that we do not thereby come nearer to a representation of the processes which occur. We may say with certainty that fluid unequally-heated metals are electrically charged, and that by simple inequalities of temperature in a conductor the conditions for transformation of heat into work are fulfilled, only that we cannot obtain the work in the convenient form of a closed current. Perhaps this excitation of electricity takes place in all fluids, and generally in all substances, *i. e.* also in the so-called vacuum; and the further question, important in principle, may arise, what influence this hitherto disregarded circumstance has upon physical processes.

If we return to the facts detailed above, everything indicates that thermoelectric excitement is an intermolecular process dependent upon the number or arrangement of atoms in the molecule. It is thus suggested that all the rough mechanical changes, which exert so much influence on the thermoelectric behaviour of a solid body, and which consist in drawing, bending, hardening, annealing, &c., are also connected with intermolecular, and thus in a measure chemical, changes. This would be in harmony with the one already known fact, that in steel, upon passage from the soft into the hard condition and *vice versâ* (also by simple drawing), the quantity of chemically combined carbon changes. Hard and soft steel give, as is well known, a tolerably powerful thermoelectric action.

LVII. *On the Expression for the Complete Elliptic Integral of the Second Kind as a Series proceeding by Sines of Multiples of the Modular Angle.* By J. W. L. GLAISHER, M.A., F.R.S.*

Series for K and E, §§ 1-4.

§ 1. **I**N vol. xix. pp. 51, 52 of Crelle's Journal, Gudermann has given the following interesting expression for K in a series proceeding by multiples of the modular angle θ :—

$$\frac{K}{\pi} = \sin \theta + \frac{1^2}{2^2} \sin 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 9\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 13\theta + \&c. \quad (1)$$

The coefficients are the same as in the well-known fundamental formula

$$\frac{2K}{\pi} = I + \frac{1^2}{2^2} \sin^2 \theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin^4 \theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin^6 \theta + \&c. \quad (2)$$

* Communicated by the Author.

Of course for the actual calculation of the numerical value of K corresponding to a given value of θ , the second formula is much to be preferred; for the trigonometrical factors diminish rapidly, whereas in the first series they have no tendency to diminish, and the gradual convergence of the series is due solely to the numerical coefficients. Thus in (1) the n th term is of the order $\frac{1}{n}$; but the corresponding term in (2) is of the order $\frac{1}{n} \sin^{2n-2}\theta$. When $\theta=0$, the series in (1) is of the form $0 \times \infty$, the true value being $\frac{1}{2}$; when $\theta=\frac{1}{2}\pi$, the series is infinite in value, as it should be.

§ 2. Gudermann's process is in effect as follows. He shows that the transformation which converts k into $e^{-2i\theta'}$ converts K into $\frac{1}{2}e^{i\theta'}(K-iK')$, where θ' denotes $\frac{1}{2}\pi-\theta$. Starting with (2), he thus finds

$$\frac{K-iK'}{\pi} = e^{-i\theta'} + \frac{1^2}{2^2}e^{-5i\theta'} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}e^{-9i\theta'} + \&c.,$$

from which (1) is immediately deducible by equating the real parts.

At the end of the investigation Gudermann, after referring to the slow convergence of (1), which renders it of no practical value, adds:—"Aus diesem Grunde übergehen wir auch die Herleitung einer ähnlichen Reihe für den Quadranten E."

The main object of this note is to give the series for E which corresponds to (1). It does not admit of quite such simple derivation as the series for K .

§ 3. Gudermann's transformation corresponds to the change of q into $iq^{\frac{1}{2}}$; and this change may be supposed to be produced by the change of q into $q^{\frac{1}{2}}$, followed by the change of q into $-q$.

By the change of q into $iq^{\frac{1}{2}}$, k' is changed into $e^{-2i\theta}$, and K' is changed into $\frac{1}{2}e^{i\theta}(K'-iK)$. Starting with the series

$$\frac{2K'}{\pi} = 1 + \frac{1^2}{2^2}k'^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k'^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k'^6 + \&c.,$$

we thus find

$$\frac{K'-iK}{\pi} = e^{-i\theta} + \frac{1^2}{2^2}e^{-5i\theta} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}e^{-9i\theta} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}e^{-13i\theta} + \&c.$$

§ 4. In order to apply the same process to the series for E' , viz.

$$\frac{2E'}{\pi} = 1 - \frac{1}{2^2}k'^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2}k'^4 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2}k'^6 - \&c.,$$

it is necessary to determine the quantity into which E' is converted by the change of q into $iq^{\frac{1}{2}}$.

It can be shown that, by the change of q into $q^{\frac{1}{2}}$,

$$E' \text{ becomes } \frac{E' + kK'}{1 + k};$$

and that, by the change of q into $-q$,

$$k \text{ becomes } \frac{ik}{k'},$$

$$K' \quad ,, \quad k'(K' - iK),$$

$$E' \quad ,, \quad \frac{E' - k^2 K' + i(E - k'^2 K)}{k'}.$$

Thus, by the change of q into $iq^{\frac{1}{2}}$, E' becomes

$$e^{-i\theta} \{E' - k^2 K' + kk' K + i(E - k'^2 K + kk' K')\}.$$

It follows, therefore, that

$$\begin{aligned} & \frac{2\{E' - k^2 K' + kk' K + i(E - k'^2 K + kk' K')\}}{\pi} \\ &= e^{i\theta} - \frac{1}{2^2} e^{-3i\theta} - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} e^{-7i\theta} - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} e^{-11i\theta} - \&c.; \end{aligned}$$

whence, equating imaginary parts,

$$\frac{2(E - k'^2 K + kk' K')}{\pi} = \sin \theta + \frac{1}{2^2} \sin 3\theta + \frac{1^2 \cdot 3}{2^2 \cdot 4^2} \sin 7\theta + \&c.$$

The left-hand member of this equation

$$= \frac{2E - K}{\pi} + \frac{K' \sin 2\theta - K \cos 2\theta}{\pi};$$

and therefore the equation itself may be written

$$\begin{aligned} \frac{2E}{\pi} &= \frac{K}{\pi} + \frac{K \cos 2\theta - K' \sin 2\theta}{\pi} \\ &+ \sin \theta + \frac{1}{2^2} \sin 3\theta + \frac{1^2 \cdot 3}{2^2 \cdot 4^2} \sin 7\theta + \&c. \end{aligned}$$

Now

$$\frac{K}{\pi} = \sin \theta + \frac{1^2}{2^2} \sin 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 9\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 13\theta + \&c.,$$

$$\frac{K'}{\pi} = \cos \theta + \frac{1^2}{2^2} \cos 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cos 9\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cos 13\theta + \&c.;$$

whence

$$\frac{K \cos 2\theta - K' \sin 2\theta}{\pi} \\ = -\sin \theta + \frac{1^2}{2^2} \sin 3\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 7\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 11\theta + \&c.$$

Thus

$$\frac{2E}{\pi} = \frac{K}{\pi} + \frac{1}{2} \sin 3\theta + \frac{1^2 \cdot 3}{2^2 \cdot 4} \sin 7\theta + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \sin 11\theta + \&c.;$$

or, substituting for $\frac{K}{\pi}$ its value,

$$\frac{2E}{\pi} = \sin \theta + \frac{1}{2} \sin 3\theta + \frac{1^2}{2^2} \sin 5\theta + \frac{1^2 \cdot 3}{2^2 \cdot 4} \sin 7\theta \\ + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 9\theta + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \sin 11\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 13\theta + \&c.,$$

which is the series for E corresponding to (1).

Formulæ for K, E, I, U, V, W, §§ 5-9.

§ 5. The results obtained in the previous sections form part of a general system of formulæ which may be stated as follows :—

Consider two quantities I and G defined by the equations

$$I = E - K,$$

$$G = E - k'^2 K ;$$

and let U, V, W denote

$$\frac{1}{2}(I + G), \quad \frac{1}{2}(G + E), \quad \frac{1}{2}(E + I)$$

respectively.

The values of the six quantities, expressed in terms of E and K , and of I and K , are therefore

$$E = E \qquad \qquad \qquad = I + K,$$

$$I = E - K \qquad \qquad \qquad = I,$$

$$G = E - k'^2 K \qquad \qquad \qquad = I + k'^2 K,$$

$$U = E - \frac{1}{2}(1 + k'^2)K \qquad \qquad \qquad = I + \frac{1}{2}k'^2 K,$$

$$V = E - \frac{1}{2}k'^2 K \qquad \qquad \qquad = I + \frac{1}{2}(1 + k'^2)K,$$

$$W = E - \frac{1}{2}K \qquad \qquad \qquad = I + \frac{1}{2}K.$$

It can be shown that

$$\begin{aligned}\frac{2K}{\pi} &= 1 + \frac{1^2}{2^2} k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} k^6 + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} k^8 + \&c., \\ \frac{2E}{\pi} &= 1 - \frac{1}{2^2} k^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4^2} k^4 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6^2} k^6 - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} k^8 - \&c., \\ \frac{2I}{\pi} &= -\frac{1}{2} k^2 - \frac{1^2 \cdot 3}{2^2 \cdot 4} k^4 - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} k^6 - \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} k^8 - \&c., \\ \frac{2G}{\pi} &= \frac{1}{2} k^2 + \frac{1^2}{2^2 \cdot 4} k^4 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6} k^6 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} k^8 + \&c., \\ \frac{4U}{\pi} &= -\frac{1^2}{2 \cdot 4} k^4 - \frac{1^2 \cdot 3^2}{2^2 \cdot 4 \cdot 6} k^6 - \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6 \cdot 8} k^8 - \&c., \\ \frac{4V}{\pi} &= 1 + \frac{1}{2^2} k^2 + \frac{1^2}{2^2 \cdot 4^2} k^4 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} k^6 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} k^8 + \&c., \\ \frac{4W}{\pi} &= 1 - \frac{1 \cdot 3}{2^2} k^2 - \frac{1^2 \cdot 3 \cdot 5}{2^2 \cdot 4^2} k^4 - \frac{1^2 \cdot 3^2 \cdot 5 \cdot 7}{2^2 \cdot 4^2 \cdot 6^2} k^6 - \&c.\end{aligned}$$

§ 6. Denoting by accented letters the same functions of k'^2 that the unaccented letters are of k^2 , it can also be shown that, by the change of q into $q^{\frac{1}{2}}$,

$$\begin{aligned}k' &\text{ becomes } e^{-2i\theta}, \\ K' &,, \quad \frac{1}{2}e^{i\theta}(K' - iK), \\ I' &,, \quad e^{-i\theta}(W' + iW), \\ V' &,, \quad e^{-i\theta}(G' + iG); \end{aligned}$$

and

$$\begin{aligned}E' &\text{ becomes } e^{-i\theta} \{ G' + kK' + i(G + kK') \}, \\ G' &,, \quad e^{-i\theta} \{ G' - kK' + i(G - kK') \}, \\ U' &,, \quad \frac{1}{2}e^{-i\theta} \{ G' + W' - kK' + i(G + W - kK') \}, \\ W' &,, \quad \frac{1}{2}e^{-i\theta} \{ G' + W' + kK' + i(G + W + kK') \}.\end{aligned}$$

§ 7. Starting with the series for $\frac{2K'}{\pi}$, $\frac{2I'}{\pi}$, $\frac{2V'}{\pi}$, in powers of k'^2 , and changing k' into $e^{-2i\theta}$, we find, by equating the real, and the imaginary, parts in the resulting formulæ,

$$\begin{aligned}\frac{K}{\pi} &= \sin \theta + \frac{1^2}{2^2} \sin 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 9\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 13\theta + \&c., \\ \frac{K'}{\pi} &= \cos \theta + \frac{1^2}{2^2} \cos 5\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cos 9\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \cos 13\theta + \&c., \\ \frac{2W}{\pi} &= \frac{1}{2} \sin 3\theta + \frac{1^2 \cdot 3}{2^2 \cdot 4} \sin 7\theta + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \sin 11\theta + \&c.,\end{aligned}$$

$$\frac{2W'}{\pi} = -\frac{1}{2} \cos 3\theta - \frac{1^2 \cdot 3}{2^2 \cdot 4} \cos 7\theta - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \cos 11\theta - \&c.,$$

$$\frac{4G}{\pi} = \sin \theta - \frac{1}{2^2} \sin 3\theta - \frac{1^2}{2^2 \cdot 4^2} \sin 7\theta - \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 11\theta - \&c.,$$

$$\frac{4G'}{\pi} = \cos \theta + \frac{1}{2^2} \cos 3\theta + \frac{1^2}{2^2 \cdot 4^2} \cos 7\theta + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2 \cdot 6^2} \cos 11\theta + \&c.,$$

§ 8. It appears from § 5 that the quantities I' and V' resemble K' in being transformable, by the change of modulus from k' to $e^{-2i\theta}$, into expressions of the form $e^{\pm i\theta}(P' \mp iP)$, where P' is the same function of k'^2 that P is of k^2 . Perhaps the simplest method of obtaining the sine-series for E is by deducing the sine-series for W from the k' -series for I' and then adding the sine-series for $\frac{1}{2}K$; and it was in this manner that I was first led to the series for E^* . It will be noticed that the process followed in § 4 also gives in the first instance the sine-series for $E - \frac{1}{2}K$ or W , and that the series for E is deduced from it by the addition of that for $\frac{1}{2}K$.

By the change of k' into $e^{-2i\theta}$,

$$k'K' \text{ becomes } \frac{1}{2}e^{-i\theta}(K' - iK),$$

and therefore

$$I' - k'K' \text{ becomes } e^{-i\theta}(I' + iE).$$

If then we start with the k' -series for $I' - k'K'$ and transform I' into $e^{-2i\theta}$, we obtain directly the sine-series for E by equating the imaginary parts in the resulting formula. The transformation therefore (of k' into $e^{-2i\theta}$) which converts

$$K' \text{ into } \frac{1}{2}e^{i\theta}(K' - iK)$$

converts also

$$K' - E' + k'K' \text{ into } e^{-i\theta}(K' - E' - iE);$$

and we may therefore derive the sine-series for E from the k' -series for $K' - E' + k'K'$ by a process exactly analogous to that by which the sine-series for K is derivable from the k' -series for K' .

The six quantities E, I, G, U, V, W are considered in detail in the Quarterly Journal of Mathematics, vol. xx. pp. 313-361, and in the Proceedings of the Cambridge Philosophical Society, vol. v. pp. 184-208. The formulæ in these papers show that E forms one of a triad of corresponding fundamental quantities of which the other two members are I and G .

§ 9. It may be remarked that $I = -J$, where J is the

* Proc. Camb. Phil. Soc. vol. v. p. 204.

quantity so denoted by Weierstrass, and $=K-E$. We thus have

$$\begin{aligned}\frac{2E}{\pi} &= \sin \theta + \frac{1}{2} \sin 3\theta + \frac{1^2}{2^2} \sin 5\theta + \frac{1^2 \cdot 3}{2^2 \cdot 4} \sin 7\theta \\ &\quad + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 9\theta + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \sin 11\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 13\theta + \&c., \\ \frac{2J}{\pi} &= \sin \theta - \frac{1}{2} \sin 3\theta + \frac{1^2}{2^2} \sin 5\theta - \frac{1^2 \cdot 3}{2^2 \cdot 4} \sin 7\theta \\ &\quad + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin 9\theta - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \sin 11\theta + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin 13\theta - \&c.\end{aligned}$$

LVIII. *Notices respecting New Books.*

An Elementary Treatise on Dynamics, containing Applications to Thermodynamics. By B. WILLIAMSON, M.A., &c., and F. A. TARLETON, LL.D. London: Longmans, Green, and Co. 459 pp.

THIS is unquestionably an important work on Dynamics, comprising within the compass of 459 pages the Dynamics of a Particle (218 pp.), the Kinematics and Kinetics of a Rigid Body (207 pp.), including two chapters on Energy (40 pp.) and one on Small Oscillations (43 pp.), and a final chapter on Thermodynamics (29 pp.). Except for a few applications to elastic material and some to the thermodynamics of gases (as illustrations of the Theory of Energy), the work is confined to the Motion of Rigid Bodies, and is on the whole a most carefully thought-out treatise on so large a subject.

It professes to "start from the most elementary conceptions, so that any student who is acquainted with the Calculus can commence the Treatise without requiring the previous study of any other work on the subject" (see preface). It is perhaps a natural result of this that the early parts are easy and in considerable detail, whilst the later parts are difficult from their conciseness. It may, indeed, be doubted whether any student could really read this Treatise through (and work out the examples) without external help in the later parts. The examples have been chosen with great care so as to form excellent illustrations of the text; of a great many, however, it must be said that they are rather instructive problems which have been solved by accomplished mathematicians than exercises for students; more or less full details of the working of these are given:—*e.g.* Ex. 15 to 27, on Kinetics of a Rigid Body, are "the substance of Prof. MacCullagh's Lectures on Rotation."

It is singular that though the work bears the names of two mathematicians on the titlepage, and seems to have received at least partial revision from a third (see preface), a decided looseness