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 integration with respect to x . The result may be written

$$2\pi^2 \iint dx dy \quad . \quad . \quad . \quad . \quad . \quad (75)$$

upon the understanding that, while the integration for y ranges over the whole vertical aperture, that for x is limited to such values of x as bring $x \pm s$ (as well as x itself) within the range of the horizontal aperture. The coefficient of the Fourier component of the intensity involving $\cos sp$, or $\cos (2r\pi p/p_1)$, is thus proportional to a certain part of the area of the aperture. Other parts of the area are inefficient, and might be stopped off without influencing the result.

The limit to resolution, corresponding to $r=1$, depends only on the width of the aperture, and is therefore for all forms of aperture the same as for the case of the rectangular aperture already fully investigated.

If the object be a row of points instead of a row of lines, $q=0$, and there is no integration with respect to it. The process is nearly the same as above, and the result for the coefficient of the r th term in the Fourier expansion is proportional to $\int y^2 dx$, instead of $\int y dx$, the integration with respect to x being over the same parts of the aperture as when the object was a grating. The application to a circular aperture would lead to an evaluation of

$$\int_{-\infty}^{+\infty} \frac{J_1^2(u) \cos su}{u^2} du.$$

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ACCORDING to Sylow's theorem these groups contain $kp+1$ conjugate subgroups of order p and $b \geq 1$ in the equation

$$\frac{8p}{kp+1} = bp.$$

Hence they must contain a self-conjugate subgroup of this order when $p > 3$ and $p \neq 7$. We shall first consider all the possible groups that contain such a self-conjugate subgroup.

* M. Levavasseur gives an enumeration of these groups, without explaining how they were obtained, in *Comptes Rendus*, March 2, 1896. His enumeration is not quite correct. He states that there are three groups which exist only when $p-1$ is a multiple of 4 without being also a multiple of 8. We shall prove that there are only two such groups.

† Communicated by the Author.

The few groups which do not contain such a subgroup will be considered afterwards.

The eight systems of intransitivity* of the given self-conjugate subgroup are systems of nonprimitivity of the required groups. Hence each one of these groups must have a $p, 1$ isomorphism to some group of order 8. As all of the latter contain subgroups of order 4, all of the former must contain subgroups of order $4p$.

Since the average number of elements in all the substitutions of a group is $n - \alpha \dagger$, n being the degree and α the number of systems of intransitivity of the group, every subgroup of a regular group must be intransitive; and an intransitive subgroup of half the order of a transitive group must contain two and only two systems of intransitivity, which are also systems of nonprimitivity of the transitive group.

Each one of the groups under consideration must therefore contain a subgroup of order $4p$, which may be formed by making some regular group of this order simply isomorphic to itself‡. Since the groups of order $4p$ are known§, our problem is reduced to the construction of the nonprimitive groups containing as heads one of the five regular groups of order $4p$ in 1, 1 correspondence to itself.

In what follows we shall consider $p > 2$, as the groups of order 16 are well known||. We shall first construct all the groups which contain as heads one of the two commutative groups of order $4p$ in 1, 1 correspondence to itself. The cyclical head will be denoted by H cyc., and the non-cyclical by H.

Groups containing H cyc.

Since there are $2(p-1)$ positive integers which are less than $4p$ and prime to $4p$, H cyc. contains $2(p-1)$ substitutions of order $4p$. The largest group which transforms H cyc. into itself without interchanging its systems transforms these substitutions according to a regular commutative group (L)

* Every operation group of a given order may be represented by a regular substitution group of the same order. Cf. Cayley, 'American Journal of Mathematics,' vol. i. p. 52; also Dyck, *Mathematische Annalen*, vol. xxii. p. 84.

† Cf. Frobenius, *Crelle*, ci. p. 287.

‡ Cf. Netto's 'Theory of Substitutions' (Cole's translation), § 98.

§ Cf. Hölder, *Mathematische Annalen*, vol. xliii. p. 411; also Cole and Glover, 'American Journal of Mathematics,' vol. xv. pp. 202-214.

|| Young, 'American Journal of Mathematics,' vol. xv. p. 160; Hölder, *Mathematische Annalen*, vol. xliii. p. 409; Miller, *Comptes Rendus*, Feb. 17, 1896.

of order $2(p-1)$, containing 3 and only 3 substitutions of order 2. These correspond to the substitutions which transform those of order $4p$ in H cyc. into their

$$2p-1, \quad 2p+1, \quad 4p-1$$

powers. We have therefore to examine four types of tails that may be added to H cyc., viz. those which are commutative to the substitutions of H cyc., and those which transform these substitutions into one of the three given powers.

Since half of the substitutions of H cyc. are the squares of its substitutions there can be only two commutative groups; viz. the cyclical group (G_1), and the group (G_2) obtained by adding to H cyc. a substitution (t) which simply interchanges its systems. The squares of the substitutions in the tail of G_2 are also the squares of the substitutions of H cyc.

We represent the three substitutions* of the second order which are commutative to t and transform the substitutions of H cyc. into the three given powers by s_1, s_2, s_3 . s_1t, s_2t, s_3t may be used to construct three distinct tails. The first of these contains $2p$ substitutions of order 2 and $2p$ of order 4, the second contains 4 of order 2 and $4(p-1)$ of order $2p$, the third contains only substitutions of order 2. We represent the groups containing these tails respectively by G_3, G_4, G_5 .

Since s_1 is commutative only with the subgroup of order 4 in H cyc., and half of the substitutions of this subgroup are squares of its substitutions, there is only one more tail of this type. This contains only substitutions of order 8. Similarly we see that there is only one additional tail of each of the other two types; and that the former of these contains 4 substitutions of order 4 and $4(p-1)$ of order $4p$, while the latter contains only substitutions of order 4. We represent the three groups containing these tails respectively by G_6, G_7, G_8 .

Hence, when $p > 2$, there are always 8 groups and only 8 that contain a cyclical subgroup of order $4p$. In what follows we need not consider the groups in which such a subgroup occurs. When $p=2$, $2p-1$ and $2p+1$ are not congruent to 1 and 3 respectively with respect to mod 4, as is the case

* That such substitutions can always be found follows from the fact that we may transform a generating substitution of a transitive cyclical group into any other generating substitution by a substitution whose degree is at least one less than the degree of the group. Since the first power of this substitution which is commutative to the group must be contained in the group, its order must be equal to the exponent to which the power into which it transforms the substitutions belongs with respect to mod α , α being the order of the given cycle.

for the other values of p . Hence some of our remarks do not apply to this case. In fact there are only 6 groups of order 16 which contain a cyclical subgroup of order 8.

Groups containing H.

By adding t to this head we obtain the third and last commutative group (G_9) of order $8p$. There are, therefore, three and only three commutative groups of this order for every value of $p > 2$. When $p = 2$ there are five such groups. The tail of G_9 contains 4 substitutions of order 2 and $4(p-1)$ of order $2p$. The squares of these substitutions are clearly the same as the squares of those of H. It remains to find the non-commutative groups that contain H.

The isomorphic group of order 8 contains at least three substitutions of the second order. If this group is commutative the corresponding tail must transform the substitutions of H into their -1 powers, as $2p$ has primitive roots*. We can easily find a substitution (s') of the second order which transforms one of the 4 cycles of a substitution of order $2p$ in H into its -1 power. By making s' symmetrical in the elements of the other cycles of the same substitution we obtain a substitution (s) which evidently transforms H into itself.

The tail of the group (G_{10}) generated by st and H contains only substitutions of the second order. Since s is commutative to the substitutions of the second order in H, we may construct a group (G_{11}) whose tail contains only substitutions of the fourth order by using, in place of s , the product of a substitution of the second order in one of the systems of H into s . The other two groups which may be constructed in the same way as G_{11} are conjugate to it with respect to (s_2), the p th power of two cycles commutative to t , these cycles being contained in some substitution of H whose order is $2p$. It remains to examine the case when the isomorphic group of order 8 is not commutative.

Since this group of order 8 contains at least 3 substitutions of the second order and is non-commutative, it is fully determined. The corresponding tail must interchange two of the divisions of the head and transform the substitutions of the other two divisions into their -1 powers. The group (G_{12}) generated by H and ss_2t clearly satisfies these conditions. Its tail contains $2p$ substitutions of order 2 and $2p$ of order 4. The other possible group is conjugate to G_{12} with respect to a substitution of the second order in one of the systems of H.

* Cf. Serret's *Cours d'Algèbre Supérieure*, vol. ii. p. 82.

We have now found 12 groups of order $8p$ which exist for every value of $p > 2$. As the remaining groups cannot contain a commutative group of order $4p$, they must transform the substitutions of the self-conjugate subgroup of order p according to a cyclic group of order 4 or 8. Such groups can exist only when $p-1$ is a multiple of 4 or 8. We shall examine these two cases separately.

Groups which exist only when $p-1$ is a multiple of 4.

We shall first consider the case when $p-1$ is not also a multiple of 8. The substitutions which are commutative to those of the self-conjugate subgroup of order p form a commutative group of order $2p$. This cyclical group is therefore also a self-conjugate subgroup of the required groups, and its four systems of intransitivity are four systems of nonprimitivity of the required groups. Hence we may regard it as the head (H_1) of the required groups.

The tail to these groups contains $4p$ substitutions which transform the substitutions of the head into a power α which belongs to the exponent 4, mod $2p$. We can easily find a substitution s which transforms the substitutions of the head into their α power, and is of order 4 and commutative to t , t representing a substitution of the 4th order which simply interchanges the 4 systems of the head cyclically. H_1 and st generate a group (G_{13}) whose tail contains $2p$ substitutions of the second order and $4p$ of the fourth order.

As s is commutative only with the subgroup of order 2 contained in H_1 there can be only one more group of this type. This (G_{14}) may be obtained by using the product of a substitution of the second order in one of the systems of H_1 into s in place of s . The tail of G_{14} contains $2p$ substitutions of order 4, and $4p$ of order 8. It remains to consider the case when $p-1$ is a multiple of 8.

The preceding 14 groups are all present in this case. If there is an additional group it must transform the substitutions of the self-conjugate subgroup of order p according to a cyclical group of order 8. We can easily find a substitution (s_1) which is symmetrical in the systems of the given self-conjugate subgroup and transforms its substitutions into their β powers, β belonging to the exponent 8, mod p .

H_1 and s_1t (t being a substitution of order 8 which merely interchanges the given systems cyclically) generate a group (G_{15}) whose tail contains p substitutions of order 2, $2p$ of order 4, and $4p$ of order 8. As the tail of G_{15} is not commutative to any substitution in the subgroup of order p , with

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the exception of identity, there can be only one group of this type. Hence *there are always 12 groups of order $8p$ ($p > 2$) which contain a self-conjugate subgroup of order p ; when $p-1$ is a multiple of 4 or 8 there are respectively 14 or 15 such groups.* It remains to consider the

*Groups of order $8p$ which do not contain a self-conjugate
subgroup of order p .*

When $p=7$, the equation

$$\frac{8p}{kp+1} = bp$$

is satisfied by $k=b=1$ as well as by $k=0, b=8$. If the subgroup of order p is not self-conjugate there must be 8 such subgroups. These contain $8 \times 6 = 48$ substitutions besides identity. The subgroup of order 8 must therefore be self-conjugate, and its 7 systems of intransitivity must be systems of nonprimitivity of the required groups.

Since the substitutions of the group (H_2) of order 8 cannot be commutative to the entire group, they must be transformed according to a group of order 7. Hence all these substitutions are of the second order, and H_2 is fully determined. If we add to H_2 a substitution (t) of order 7 which simply permutes its 7 systems cyclically, we obtain a group (G_{16}) whose tail contains only substitutions of order 7. As no substitution of H_2 , besides identity, is commutative to t , there can be no other group of this type.

Hence there are 13 groups of order 56; 12 of these contain a self-conjugate subgroup of order 7. The remaining one contains 8 conjugate subgroups of order 7 and a self-conjugate subgroup of order 8. The last group occurs for the first time as a group of degree 8*.

The only other value of $p > 2$ for which there can be groups which do not contain a self-conjugate subgroup of order p is 3. In this case it is known that there are three such groups†. Hence all the groups of order $8p$ are completely determined.

Paris, June 1896.

* Cf. Cole, 'Bulletin of the New York Mathematical Society,' vol. ii. p. 189.

† Cf. Levassasseur, *Comptes Rendus*, March 2, 1896.