John Bolyai's "Science Absolute of Space." II

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JOHN BOLYAI'S "SCIENCE ABSOLUTE OF SPACE."

II.

WE have already explained pp. (25-31) that there are two types of uniform three-dimensional space, viz., finite or elliptic space, in which the geodesics or straight lines are closed and of constant length, and infinite or hyperbolic space, in which the geodesics are unclosed and of infinite length. Euclidean or parabolic space may be regarded as a special third type, or as the dividing case between the two general types; but it is best to regard it as a variety of hyperbolic space, owing to the reality of the infinite. In elliptic space all geodesics through a point either return without intersecting again, or have at most one other fixed point of intersection. In hyperbolic space it does not appear necessary that a geodesic should consist of only a single branch, though this is of course the simplest case; if the geodesics have several branches, all geodesics through a point will have a fixed point on each branch, and two geodesics which intersect will have as many points of intersection as they have branches. When we consider two-dimensional space we find that, even in the elliptic type, the geodesics may have any number of intersections, and may be unclosed and infinite. The proof to the contrary on p. 31 does not apply when the geometry is confined to a surface.

The distinctive property of uniform space, following at once from the principle of superposition, is that the sides and angles of any triangle are connected by three fixed independent relations. Directly connected with this is the remarkable theorem that the fixed relations (if any) which exist for any triangle in space can be determined, provided only it is known that there are three of them. They cannot, in fact, be any other than the well-known formulae of Euclidean spherical trigonometry, or their equivalents in hyperbolic functions. The former lead to elliptic and the latter to hyperbolic geometry. Hence it follows that there are no other types of uniform space than

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the elliptic and hyperbolic, whatever varieties there may be of each type.

The analytical geometry of great circles on spheres with finite radii is the same in all kinds of uniform space, forming the most intimate connecting link between them. Small spheres have approximately the same shape in all space, whether uniform or not; but as the radii become greater they diverge more and more from the shape of Euclidean spheres. In uniform space the locus of all points at a given distance from a plane (geodesic surface) on both sides is a sphere, or else consists of two antipodal spheres which may be regarded as forming one complete sphere. A plane is therefore formed by the coincidence of a sphere with itself, back to back. Plane hyperbolic geometry is the same as that which would have place on a Euclidean sphere with purely imaginary radius; and holds in all essential respects, as shown by Beltrami, on any real Euclidean surface of constant negative curvature.

The properties of simple elliptic space are in some respects less easily grasped than those of other uniform spaces. Any plane encloses the whole of space, *i.e.* the whole of space is inside it, and it has no outside. If the plane be regarded as having infinitesimal thickness, the outside consists only of the material of which the plane is made. A line can pass from one side of the plane to the other without crossing it, as may be seen by following the course of any geodesic which does not lie in the plane. Also a geodesic which makes a single circuit in the plane ends on the other side from which it started; and the plane is not divided into two parts by cutting it along a geodesic. The special difficulty is in forming a mental picture of planes or any surfaces on which closed lines can be drawn with only one point of intersection. This property can be illustrated on Euclidean surfaces possessing singularities. If, for example, a cylinder be bent until the two arms lie in contact along a generating line, we get a surface of the kind described. Each of the closed lines must pass over the bent portion of the cylinder, and extend to both arms, passing from one on to the other by crossing the line of contact.

There is also a very interesting way of practically converting Euclidean into elliptic or hyperbolic space. This consists in assuming a new definition of distance. Suppose, for example, the distance between any two points P , Q in Euclidean space to be defined as $PQ \div OP \cdot OQ$, where O is any fixed point; and let this distance be denoted by $\overline{PQ}$. Take any two fixed points, A , B , and consider the locus of a point P such that $\overline{AP} + \overline{PB} = \overline{AB}$, or $AP \cdot OB + PB \cdot OA = AB \cdot OP$. The locus of P is not a surface, as would generally happen from a single condition, but a line, *viz.* the circle OAB . Also if P , Q , R be any three

points on the circle OAB we have $\overline{PQ} + \overline{QR} = \overline{PR}$. These facts suffice to make the above definition of distance an appropriate one for some system of geometry, in which the geodesics or shortest lines are circles through O . Further, if we examine the relations connecting the sides and angles of any triangle ABC formed by such geodesics, we find that the sum of the angles is two right angles, and that the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ are proportional to the sines of the opposite angles. Hence it follows that the geometry which applies to these geodesics is Euclidean. The geodesic surfaces are spheres through O . To get the idea of rotation we have to imagine an infinite series of spheres drawn through the same fixed circle, each sphere becoming visible in succession while the rest disappear. Every property of Euclidean straight lines and planes will be true of circles and spheres through O , provided the distance between any two points is defined as above. All this is only equivalent to ordinary inversion; but the method can be extended. If for geodesics we take circles whose power with respect to a fixed point O is constant (instead of zero as above), there is a corresponding appropriate definition of distance, and the geodesic surfaces are in this case spheres whose power with respect to O is the same as that of the circles. The geometry which applies to these is hyperbolic or elliptic according as the power is positive or negative. If the power is positive the sphere with centre O and radius equal to the square root of the power is cut orthogonally by all the geodesic circles and spheres; and the extent of space to which the geometry applies is confined either to the interior or to the exterior of the orthogonal sphere. This is the only way, by merely taking a new definition of distance, of making elliptic and hyperbolic geometry applicable to Euclidean three-dimensional space; but the same thing can be effected on a Euclidean plane in an infinite number of ways.

In respect to uniform space Bolyai was right in assuming that his geometry was the "Science Absolute"; for all real curves and surfaces in elliptic and Euclidean space, and many imaginary ones, have real equivalents in hyperbolic space. In describing the properties of hyperbolic space, we shall, following Bolyai, speak of straight lines instead of geodesics, in the same sense that we have already used the term plane as equivalent to geodesic surface.

The assumptions made by Bolyai may be stated as follows: (i.) that space is uniform, so that proof by superposition is valid; (ii.) that straight lines are unclosed and infinite; (iii.) that two straight lines cannot enclose a space; (iv.) that a plane, defined as the surface generated by the rotation of a straight line about a perpendicular straight line, has the property that the straight line joining any two points on the plane lies wholly within it.

We have attempted to show (p. 31) that (iii.) and (iv.) follow from (i.) and (ii.); but, whether this is actually the case or not, it is evident that (iii.) and (iv.) must be classed as distinct assumptions so long as they are not proved. These assumptions being granted, it follows that the first twenty-eight propositions of Euclid Book I., and the first nineteen propositions of Book III., will also hold in Bolyai's geometry, since the same methods of proof will be applicable. Also some propositions of Book XI. will hold. Thus two planes which meet intersect in a straight line. A straight line PQ perpendicular to two straight lines PA , PB is normal to the plane PAB , *i.e.* is perpendicular to every straight line in the plane through P . Again, any two normals to a plane lie in one plane, although they are not parallel. This is used by Bolyai more than once, and may be proved as in Euclid XI. 6. Hence it follows that if two planes are at right angles at one point, *i.e.* if each contains the normal to the other at that point, they will be at right angles at every point in their line of intersection. The angle between two planes is taken to be the same as in Euclid, *viz.* the angle between the lines in the two planes perpendicular to their line of intersection at any point. It is assumed that this angle is the same at every point on the line of intersection; this is by no means evident, but may be proved from considerations of symmetry, and in other ways.

Bolyai defines a parallel in his first paragraph (§ 1) as follows: "If the line BP is not cut by AR , situated in the same plane, but is cut by every straight line within the angle BAR , AR is called parallel to BP ." It is clear that there is only one parallel through A to BP , and that the angles PBA , RAB must be together equal to or less than two right angles; if equal, it is proved (§ 13) that the same property must hold for any pair of parallel lines crossed by any third line, and that consequently all the geometry of Euclid is applicable. Bolyai therefore, like Lobatchewsky, makes the contrary supposition, *viz.* that the sum of the angles PBA , RAB is less than two right angles, and proves (§ 14) that the same property must then hold for any pair of parallel lines. On this hypothesis it is to be noticed that the parallel AS to BQ or PB produced is not in a straight line with AR , but makes with it, on the side on which PBQ is, an angle less than two right angles; and that a straight line drawn in either of the external angles between AR and AS will not cut PQ at all. Thus some of the fundamental properties of Euclidean lines and curves have no place in the geometry of Bolyai. Parallel planes do not exist, since the parallels through a given point

to all lines on a plane do not themselves lie on a plane, but on a cone with the given point as vertex.

From Bolyai's second to his fifth paragraph (§§ 2-5) there is nothing which is not self-evident; these lead up to § 6, which proves that two lines which are parallel maintain their property of parallelism for the same direction of extension at all their points. In § 7 we have the interesting theorems, very different to the corresponding theorems in Euclid, that two lines which are parallel to the same line are parallel to one another, and two planes through two parallel lines intersect in a third line parallel to both. From this it may be assumed in theory that any system of parallel lines meet in the same point at infinity; and that a straight line meets infinity in opposite directions in two distinct points. § 9 contains a most important and remarkable theorem, viz., that two planes which cut a third plane along parallel lines must intersect if they make with it two angles on the same side which are together less than two right angles. It follows as an immediate consequence that the sum of the three interior angles of the three intersecting planes is equal to two right angles; but Bolyai reserves this in order to introduce it with greater effect later on (§ 21).

The first ten pages (§§ 1-15) make it abundantly clear that Euclidean is but a particular case of the more general geometry founded on the rejection of the parallel axiom. From this point Bolyai begins to develop the latter to its full extent. Starting with a plane, and any straight line AM on it, he defines the *L-line with axis AM* as the locus of all points B in the plane such that AM and the parallel through B to AM make equal angles with AB and BA respectively; and the *F-surface with axis AM* as the locus of all points B in space possessing a like property. These *L-lines* and *F-surfaces* do not exist in elliptic space, and become straight lines and planes in Euclidean space. Bolyai proves that an *L-line* has the properties of a circle, and an *F-surface* those of a sphere, their centres being at infinity. Every plane through AM cuts the *F-surface* with axis AM in an *L-line* with axis AM ; and every plane through A oblique to AM cuts the *F-surface* in a finite circle. L is a uniform line, and F a uniform surface. Again, AM is a normal to the *F-surface* and the parallel to AM through any point B on the surface is both a normal and an axis of the *F-surface*, just as AM is. Finally follows a theorem (§ 21), which is undoubtedly the most remarkable property of hyperbolic space, that the sum of the angles of any triangle formed by *L-lines* on an *F-surface* is equal to two right angles. On this theorem Bolyai remarks: "From this it is evident that Euclid's twelfth Axiom and all things which are claimed in geometry and plane trigonometry hold good absolutely in *F*, *L-lines* taking the place of straight lines. Therefore the

trigonometric functions are defined here to have the same values as in Euclidean geometry; and the perimeter of the circle on F with L form radius r is $2\pi r$; and likewise the area of the same circle on F is πr^2 ; where $\pi = 3.1415926\dots$ The whole of Euclidean geometry is true for uniform curves no less than for straight lines, provided that, as is the case with L -lines on an F -surface, the assumptions made by Euclid hold with respect to them. In particular, two equiangular triangles need not be of the same size, but have their sides proportional; and on this property is based the definitions of the trigonometric functions. Bolyai, in fact, deduces his geometry of hyperbolic space from the Euclidean geometry of the F -surfaces. It should be noticed that a circle on an F -surface, like a circle on a sphere, is also a plane circle with a shorter radius.

From the last theorem two other fundamental theorems are deduced in §§ 25, 26, viz., that the sines of the angles of a rectilineal triangle are proportional to the perimeters of the plane circles described with the opposite sides as radii; and that the sines of the angles of a triangle formed by great circles on a sphere are proportional to the sines of the angles which the opposite sides subtend at the centre of the sphere. The trigonometry of great circles on finite spheres in any uniform space is the same as in Euclidean space.

The proper place for the last-mentioned §§ 25, 26 would appear to be immediately after § 21; then should follow §§ 22-24, 28, 27, 29-31, in the order named. These eight paragraphs are in themselves very involved, and their difficulty is increased by the

order in which they are given. § 22 proves that parallels to an L -line are also L -lines; i.e. if equal lengths $AC, BD, \dots$ be taken on the normals to an L -line AB , the locus of the points $C, D, \dots$ is an L -line having the same system of normals as AB . § 23 proves that the ratio of the arcs AB, CD is a function of the distance AC between the L -lines, and is independent of the length of the arc AB . If x denotes the length of AC , the ratio $AB:CD$, which is a function of x , is denoted by the corresponding capital letter X . It is proved in § 24 that $\log X : \log X' = x : x'$; in § 28, that $X = \sin ADB : \sin DAC$; and in § 30, that $X = e^x$.

In § 27 Bolyai considers the locus of a point whose perpendicular distance from a straight line AB is given. Any such line is a uniform curve, concave to AB , whose curvature is greater than that of a straight line, and less than that of an L -line. The complete locus on both sides of the line may be regarded as a circle without a centre, and may be called a hyperbolic circle, since it has two branches. There are thus four kinds of uniform lines, viz., straight

lines, hyperbolic circles with two branches, *L*-lines or parabolic circles with infinite radii, and elliptic circles with finite radii.

In § 29 it is proved that if a parallel *AC* is drawn to a given line *BD*, at a distance *BA* = *x* from *BD*, then the angle *CAB* is given by the formula $\cot \frac{1}{2}u = X$, or $\operatorname{cosec} u = \cosh x$ (by § 30). In § 30 an expression is sought for the perimeter of a plane circle in terms of the radius *x*. Let the *L*-line with axis *AC* cut *BD* in *E*; and let the arc *AE* be denoted by *r*. Then if *EACD* be rotated round *ED*, the arc *EA* describes a portion of the *F*-surface having *ED* and *AC* for axes; and the point *A* describes a circle on the *F*-surface with centre *E* and radius *r*, and at the same time the plane circle with centre *B* and radius *x*. Hence the perimeter of the plane circle with radius *x* is $2\pi r$ (by § 21). It is shown in § 30 that the ratio $r : \cot u$ is a constant independent of *r*; denoting this constant by *i*, so that $r = i \cot u$, it is next shown that the corresponding ratio *I* (the ratio of *L*-lines at a distance *i* apart) is no other than the base *e* of natural logarithms, "which noted quantity shines forth here also." This constant length *i* serves as an absolute unit of measurement in the hyperbolic space considered; hence in what follows we take $i = 1$. By § 24, we have $\log X : \log I = x : i$, or $X = e^x$, since $I = e$, $i = 1$. Hence the perimeter of the plane circle with radius *x* is given by

$$\begin{aligned} 2\pi r &= 2\pi i \cot u = 2\pi \cot u = \pi(\cot \tfrac{1}{2}u - \tan \tfrac{1}{2}u) \\ &= \pi(e^x - e^{-x}) = 2\pi \sinh x. \end{aligned}$$

Also by § 29, since $\cot \frac{1}{2}u = X = e^x$, we have

$$\operatorname{cosec} u = \tfrac{1}{2}(\cot \tfrac{1}{2}u + \tan \tfrac{1}{2}u) = \cosh x.$$

Having thus found the perimeter of the circle, Bolyai obtains (§ 31) the trigonometric formulae for plane right-angled triangles, "whence the solution of all triangles is easy." The following formulae are proved, *C* being the right angle,

$$\begin{aligned} 1 : \sin A &= \sinh c : \sinh a, \quad \cos A : \sin B = \cosh a, \\ \cosh c &= \cosh a \cdot \cosh b = \cot A \cdot \cot B. \end{aligned}$$

These are the same as the formulae for Euclidean spherical right-angled triangles, except that hyperbolic functions of the sides appear in the place of trigonometric functions. A wrong construction is given for the proof of $\cosh c = \cosh a \cdot \cosh b$; instead of drawing " $\beta\beta'$ and $\gamma\gamma'$ parallel to aa' ," it should be "draw aa' perpendicular to $\beta a\gamma$, $\beta'a'\gamma'$ perpendicular to aa' , and $\beta\beta'$, $\gamma\gamma'$ perpendicular to $\beta'a'\gamma'$."

The equation to the hyperbolic circle at distance δ from $xx' + yy' = 1$ is

$$\alpha(1 - x^2 - y^2) = (1 - xx' - yy')^2, \text{ where } \alpha = (x'^2 + y'^2 - 1) \sinh^2 \delta. \dots\dots(4)$$

These are also formulae giving the distance between the points x, y and x', y' , and the distance of the point x, y from the straight line $xx' + yy' = 1$. In taking the square root of (3) the positive sign should be prefixed, since all the factors as written are necessarily positive.

The equation to the L -line, or parabolic circle, with the point $\cos \alpha, \sin \alpha$ at infinity for centre, all lines through this point being normal axes, is

$$\alpha(1 - x^2 - y^2) = (1 - x \cos \alpha - y \sin \alpha)^2. \dots\dots\dots(5)$$

The condition that $xx' + yy' = 1$, $xx'' + yy'' = 1$ are perpendicular, is

$$x'x'' + y'y'' = 1. \dots\dots\dots(6)$$

Transformation of equations for any change of rectangular axes can be effected without much difficulty. If the origin be unchanged and the axes be turned round an angle α , we have

$$x = r \cos(\theta + \alpha) = x' \cos \alpha - y' \sin \alpha, \quad y = x' \sin \alpha + y' \cos \alpha. \dots\dots\dots(7)$$

The origin may be changed to any point in two steps, by first changing to a

point O' on OX , and then to a point on the new axis $O'Y'$. If $OO' = \gamma$, we have

$$x = \tanh OM = \tanh(OO' + O'M) = \frac{x' + \tanh \gamma}{1 + x' \tanh \gamma}; \dots\dots\dots(8)$$

and $\tanh PM = \tanh POM \cdot \sinh OM = \frac{y}{x} \cdot x \cosh OM = y \cosh OM = y' \cosh O'M,$

$$\therefore y = \frac{y' \cosh O'M}{\cosh(OO' + O'M)} = \frac{y' \operatorname{sech} \gamma}{1 + x' \tanh \gamma}. \dots\dots\dots(9)$$

These formulae show that the degree of an equation in x, y is not altered by any transformation of rectangular axes, as might be inferred from the fact that a straight line has always a linear equation. We may then define a conic as a curve whose equation is of the second degree. The equation of a real conic, or any real curve, must be such that real values x, y of the variables can satisfy the relation $x^2 + y^2 < 1$. A straight line cannot cut a conic in more than two real points; and it may cut it in one real point without cutting it in a second.

Any real conic (with one special exception) has at least one real axis of symmetry, such that any chord perpendicular to the axis is bisected by the axis. One proof of this depends on the property of a cubic equation stated in Problem 119, p. 70. Also one axis of a conic cuts the curve in a real vertex A . If this axis cuts the conic in a second real vertex A' , the line at right angles to AA' through its middle point C is also an axis; and every chord through C is bisected at C . Thus conics may be divided into central conics and non-central conics, or parabolas.

Central conics may be divided into ellipses, which cut the absolute $x^2 + y^2 = 1$ in four imaginary points; and hyperbolas, which cut the absolute in four real points, and have four real asymptotes. Parabolas in general cut the absolute in two real and two imaginary points, and have two real asymptotes; but they include also all conics which have contact at a single point with the absolute. Finally, circles are conics which have double contact with the absolute. The analytical theory of conics is in many respects the same as in Euclidean geometry, the great difference being that the coordinates of a real point may not have unrestricted real values. The formulae for the straight line joining two given points, for the tangent at a given point on a conic, for the polar of a given point, and the pole of a given line, are of precisely the same form, and obtained in the same way, as in Euclidean geometry.

The equation of any central conic may be reduced to the form $\frac{x^2}{a} + \frac{y^2}{b} = 1$, where a is greater than b . Taking $c^2 = \frac{a-b}{1-b}$, and $e^2 = \frac{a-b(1+a)}{a}$, the various kinds of conics may be described in greater detail as follows:

I. *Ellipses* ($0 < b < a < 1$). Ellipses are closed curves, with real foci S, S' ($0, \pm c$), and real vertices A, A' ($0, \pm \sqrt{a}$), such that, P being any point on the curve,

$$PS + PS' = AA'.$$

(i.) If $e^2 > 0$, the ellipse has real directrices ($cx = \pm a$), possessing the property that if PM be perpendicular to the directrix XM ($cx = a$), then

$$\sinh PS : \sinh PM = e;$$

i.e. the perimeters of the circles with PS, PM as radii are in a constant ratio, less than unity.

(ii.) If $e'^2 = -e^2 > 0$, the directrices are imaginary; but if S_1 is the point $0, \frac{c}{a'}$, the relation in (i.) gives place to

$$\sinh PS : \cosh PS_1 = e'.$$

The limiting case of an ellipse is given by $a = 1$, when the vertices and foci are at infinity, and the ellipse becomes a hyperbolic circle.

II. *Convex Hyperbolas* ($0 < b < 1 < a$). The vertices and foci have disappeared in the imaginary region beyond infinity; and the curve has now two branches, one lying between the lines $y = \sqrt{b}, y = \sqrt{\frac{b(a-1)}{a-b}}$, and both lying between the lines $x = \pm \sqrt{\frac{a(1-b)}{a-b}}$.

(i.) If $e^2 > 0$, and S_1 be the point $0, \frac{c}{a}$, and

PM_1 perpendicular to the line $cx = 1$, then

$$\cosh PM_1 : \cosh PS_1 = e.$$

(The points S, X, M in the figure should be S_1, X_1, M_1 .) The limiting case is given by $a = \infty$, when the conic becomes the two straight lines $y = \pm \sqrt{b}$.

(ii.) If $e'^2 = -e^2 > 0$, and PM, PM_1 perpendiculars to the lines $cx = a$, $cx = 1$, then

$$\cosh PM_1 : \sinh PM = e'.$$

A similar relation to this last, for two lines perpendicular to OY , holds for any convex hyperbola.

III. *Concave Hyperbolas* ($b < 0 < a < 1$). The vertices, foci $(0, \pm c)$, and directrices ($ax = \pm a$) are all real; and we have

$$PS \sim PS' = AA', \quad \sinh PS : \sinh PM = e.$$

IV. *Conics with special properties.*

(i.) Conics for which $e^2 = 0$. The relation $\sinh PS : \sinh PS_1 = \text{constant}$ gives an ellipse for which $e = 0$; $\cosh PM : \cosh PM_1 = \text{constant}$, a convex hyperbola; and $\sinh PS : \cosh PM = \text{constant}$, an ellipse or convex hyperbola. In all such conics the fixed points and lines S, S_1, XM, X_1M_1 may to some extent be chosen arbitrarily. In the ellipse the axis AA' subtends a right angle at any point on the curve; the directrices are at infinity; and S, S_1 are any pair of points $(0, c), (0, c')$ on the axis, such that $cc' = a$.

(ii.) Each of the relations

$$\begin{aligned} \cosh PS : \cosh PS_1 &= \text{constant}, \quad \sinh PM : \sinh PM_1 = \text{constant}, \\ \cosh PS : \sinh PM &= \text{constant}, \end{aligned}$$

gives a pair of straight lines.

V. *Parabolas.* The equation of a parabola can in general be reduced to the form $\frac{x^2}{a} + \frac{y^2}{b} = x$, where a is positive and greater than 1. By changing the origin to $0, \frac{1}{a}$ the equation assumes the form $\frac{y^2}{b} = x + \frac{1}{a}$.

(i.) Parabolas in which $b < a$. This includes the case in which b is negative, the positive direction of the axis OX being then outwards. For parabolas of this kind there are two fixed points S, S_1 on the axis, and two fixed lines XM, X_1M_1 perpendicular to the axis, such that

$$\sinh PS : \sinh PM = \cosh PM_1 : \cosh PS_1 = \left(1 - \frac{b}{a}\right)^{\frac{1}{2}}.$$

If we consider the conic with focus S , directrix XM , and vertex A on SX , such that $\sinh PS : \sinh PM = \sinh AS : \sinh AX$, it is clear that when A is near to S , the conic is an ellipse, and when near to X , a hyperbola. For a certain position of A the centre of the ellipse is at infinity; and if XA' be made equal to AS , the conic with vertex A' is a hyperbola with centre at infinity. For any position of the vertex between these extremes the conic is a parabola ($b < a$), which cannot in any sense be said to have a centre. If the vertex of the parabola bisects SX , its equation is $y^2 = bx$.

(ii.) Parabolas in which $b > a$. In this case the relations in (i.) give place to

$$\sinh PS : \cosh PS_1 = \cosh PM_1 : \sinh PM = \left(\frac{b}{a} - 1\right)^{\frac{1}{2}}.$$

[In both (i.) and (ii.), $\frac{x^2}{a} + \frac{y^2}{b} = x$ being the equation of the curve, and $(0, c), (0, c_1)$ the points S, S_1 , and $dx = 1, d_1x = 1$ the lines XM, X_1M_1 , then $c(c^2 < 1)$ and $d_1(d_1^2 > 1)$ are determined without ambiguity as the roots of $ab(1 - x^2) = 4x(a - x)$, and c_1, d as the roots of

$$a^2b(1 - x^2) = 4(b - a)(1 - ax).]$$

Parabolas in which $b = a$ are included in IV. (i.).

(iii.) *Contact Parabolas.* The equation of a parabola which touches the absolute can in general be reduced to one of the forms $y^2 = bx(1-x)$ or $y^2 = b(x+1)$, but not to both forms. The only real vertex of the parabola $y^2 = b(x+1)$ is at infinity.

The equation to the parabola which has contact of the second order with the absolute at the point 0, 1 can be reduced to the form $1 - x^2 - y^2 = (1-x)y$, both axis and vertex being at infinity. This parabola is of fixed shape and size, and is remarkable as being the only conic without an axis of symmetry.

A parabola which has contact of the third order with the absolute is an L -line, or parabolic circle.

VI. Circles, or conics which have double contact with the absolute, include all uniform lines, as may be seen at once from formulae (3), (4), (5), p. 57.

Two non-intersecting straight lines are a limiting form of a convex hyperbola, two intersecting lines of a concave hyperbola, two parallel lines of a contact parabola, two coincident lines of a hyperbolic circle, and a single line of an ordinary parabola in which $b > a$.

EDITOR.

NOTES ON ELEMENTARY DYNAMICS. II.

IN problems where more than two bodies are simultaneously in collision, the rules usually stated in the text books are not, in general, sufficient for their solution. It has been suggested to me that I should continue the subject of Collision by a discussion of the method required in such cases.

In what follows I confine myself to the case of smooth spheres impinging on one another, or on fixed bodies, so that rotations do not fall to be considered.

I begin by explaining the method which seems to be at present in use, but which I have not seen explained in any text book; and thereafter I propose to discuss the limits of the validity of the method, taking up incidentally some other points connected with the subject, *e.g.*, its nomenclature.

The method in question is based on an assumed Law, which is a generalization of Newton's Experimental Law (*Principia*: Scholium on Cor. VI. to the Axiomata.)

Newton's Law is that the relative velocity of bodies, after directly impinging, bears a constant ratio to their relative velocity of approach. In the case of *oblique* impact, which is not referred to by Newton, it is usual to assume that the Law holds good for the components of the velocities normal to the surfaces of contact, the tangential components being unaltered.

Now this Law is equivalent (as is shewn in the text books) to the statement that the Impulse on either body *after* the instant of greatest compression (hereafter for brevity called