For the majority of substances, then, an increase in the dielectric constant similar to that which I have found in the substances investigated might be expected; and it appears to me that in experimenting with slow swings there is more chance of success, since in that case, proceeding as Schiller or Cohn and Arons have done, the causes of error can be more easily eliminated from the experiments or allowed for in the observations.

Vienna, May 1890.

XXIII. *On the Flexure of a Flat Elastic Spring.*

By HORACE LAMB, M.A., F.R.S.*

IT is a well-known consequence of the ordinary theory of flexure that when a bar of (say) rectangular section is slightly bent by opposing couples in a plane parallel to one pair of faces, so as to form a circular arc of radius ρ , the remaining faces take an anticlastic curvature σ/ρ , where σ is "Poisson's ratio;" and this fact has been made the basis of ingenious experimental methods of determining σ , by Cornu† and Mallock‡. It appears to have been first remarked by Thomson and Tait ('Natural Philosophy,' § 717) that this tendency to anticlastic curvature imposes in certain cases a limit to the degree of flexure, beyond which the theory in question will not apply. "For unless the *breadth* of the bar (or diameter perpendicular to the plane of flexure) be very small in comparison with the mean proportional between the radius $[\rho/\sigma]$ and the thickness, the distances [in any cross section] from [a line through the axis perpendicular to the plane of flexure] to [one pair of corners] would fall short of the half-thickness, and the distances to [the other pair] would exceed it by differences comparable with its own amount. This would give rise to sensibly less

10 per cent. of sulphuric acid has been added. With Hertz oscillations on the other hand, the distilled water, used in the same trough, behaved as if it had a dielectric constant of about 300. If a few drops of sulphuric acid were then added to it, the oscillations either entirely ceased or at any rate were no longer visible by my method. The insertion of such thick layers would be of advantage, inasmuch as one could compare the large dielectric constants not with air, but with a dielectric constant ∞ . This mode of procedure cannot unfortunately be directly applied to Hertz oscillations, since the insertion of a conductor 27 centim. long alters the length of the path of the oscillations by too great an amount.

* From the Proceedings of the Manchester Literary and Philosophical Society, April 29th, 1890. Communicated by the Author.

† *Comptes Rendus*, August 2, 1869.

‡ Proc. Roy. Soc. June 19, 1879.

and greater shortenings and stretchings in the filaments towards the corners than those expressed in our formulæ, and so vitiate the solution. Unhappily, mathematicians have not hitherto succeeded in solving, possibly not even tried to solve, the beautiful problem thus presented by the flexure of a broad very thin band (such as a watch-spring) into a circle of radius comparable with a third proportional to its thickness and its breadth."

The difficulty thus arising in the case of a straight band, when the flexure exceeds a certain amount, makes its appearance *ab initio*, even for infinitely small changes of curvature, in the case of a band originally curved, and is in this way closely connected with the circumstance, first pointed out by Mr. Love*, that it is in general impossible to satisfy the boundary conditions for a curved plate or shell by a deformation such that the middle surface is absolutely unextended. In considering this latter question I had occasion, recently†, to work out the uniform flexure of a cylindrical plate, but I did not notice at the time that the same analysis, with the proper change of meaning of a coefficient, gives the solution of the problem proposed by Thomson and Tait. As the matter is of independent interest, and as so much importance has been attached to it by these writers, I take the liberty of reproducing the investigation (as suitably modified) in a separate form. The main results are such as might be anticipated from the above quotation. The extensions and contractions of the middle surface of the band, which are called into play by the tendency to the contrary curvature in the direction of the breadth, keep this curvature in check, so that the strained form never deviates appreciably from that of a cylinder. When the radius ρ is large compared with the third proportional aforesaid, the couple required to maintain the flexure has the value given by the ordinary theory; whilst in the opposite extreme it tends to the value appropriate to a *plate*. For intermediate cases we must have recourse to the general formula (18) given below.

Considering, then, a straight flat spring whose breadth $2b$ is large compared with the thickness $2h$, let us in the first instance suppose that by a proper application of force to the two ends (to be afterwards determined) it is bent so that the strained form is one of revolution. As regards the amount of the bending, we shall suppose only that ρ the radius of curvature of the medial line or axis, is large

* Phil. Trans. 1881 (A), pp. 521, 524.

† Proc. Lond. Math. Soc. Jan. 1890. See also § 7 of a previous paper "On the Flexure of an Elastic Plate," December 1889.

where λ, μ are the elastic constants of Lamé, equivalent to $m-n$ and n , respectively, in the notation of Thomson and Tait. If, further, the plate experience curvatures $1/\rho_1$ and $1/\rho_2$ parallel to its edges, the corresponding flexural couples are

$$\left. \begin{aligned} G &= \frac{4(\lambda + \mu)\mu}{\lambda + 2\mu} \left(\frac{1}{\rho_1} + \frac{\sigma}{\rho_2} \right) \cdot \frac{2h^3}{3} \\ H &= \frac{4(\lambda + \mu)\mu}{\lambda + 2\mu} \left(\frac{1}{\rho_2} + \frac{\sigma}{\rho_1} \right) \cdot \frac{2h^3}{3} \end{aligned} \right\} \dots \dots (5)$$

Now let w denote the deviation at any point of the middle surface from the cylindrical surface (of radius ρ) drawn through the medial line, in the strained condition; and let σ_0 be the extension of this medial line. We have evidently, in the present case,

$$\left. \begin{aligned} \sigma_1 &= \sigma_0 + w/\rho, \\ \rho_1 &= \rho + w, \\ 1/\rho_2 &= -d^2w/dx^2. \end{aligned} \right\} \dots \dots (6)$$

The equations (1) and (4) show that

$$\sigma_2 = -\sigma\sigma_1,$$

and substituting from (4), (5), and (6), in (2), we get

$$\frac{h^2}{3} \frac{d^2}{dx^2} \left(\frac{d^2w}{dx^2} - \frac{\sigma}{\rho + w} \right) + \frac{1 - \sigma^2}{\rho} \left(\sigma_0 + \frac{w}{\rho} \right) = 0.$$

An error of the order w/ρ in the values of the flexural couples is clearly unimportant*, so that we may write for the last equation

$$\frac{h^2\rho^2}{3} \frac{d^4w}{dx^4} + (1 - \sigma^2)(w + \sigma_0\rho) = 0. \dots (7)$$

The boundary conditions (3) give

$$\frac{d^2w}{dx^2} - \frac{\sigma}{\rho + w} = 0,$$

$$\frac{d}{dx} \left(\frac{d^2w}{dx^2} - \frac{\sigma}{\rho + w} \right) = 0,$$

or, by a similar approximation,

* This approximation may also be justified *à posteriori*. It will appear that the terms neglected are of the order h/ρ compared with those retained.

$$\left. \begin{aligned} \frac{d^2 w}{dx^2} - \frac{\sigma}{\rho} &= 0, \\ \frac{d^3 w}{dx^3} &= 0, \end{aligned} \right\} \dots \dots \dots (8)$$

Let us write for shortness

$$m^4 = \frac{3(1-\sigma^2)}{4h^2\rho^2} \dots \dots \dots (9)$$

The proper solution of (7) is then

$$w + \sigma_0 \rho = A \cos mx \cosh mx + B \sin mx \sinh mx; \dots (10)$$

and the conditions (8) to be satisfied at the edges $x = \pm b$ give

$$\left. \begin{aligned} -A \sin mb \sinh mb + B \cos mb \cosh mb &= \frac{\sigma}{m^2 \rho} \\ A (\cos mb \sinh mb + \sin mb \cosh mb) \\ + B (\sin mb \cosh mb - \cos mb \sinh mb) &= 0 \end{aligned} \right\} \dots (11)$$

leading to

$$\left. \begin{aligned} A &= -\frac{\sigma}{m^2 \rho} \cdot \frac{\sin mb \cosh mb - \cos mb \sinh mb}{\sinh 2mb + \sin 2mb} \\ B &= \frac{\sigma}{m^2 \rho} \cdot \frac{\sin mb \cosh mb + \cos mb \sinh mb}{\sinh 2mb + \sin 2mb} \end{aligned} \right\} \dots (12)$$

The condition that $w=0$ when $x=0$ gives

$$\sigma_0 \rho = A, \dots \dots \dots (13)$$

and the value of w is thus completely determined.

The distribution of applied force and couple over the ends (supposed straight) necessary to produce the strain in question is given by

$$\begin{aligned} P &= \frac{8(\lambda + \mu)\mu}{\lambda + 2\mu} (1 - \sigma^2) h \cdot \sigma_1 \\ &= \frac{2(3\lambda + 2\mu)\mu}{\lambda + 2\mu} h \left(\sigma_0 + \frac{w}{\rho} \right), \dots \dots (14)^* \end{aligned}$$

$$G = \frac{8(\lambda + \mu)\mu}{3(\lambda + 2\mu)} h^3 \left(\frac{1}{\rho} - \sigma \frac{d^2 w}{dx^2} \right) \dots \dots \dots (15)$$

This distribution is somewhat artificial, but the theory of "local perturbations" developed by St. Venant, Boussinesq,

* Since $\sigma = \lambda/2(\lambda + \mu)$.

and others, warrants us in asserting that if it be replaced by any other distribution having the same force- and couple-resultants, the form assumed by the band will not be sensibly altered, except within a distance from the ends comparable with the breadth. It appears from (7) and (8) that

$$\int_{-b}^b P dx = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

so that the forces on either end reduce to a couple

$$\int_{-b}^b G dx = \frac{8}{3} \frac{(\lambda + \mu)\mu}{\lambda + 2\mu} h^3 \left\{ \frac{2b}{\rho} - 2\sigma \left(\frac{dw}{dx} \right) \right\}_{x=b}. \quad . \quad . \quad . \quad (17)$$

On substitution from (10) and (12) this becomes, after a little simplification,

$$\int_{-b}^b G dx = \frac{8}{3} \frac{(\lambda + \mu)\mu}{\lambda + 2\mu} h^3 \cdot \frac{2b}{\rho} \left\{ 1 - \frac{\sigma^2 \cosh 2mb - \cos 2mb}{mb \sinh 2mb + \sin 2mb} \right\} \quad (18)$$

The form assumed by the cross section of the band, and the value of the flexural couple, depend on the magnitude of mb , which is comparable with $b/\sqrt{(h\rho)}$. For sufficiently small curvatures, *i.e.* so long as ρ is large compared with b^2/h , mb is small, and we find without difficulty

$$w = \frac{\sigma}{2} \frac{x^2}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

whilst the formula (18) for the flexural couple reduces to

$$\frac{4}{3} \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu} \frac{h^3 b}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

which is in fact the value given by the ordinary theory for a bar of breadth $2b$ and depth $2h$, the coefficient $(3\lambda + 2\mu)\mu/(\lambda + \mu)$ being Young's modulus.

As the curvature increases beyond the limit above indicated, the flexural couple increases in a greater ratio, until in the other extreme, when ρ is small compared with b^2/h , and mb is consequently large, the expression for the couple becomes

$$\frac{16}{3} \frac{(\lambda + \mu)\mu}{\lambda + 2\mu} \frac{h^3 b}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)^*$$

the same as for a *plate*.

* This exceeds the value (20), given by the ordinary theory, in the ratio $1/(1-\sigma^2)$.

We have also in this case

$$\left. \begin{aligned} A &= \sqrt{2} \frac{\sigma}{m^2 \rho} e^{-mb} \cos\left(mb + \frac{\pi}{4}\right) \\ B &= \sqrt{2} \frac{\sigma}{m^2 \rho} e^{-mb} \sin\left(mb + \frac{\pi}{4}\right) \end{aligned} \right\} \dots \quad (22)$$

approximately, so that the value of w given by (10) and (13) is insensible except close to the edges. Near the edge $x=b$ we have, for example,

$$w = \frac{1}{\sqrt{2}} \frac{\sigma}{m^2 \rho} e^{-m(b-x)} \cos\left\{m(b-x) + \frac{\pi}{4}\right\} \dots \quad (23)$$

At the edge itself the deviation from the cylindrical form is comparable with h , but it rapidly diminishes as we pass inwards, at the same time fluctuating in sign. The latter result may perhaps be unexpected, but a little consideration will show that it is intimately bound up with the supposition we have made, that there is no resultant *force* on the ends of the band. The amplitudes of the fluctuations diminish, however, so rapidly that it is not likely that this feature of the strain could ever be made the subject of observation.

XXIV. *Specific Inductive Capacity of Electrolytes.* By
EDWARD B. ROSA, *Fellow of the Johns Hopkins University* *.

[Plate VI.]

I. **T**HE study of the specific inductive capacity of water and other conducting liquids was suggested to me about a year ago by Professor Rowland. Cohn and Arons and others had recently found values of K for water, ranging from 75 to 85 and for alcohol from 25 to 30; these exceeded so many times the value of K for most non-conductors that it seemed possible that there might be a relation between the conductivity and inductivity †; that these high values were directly due either to comparatively large conductivity, or

* Communicated by the Author. An abstract of this paper was read before the American Association for the Advancement of Science, Indianapolis, August, 1890.

† This seems to be a somewhat prevalent opinion. Thus, Andrew Gray ('Absolute Measurements in Electricity and Magnetism,' p. 491) says, referring to Cohn and Arons' determination of K for water and alcohol: "These substances have, however, considerable conductivity, which would tend, of course, to give an apparently high specific inductive capacity."