Hence equations (13) and (14) contain the symbolical solution for the cases now under consideration.

The calculation of p' which follows equation (14) does not, of course, apply to this case. The calculation could, however, be effected in an exactly similar way, either for the potential function now under discussion or for any other potential function.

8. It will now be seen that the method adopted in these two cases is, in its main principles, quite general.

The first step towards the discussion of any given shape of current-sheet consists of the construction of a Riemann's space of two regions,* one of which is to be identical with the ordinary space containing the inducing system, and the other of which will contain certain images of the system.

The current-sheet itself must be the branch membran† of the space, and the boundary of the current-sheet its branch line.‡

If we can find three systems of mutually orthogonal surfaces, each filling all space, such that all the surfaces of one system are bounded by the edge of the current-sheet, and one of them coincides with the current-sheet, then the calculation of the potential function for this space will consist of a series of known processes in pure mathematics. The mode of procedure is given in § 6 of Sommerfeld's paper.

Independently, however, of the form which the potential function assumes for any particular space, and independently even of the

* Exemplar.

† Verzweigungsmembrane.

‡ Verzweigungslinie.

possibility of calculating it, the symbolical work of §§ 4 and 5 will hold good for the most general case.

For the symbolical equation

$$\Pi \{f(x, y, z)\} = P \{f(x, y, z)\} + P \{f(x', y', z')\}$$

is true for any Riemann's space of two regions.

If a formal proof of this statement is required, it is sufficient to remark that throughout real space each member of the equation satisfies the four following conditions:—

- (i.) It is a solution of Laplace's equation.
- (ii.) It becomes infinite at the poles of $f(x, y, z)$ to the same order and with the same residues as $\Pi \{f(x, y, z)\}$.
- (iii.) It vanishes at infinity.
- (iv.) It is continuous everywhere, including the branch-membrane and the branch-line of the Riemann's space.

Those conditions are sufficient to uniquely determine a function of position in real space, and therefore the validity of the above equation is established.

It only remains to remark that the only quantity entering the original equations which is not entirely symbolical is z , that differentiation with respect to z remains the same in our new system of coordinates, and, lastly, that nowhere in the course of the symbolical work of §§ 4 and 5 has any account been taken of the form of the special functions which the symbolical notation is there supposed to suggest.

This establishes the generality of the method, and therefore we conclude, since $\Pi \{f(x, y, z)\}$ has the same meaning in connexion with every problem, that, if the right interpretation be assigned to the symbol P , equations (13) and (14) will contain the solution of our problem for a plane current-sheet bounded by any given curve in the plane of xy .