where

$$l = A_1^2 v^2, \quad n = A_1 (R_1 - P_1) F^2,$$

$$m = FA_1 [P_1 F - L_1 v + N_1 (v - \beta) - R_1 (F - \alpha)].$$

In Klein's case the quantity n is zero—a fact which considerably reduces the number of types.* There is no difficulty in tracing the curve for any particular case; one device is, perhaps, worth mentioning. Since $l > 0$, it follows that $(y^2 - 2mx - nx^2)$ and $x/(2-x)$ are of the same sign; hence the plane is to be divided into regions by the conic $y^2 - (2mx + nx^2) = 0$ and the lines $x = 0$, $x = 2$; then the curve is either outside the conic and between the lines, or inside the conic and outside the lines.

In this way we find precisely the same results as those already indicated by the use of infinite series; this provides a check on the accuracy of the former work.

The Distribution of Velocity and the Forms of the Stream Lines due to the Motion of an Ellipsoid in Fluid, Frictionless or Viscous. By THOMAS STUART. Communicated by Dr. J. LARMOR, February 14th, 1901. Received, in revised form, May 17th, 1901.

1. In the *Quarterly Journal of Mathematics*, Vol. xxvi., pp. 70-74, D. Edwardes has investigated the motion due to an ellipsoid which is rotating with a small angular velocity ω round a principal axis in an infinite mass of incompressible viscous liquid, and by an ingenious, though indirect, method he deduces the component velocities of the fluid at any point, and hence the whole circumstances of the motion.

We begin by developing a more direct method of treating this problem, by analogy with Oberbeck's solution (*Borchardt*, Vol. LXXXI., p. 62) for an ellipsoid moving parallel to a principal axis, and the corresponding solutions in a perfect liquid. The results are readily

* There are three types given by $n = 0$ (one, for which $l + 4m = 0$, is not drawn by Klein), and I have found twelve others; the discriminating quantities are the signs of $(l + 4m)$, m , n , $(n - m)$, $(n + m)$.

extended to rotation round any axis through the centre of the ellipsoid.

Let the semiaxes of the ellipsoid be a, b, c , and let $2\pi abc\Omega$ denote the gravitational potential at an external point of this ellipsoid, supposed of unity density, so that

$$\Omega = \frac{1}{2} (A_\lambda x^2 + B_\lambda y^2 + C_\lambda z^2) - H_\lambda, \quad (1)$$

where

$$A_\lambda = \int_\lambda^\infty \frac{d\psi}{(a^2 + \psi) \Delta_\psi}, \quad B_\lambda = \int_\lambda^\infty \frac{d\psi}{(b^2 + \psi) \Delta_\psi}, \quad C_\lambda = \int_\lambda^\infty \frac{d\psi}{(c^2 + \psi) \Delta_\psi},$$

and

$$H_\lambda = \frac{1}{2} \int_\lambda^\infty \frac{d\psi}{\Delta_\psi},$$

Δ_ψ denoting $\sqrt{(a^2 + \psi)(b^2 + \psi)(c^2 + \psi)}$.

When this ellipsoid moves in a viscous fluid, parallel to the axis of x , with a constant velocity, the pressure at any point is of the form $\alpha_0 \frac{dH_\lambda}{dx} + \beta_0$, where α_0 and β_0 are constants (Basset, *Hydrodynamics*, Vol. II., p. 274). If a sphere is rotating round an axis through its centre in a viscous fluid, the motion set up is such that the pressure at all points of the fluid is the same and equal to the pressure at infinity. Again, in a perfect liquid, the velocity potentials for motion parallel to and rotation round the axis of x are respectively proportional to $\frac{d\Omega}{dx}$ and $y \frac{d\Omega}{dz} - z \frac{d\Omega}{dy}$. This suggests that when $\frac{d}{dx}$ of any function occurs in the linear motion parallel to x we may have $y \frac{d}{dz} - z \frac{d}{dy}$ of the same function in the angular motion round the same axis.

The equations of viscous motion are, neglecting the products and squares of the velocities,

$$\left. \begin{aligned} \nabla^2 u &= \frac{1}{\mu} \frac{d\Pi}{dx} \\ \nabla^2 v &= \frac{1}{\mu} \frac{d\Pi}{dy} \\ \nabla^2 w &= \frac{1}{\mu} \frac{d\Pi}{dz} \end{aligned} \right\}, \quad (2)$$

and the equation of continuity is

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Thus

$$\nabla^2 \Pi = 0,$$

and in the problem of rotation round the axis of x we may try for the pressure the solution

$$\Pi = \Pi_0 + K_0 \left(y \frac{dH_\lambda}{dz} - z \frac{dH_\lambda}{dy} \right),$$

where Π_0 is the pressure at infinity and K_0 is a constant. Now

$$\begin{aligned} y \frac{dH_\lambda}{dz} - z \frac{dH_\lambda}{dy} &= \frac{p_1^2 yz}{\Delta_\lambda} \left(\frac{1}{c^2 + \lambda} - \frac{1}{b^2 + \lambda} \right) = (b^2 - c^2) \frac{p_1^2 yz}{(b^2 + \lambda)(c^2 + \lambda)} \\ &= -\frac{1}{2} (b^2 - c^2) \frac{d^2 \Omega}{dy dz}. \end{aligned}$$

Hence

$$\Pi = \Pi_0 - \frac{1}{2} K_0 (b^2 - c^2) \frac{d^2 \Omega}{dy dz}.$$

The second term vanishes if the ellipsoid becomes a sphere, and hence Π would be constant throughout the fluid, as it should be, in agreement with the result above stated.

Expressing our assumption in the form

$$\Pi = \Pi_0 + \mu K \frac{d^2 \Omega}{dy dz},$$

where K is a constant, we have

$$\left. \begin{aligned} \nabla^2 u &= K \frac{d^3 \Omega}{dx dy dz} \\ \nabla^2 v &= K \frac{d^3 \Omega}{dy^2 dz} \\ \nabla^2 w &= K \frac{d^3 \Omega}{dy dz^2} \end{aligned} \right\} \quad (3)$$

The particular solutions, involving Ω , of this system of equations, are evidently

$$u = \alpha x \frac{d^2 \Omega}{dy dz} + \beta y \frac{d^2 \Omega}{dz dx} + \gamma z \frac{d^2 \Omega}{dx dy}, \quad (4)$$

$$v = \alpha_1 z \frac{d^2 \Omega}{dy^2} + \beta_1 y \frac{d^2 \Omega}{dy dz}, \quad (5)$$

$$w = \alpha_2 z \frac{d^2 \Omega}{dy dz} + \beta_2 y \frac{d^2 \Omega}{dz^2}, \quad (6)$$

where $\alpha + \beta + \gamma = \alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \frac{1}{2}K$. (7)

To these values of u, v, w we must add terms from the solutions of

$$\nabla^2 u' = 0, \quad \nabla^2 v' = 0, \quad \nabla^2 w' = 0,$$

which make the results suit the surface conditions $u = 0, v = -\omega_1 z, w = \omega_1 y$. The value of u can satisfy the surface conditions without any addition, but we add to the right-hand sides of (5) and (6) the terms $\gamma_1 \frac{d\Omega}{dz}$ and $\gamma_2 \frac{d\Omega}{dy}$ respectively, which satisfy $\nabla^2 v' = 0, \nabla^2 w' = 0$, and, at the surface, take the values suitable for the boundary conditions.

The complete values of u, v, w then are

$$\begin{aligned} u &= \alpha x \frac{d^2 \Omega}{dy dz} + \beta y \frac{d^2 \Omega}{dz dx} + \gamma z \frac{d^2 \Omega}{dx dy}, \\ v &= \alpha_1 z \frac{d^2 \Omega}{dy^2} + \beta_1 y \frac{d^2 \Omega}{dy dz} + \gamma_1 \frac{d\Omega}{dz}, \\ w &= \alpha_2 z \frac{d^2 \Omega}{dy dz} + \beta_2 y \frac{d^2 \Omega}{dz^2} + \gamma_2 \frac{d\Omega}{dy}. \end{aligned}$$

The condition $u = 0$ at the surface gives

$$a^2 \alpha + b^2 \beta + c^2 \gamma = 0. \quad (8)$$

Again, $v = -\omega z$ at the surface gives

$$(\alpha_1 B + \gamma_1 C) - \frac{2p^2 y^2}{ab^2 c} \left(\frac{\alpha_1}{b^2} + \frac{\beta_1}{c^2} \right) = 0,$$

$$\text{and this requires } \left. \begin{aligned} \frac{\alpha_1}{b^2} + \frac{\beta_1}{c^2} &= 0 \\ \alpha_1 B + \gamma_1 C &= -\omega_1 \end{aligned} \right\} \quad (9)$$

Similarly, $w = \omega_1 y$ at the surface requires

$$\left. \begin{aligned} \frac{\alpha_2}{b^2} + \frac{\beta_2}{c^2} &= 0 \\ \beta_2 C + \gamma_2 B &= \omega_1 \end{aligned} \right\} \quad (10)$$

From (7), (9), and (10) we get

$$\begin{aligned} \alpha_1 &= \alpha_2 = \frac{Kb^2}{2(b^2 - c^2)}, \quad \beta_1 = \beta_2 = \frac{-Kc^2}{2(b^2 - c^2)}; \\ \gamma_1 &= -\left[\frac{\omega_1}{C} + \frac{Kb^2 B}{2C(b^2 - c^2)} \right], \quad \gamma_2 = \frac{\omega_1}{B} + \frac{Kc^2 C}{2B(b^2 - c^2)}. \end{aligned}$$

The equation of continuity requires

$$\begin{aligned} (\alpha + \beta_1 + \gamma_1 + \alpha_1 + \gamma_2) \frac{d^2\Omega}{dydz} + y \frac{d}{dz} \left(\beta \frac{d^2\Omega}{dx^2} + \beta_1 \frac{d^2\Omega}{dy^2} + \beta_2 \frac{d^2\Omega}{dz^2} \right) \\ + z \frac{d}{dy} \left(\gamma \frac{d^2\Omega}{dx^2} + \alpha_1 \frac{d^2\Omega}{dy^2} + \alpha_2 \frac{d^2\Omega}{dz^2} \right) \\ + \alpha x \frac{d^3\Omega}{dxdydz} \equiv 0. \end{aligned}$$

But $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$; hence

$$\begin{aligned} (\alpha + \alpha_1 + \beta_1 + \gamma_1 + \gamma_2) \frac{d^2\Omega}{dydz} + y(\beta - \beta_1) \frac{d^3\Omega}{dx^2dz} + z(\gamma - \alpha_1) \frac{d^3\Omega}{dx^2dy} \\ + \alpha x \frac{d^3\Omega}{dxdydz} \equiv 0. \end{aligned}$$

This requires

$$\alpha = 0, \quad \alpha_1 + \beta_1 + \gamma_1 + \gamma_2 = 0, \quad \beta = \beta_1, \quad \gamma = \alpha_1. \quad (11)$$

Now, if $\alpha = 0$ and $\beta = \beta_1$, we have, from (8),

$$b^2\beta_1 + c^2\gamma = 0,$$

and, from (9),

$$b^2\beta_1 + c^2\alpha_1 = 0,$$

and hence

$$\gamma = \alpha_1.$$

Hence all the equations are consistent.

From (11), we have $\gamma_1 + \gamma_2 = -\frac{1}{2}K$,

and, substituting values of γ_1 and γ_2 found above, we get

$$K = -\frac{2\omega_1(b^2 - c^2)}{b^2B + c^2C}.$$

Hence

$$\beta = \beta_1 = \beta_2 = -\gamma_1 = \frac{\omega_1 c^2}{b^2B + c^2C}$$

and

$$\gamma = \alpha_1 = \alpha_2 = -\gamma_2 = \frac{-\omega_1 b^2}{b^2B + c^2C}.$$

Therefore the component velocities are given by

$$\begin{aligned} u &= \sigma_1 \left[c^2 y \frac{d^2\Omega}{dzdx} - b^2 z \frac{d^2\Omega}{dxdy} \right], \\ v &= \sigma_1 \left[c^2 y \frac{d^2\Omega}{dydz} - b^2 z \frac{d^2\Omega}{dy^2} - c^2 \frac{d\Omega}{dz} \right], \\ w &= \sigma_1 \left[c^2 y \frac{d^2\Omega}{dz^2} - b^2 z \frac{d^2\Omega}{dydz} + b^2 \frac{d\Omega}{dy} \right], \end{aligned}$$

where

$$\sigma_1 = \frac{\omega_1}{b^3 B + c^3 C};$$

or by the following scheme, showing clearly the analogy with Oberbeck's solution for motion parallel to a principal axis:—

$$\left. \begin{aligned} u &= \sigma_1 \frac{d\phi_1}{dx} \\ v &= \sigma_1 \left[\frac{d\phi_1}{dy} - 2c^3 \frac{d\Omega}{dz} \right] \\ w &= \sigma_1 \left[\frac{d\phi_1}{dz} + 2b^3 \frac{d\Omega}{dy} \right] \end{aligned} \right\}, \quad (12)$$

where ϕ_1 stands for $\left(c^3 y \frac{d\Omega}{dz} - b^3 z \frac{d\Omega}{dy} \right)$,

and the mean pressure is given by

$$\Pi = \Pi_0 - 2\mu\sigma_1 (b^3 - c^3) \frac{d^2\Omega}{dydz}.$$

We now proceed to the general case of rotation about any axis. Let u_1, u_2, u_3 denote the component velocities of the fluid parallel to the axis of x , when the ellipsoid is rotating round the axes of x, y, z , with angular velocities $\omega_1, \omega_2, \omega_3$ respectively; let v_1, v_2, v_3 and w_1, w_2, w_3 have similar meanings. Then u_1, v_1, w_1 have the values given by (12), and satisfy the differential equations

$$\nabla^2 u_1 = -2\sigma_1 (b^3 - c^3) \frac{d^3\Omega}{dx dy dz},$$

$$\nabla^2 v_1 = -2\sigma_1 (b^3 - c^3) \frac{d^3\Omega}{dy^2 dz},$$

$$\nabla^2 w_1 = -2\sigma_1 (b^3 - c^3) \frac{d^3\Omega}{dy dz^2},$$

and the surface conditions $u_1 = 0, v_1 = -\omega_1 z, w_1 = \omega_1 y$.

Similarly, the solutions of

$$\nabla^2 u_2 = -2\sigma_2 (c^3 - a^3) \frac{d^3\Omega}{dz dx^2},$$

$$\nabla^2 v_2 = -2\sigma_2 (c^3 - a^3) \frac{d^3\Omega}{dx dy dz},$$

$$\nabla^2 w_2 = -2\sigma_2 (c^3 - a^3) \frac{d^3\Omega}{dz^2 dx},$$

where

$$\sigma_2 = \frac{\omega_2}{c^2 C + a^2 A},$$

subject to the conditions $u_2 = \omega_2 z$, $v_2 = 0$, $w_2 = -\omega_2 x$, are

$$u_2 = \sigma_2 \left[\frac{d\phi_1}{dx} + 2c^2 \frac{d\Omega}{dy} \right],$$

$$v_2 = \sigma_2 \frac{d\phi_2}{dy},$$

$$w_2 = \sigma_2 \left[\frac{d\phi_3}{dz} - 2a^2 \frac{d\Omega}{dx} \right],$$

where ϕ_2 stands for $\left(a^2 z \frac{d\Omega}{dx} - c^2 x \frac{d\Omega}{dz} \right)$;

and the solutions of similar differential equations for u_3 , v_3 , w_3 , subject to the surface conditions $u_3 = -\omega_3 y$, $v_3 = \omega_3 x$, $w_3 = 0$, are

$$u_3 = \sigma_3 \left[\frac{d\phi_3}{dx} - 2b^2 \frac{d\Omega}{dy} \right],$$

$$v_3 = \sigma_3 \left[\frac{d\phi_3}{dy} + 2a^2 \frac{d\Omega}{dx} \right],$$

$$w_3 = \sigma_3 \frac{d\phi_3}{dz},$$

where

$$\phi_3 = \left(b^2 x \frac{d\Omega}{dy} - a^2 y \frac{d\Omega}{dx} \right)$$

and

$$\sigma_3 = \frac{\omega_3}{a^2 A + b^2 B}.$$

When the ellipsoid is rotating round an axis through the centre of the ellipsoid with an angular velocity ω , made up of components ω_1 , ω_2 , ω_3 round the principal axes, the component velocities at any point are given by the superposition of the above values; for, if

$$u = \sum_1^3 u_r, \quad v = \sum_1^3 v_r, \quad w = \sum_1^3 w_r,$$

then u , v , w satisfy the differential equations

$$\left. \begin{aligned} \nabla^2 u &= \frac{d\psi}{dx} \\ \nabla^2 v &= \frac{d\psi}{dy} \\ \nabla^2 w &= \frac{d\psi}{dz} \end{aligned} \right\}, \quad (13)$$

where

$$\psi \equiv -2 \left[\sigma_1 (b^2 - c^2) \frac{d^2 \Omega}{dy dz} + \sigma_2 (c^2 - a^2) \frac{d^2 \Omega}{dz dx} + \sigma_3 (a^2 - b^2) \frac{d^2 \Omega}{dx dy} \right],$$

and also the surface conditions

$$u = \omega_2 z - \omega_3 y, \quad v = \omega_3 x - \omega_1 z, \quad w = \omega_1 y - \omega_2 x.$$

But the equations (13) are the equations of motion, when Π has the value $\Pi_0 + \mu\psi$, which is the value that was to be expected from our previous analysis. Hence we have at any point of the fluid

$$\left. \begin{aligned} u &= \frac{dU}{dx} + 2 \left(c^2 \sigma_2 \frac{d\Omega}{dz} - b^2 \sigma_3 \frac{d\Omega}{dy} \right) \\ v &= \frac{dU}{dy} + 2 \left(a^2 \sigma_3 \frac{d\Omega}{dx} - c^2 \sigma_1 \frac{d\Omega}{dz} \right) \\ w &= \frac{dU}{dz} + 2 \left(b^2 \sigma_1 \frac{d\Omega}{dy} - a^2 \sigma_2 \frac{d\Omega}{dx} \right) \end{aligned} \right\}, \quad (14)$$

where

$$U \equiv \sigma_1 \phi_1 + \sigma_2 \phi_2 + \sigma_3 \phi_3.$$

The components of the differential rotation ξ, η, ζ are given by

$$\xi = \sigma_1 D\Omega - a^2 \frac{dX}{dx},$$

$$\eta = \sigma_2 D\Omega - b^2 \frac{dX}{dy},$$

$$\zeta = \sigma_3 D\Omega - c^2 \frac{dX}{dz},$$

where D denotes the operator $a^2 \frac{d^2}{dx^2} + b^2 \frac{d^2}{dy^2} + c^2 \frac{d^2}{dz^2}$ and

$$X \equiv \sigma_1 \frac{d\Omega}{dx} + \sigma_2 \frac{d\Omega}{dy} + \sigma_3 \frac{d\Omega}{dz}.$$

The couple that must be applied to the ellipsoid, in order to maintain the motion, can readily be found by direct integration over the ellipsoidal surface.

Letting N_1, N_2, N_3 denote the three normal stresses, and T_1, T_2, T_3

the three tangential stresses, we have

$$\begin{aligned}
 N &= -\Pi + 2\mu \frac{dw}{dz} \\
 &= -\Pi_0 + 2\mu \left\{ \sigma_1 (b^3 - c^3) \frac{d^2\Omega}{dy dz} + \sigma_2 (c^3 - a^3) \frac{d^2\Omega}{dz dx} + \sigma_3 (a^3 - b^3) \frac{d^2\Omega}{dx dy} \right. \\
 &\quad \left. + \frac{d^3U}{dz^3} + 2 \left(b^3 \sigma_1 \frac{d^2\Omega}{dy dz} - a^3 \sigma_2 \frac{d^2\Omega}{dx dz} \right) \right\} \\
 &= -\Pi_0 + 2\mu \left\{ \Sigma \sigma_1 (b^3 - c^3) \frac{d^2\Omega}{dy dz} + \left[a^2 (\sigma_2 z - \sigma_3 y) \frac{d}{dx} + b^2 (\sigma_3 x - \sigma_1 z) \frac{d}{dy} \right. \right. \\
 &\quad \left. \left. + c^2 (\sigma_1 y - \sigma_2 x) \frac{d}{dz} \right] \frac{d^2\Omega}{dz^2} \right\},
 \end{aligned}$$

with similar values for N_1 and N_2 . Also

$$\begin{aligned}
 T_3 &= \mu \left(\frac{dv}{dx} + \frac{dw}{dy} \right) \\
 &= 2\mu \left[\frac{d^3U}{dx dy} + a^2 \sigma_3 \frac{d^2\Omega}{dx^2} + c^2 \sigma_2 \frac{d^2\Omega}{dy dz} - b^2 \sigma_3 \frac{d^2\Omega}{dy^2} - c^2 \sigma_1 \frac{d^2\Omega}{dx dz} \right] \\
 &= 2\mu \left[a^2 (\sigma_2 z - \sigma_3 y) \frac{d}{dx} + b^2 (\sigma_3 x - \sigma_1 z) \frac{d}{dy} + c^2 (\sigma_1 y - \sigma_2 x) \frac{d}{dz} \right] \frac{d^2\Omega}{dx dy},
 \end{aligned}$$

with similar values for T_1 and T_2 .

Now, in the integral $\iint (Hy - Gz) dS$, where $H = lT_2 + mT_1 + nN_3$, $G = \&c.$, we may neglect all terms containing products and powers of x, y, z which are of odd degree, as these terms clearly vanish in the integration from symmetry. We have

$$yH = y (lT_2 + mT_1 + nN_3)$$

$$\begin{aligned}
 &= -\frac{pzy}{c^2 + \lambda} \Pi_0 \\
 &\quad + \frac{2\mu pzy}{c^2 + \lambda} \left\{ \sigma_1 (b^3 - c^3) \frac{d^2\Omega}{dy dz} + \sigma_2 (c^3 - a^3) \frac{d^2\Omega}{dz dx} + \sigma_3 (a^3 - b^3) \frac{d^2\Omega}{dx dy} \right\} \\
 &\quad + 4\mu py \left\{ a^2 (\sigma_2 z - \sigma_3 y) \frac{d}{d\lambda} \left(\frac{d^2\Omega}{dx dz} \right) + b^2 (\sigma_3 x - \sigma_1 z) \frac{d}{d\lambda} \left(\frac{d^2\Omega}{dy dz} \right) \right. \\
 &\quad \left. + c^2 (\sigma_1 y - \sigma_2 x) \frac{d}{d\lambda} \left(\frac{d^2\Omega}{dz^2} \right) \right\},
 \end{aligned}$$

since

$$l \frac{d}{dx} + m \frac{d}{dy} + n \frac{d}{dz} = 2p \frac{d}{d\lambda}.$$

$$\text{But } \frac{d}{d\lambda} \left(\frac{d^2\Omega}{dz^2} \right) = -\frac{1}{(c^2+\lambda)\Delta_\lambda} - \frac{2z^2}{(c^2+\lambda)\Delta_\lambda} \\ + \frac{p^2z^2}{(c^2+\lambda)^2\Delta_\lambda} \left[\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{3}{c^2+\lambda} \right]$$

and

$$\frac{d}{d\lambda} \left(\frac{d^2\Omega}{dydz} \right) = \frac{yz}{(b^2+\lambda)(c^2+\lambda)\Delta_\lambda} \left[-2 + p^2 \left(\frac{1}{a^2+\lambda} + \frac{2}{b^2+\lambda} + \frac{2}{c^2+\lambda} \right) \right],$$

with similar values for the other derivatives with regard to λ .

Retaining only the even powers and products, we have

$$[yH]_{\lambda=0} = \frac{-4\mu p_0^3 \sigma_1 y^3 z^2}{ab^3 c^3} \left(\frac{b^2 - c^2}{c^2} \right) \\ + 4\mu p_0 y \left[c^3 \sigma_1 y \frac{d}{d\lambda} \left(\frac{d^2\Omega}{dz^2} \right) - b^3 \sigma_1 z \frac{d}{d\lambda} \left(\frac{d^2\Omega}{dydz} \right) \right]_{\lambda=0},$$

p_0 being the value of p at the surface; that is

$$[yH]_{\lambda=0} = \frac{-4\mu \sigma_1 p_0 y^2}{abc}.$$

Similarly,

$$[zG]_{\lambda=0} = \frac{4\mu \sigma_1 p_0 z^2}{abc};$$

$$\text{hence } \iint (Hy - Gz) dS = -\frac{4\mu \sigma_1}{abc} \iint p_0 (y^2 + z^2) dS \\ = -\frac{16}{3} \pi \mu \sigma_1 (b^2 + c^2).$$

Thus the required couple has for its components

$$\frac{16\pi\mu\sigma_1(b^2+c^2)}{3(b^2B+c^2O)}, \text{ \&c., \&c.}$$

This result may also be deduced in a somewhat different manner, for we have

$$\frac{dT_x}{dx} + \frac{dT_y}{dy} + \frac{dN_z}{dz} = 0,$$

and two similar equations; hence

$$\iint (Hy - Gz) dS = 0,$$

the integration extending over the ellipsoid and any exterior surface, which may be taken to be a sphere whose radius is ultimately made infinite. The values of the stresses at the spherical boundary are

easily found from the previous analysis by making λ very large, and on integration we find

$$\iint (Hy - Gz) dS \text{ over the sphere} = \frac{1}{3} \pi \mu \sigma_1 (b^3 + c^3).$$

Thus we get the same value for the couple as before. The usual method (though leading to correct results in this case) of writing down the limiting forms of the velocities when the distance from the origin is increased indefinitely and thence deducing the stresses by differentiation is open to objection, since the operations are not performed in their proper order and there is no *a priori* evidence to show that they are commutative.

2. The equations of the stream lines due to the steady motion of an ellipsoid in a perfect fluid, both when moving parallel to and rotating round a principal axis, have been obtained in an integrable form by Clebsch (*Crelle*, Vol. LII., pp. 103-132, and Vol. LIII., pp. 287-292). He used Cartesian coordinates in his investigation; while Mr. Herman (*Quart. Jour. of Math.*, Vol. XXIII., pp. 378-384), who considers the same two problems, and also the case when the ellipsoid is in steady motion parallel to a principal axis in a viscous fluid, uses ellipsoidal coordinates. Mr. Herman deduces, in each case, two integrable equations and one corresponding first integral.

The object of this paper is to show that by a proper combination of Cartesian and ellipsoidal coordinates we can deduce both the first integrals in every case. The case of an ellipsoid rotating round a principal axis in a viscous liquid is also treated.

With the same notation as in § 1, let A, B, C denote the values of $A_\lambda, B_\lambda, C_\lambda$ respectively when λ is zero, λ denoting the positive root of the equation

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1.$$

We have also $\frac{d\Omega}{dx} = A_\lambda x, \quad \frac{d\Omega}{dy} = B_\lambda y, \quad \frac{d\Omega}{dz} = C_\lambda z,$

and $A_\lambda + B_\lambda + C_\lambda = \frac{2}{\Delta_\lambda}.$

The velocity potential due to the motion of an ellipsoid in an infinite liquid, when moving parallel to the axis of x with velocity V , is

$$\phi = -\frac{V}{B+C} A_\lambda x$$

(Basset, *Hydrodynamics*, Vol. I., pp. 141, 142).

Hence the differential equations of the relative stream lines are

$$\frac{dx}{\frac{d\phi}{dx} - V} = \frac{dy}{\frac{d\phi}{dy}} = \frac{dz}{\frac{d\phi}{dz}},$$

that is,

$$\frac{dx}{B+C+A_\lambda - \frac{2p_1^2 x^2}{(a^2+\lambda)^2 \Delta_\lambda}} = \frac{dy}{-2p_1^2 xy} = \frac{dz}{-2p_1^2 xz},$$

where p_1 is the perpendicular from the origin on the tangent plane to the confocal whose primary semiaxis is $\sqrt{a^2+\lambda}$.

Each of the above expressions

$$\equiv \frac{\sum x \frac{dx}{a^2+\lambda}}{(B+C+A_\lambda) \frac{x}{a^2+\lambda} - \frac{2x}{(a^2+\lambda) \Delta_\lambda}} \equiv \frac{d\lambda}{(B+C+A_\lambda) \frac{2p_1^2 x}{a^2+\lambda} - \frac{4p_1^2 x}{(a^2+\lambda) \Delta_\lambda}},$$

since
$$\sum \frac{x dx}{a^2+\lambda} = \frac{d\lambda}{2p_1^2}.$$

Hence

$$\frac{\frac{dy}{y}}{\frac{1}{(b^2+\lambda) \Delta_\lambda}} = \frac{\frac{dz}{z}}{\frac{1}{(c^2+\lambda) \Delta_\lambda}} = \frac{-d\lambda}{B+C+A_\lambda - \frac{2}{\Delta_\lambda}} = \frac{-d\lambda}{B+C-B_\lambda-C_\lambda}.$$

Letting θ_λ denote $B+C-B_\lambda-C_\lambda$, then, since

$$\frac{d\theta_\lambda}{d\lambda} = \left(\frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} \right) \frac{1}{\Delta_\lambda},$$

we have

$$\frac{dy}{y} + \frac{dz}{z} + \frac{1}{\theta_\lambda} \frac{d\theta_\lambda}{d\lambda} d\lambda = 0.$$

Integrating, one first integral is

$$yz [B+C-B_\lambda-C_\lambda] = \text{constant},$$

that is,
$$yz \int_0^\lambda \left(\frac{1}{b^2+\psi} + \frac{1}{c^2+\psi} \right) \frac{d\psi}{\Delta_\psi} = \text{constant}.$$

This is the first integral given by Mr. Herman; the present method shows that it is the product of the integrals

$$y e^{\int_\lambda^\infty \frac{d\lambda}{(b^2+\lambda) \Delta_\lambda \theta_\lambda}} = \text{constant},$$

$$z e^{\int_\lambda^\infty \frac{d\lambda}{(c^2+\lambda) \Delta_\lambda \theta_\lambda}} = \text{constant}.$$

By squaring and adding these, we get the integral that corresponds to Stokes' current function in the case of a sphere, viz.,

$$\psi = \frac{V a^3}{2r^3} (y^2 + z^2).$$

The velocity potential, when the ellipsoid is rotating round Ox with angular velocity ω , is

$$\phi = \frac{\omega}{K} (B_\lambda - C_\lambda) yz,$$

where K denotes $A + \frac{2(b^2 B - c^2 C)}{b^2 - c^2}$. The differential equations of the relative stream lines are

$$\frac{dx}{d\phi} = \frac{dy}{\frac{d\phi}{dy} + \omega z} = \frac{dz}{\frac{d\phi}{dz} - \omega y},$$

that is,
$$\frac{x dx}{\frac{2(b^2 - c^2)}{\Delta_\lambda^3} p_1^2 x^2} = \frac{y dy}{\frac{2(b^2 - c^2)}{(b^2 + \lambda)^2 (c^2 + \lambda)^2 \Delta_\lambda^3} p_1^2 y^2 + B_\lambda - C_\lambda + K}$$

$$= \frac{z dz}{\frac{2(b^2 - c^2)}{(b^2 + \lambda)(c^2 + \lambda)^2 \Delta_\lambda^3} p_1^2 z^2 + B_\lambda - C_\lambda - K}.$$

Each of these expressions

$$\equiv \frac{\sum \frac{x dx}{a^2 + \lambda}}{\frac{2(b^2 - c^2)}{(b^2 + \lambda)(c^2 + \lambda) \Delta_\lambda} + (B_\lambda - C_\lambda) \left(\frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) + K \left(\frac{1}{b^2 + \lambda} - \frac{1}{c^2 + \lambda} \right)}$$

$$\equiv \frac{d\lambda}{\frac{2\rho_1^2}{(b^2 + \lambda)(c^2 + \lambda)} \left\{ (b^2 - c^2) \left(\frac{2}{\Delta_\lambda} - K \right) + (b^2 + c^2 + 2\lambda)(B_\lambda - C_\lambda) \right\}}$$

Denoting $(b^2 - c^2) \left(\frac{2}{\Delta_\lambda} - K \right) + (b^2 + c^2 + 2\lambda)(B_\lambda - C_\lambda)$ by G_λ , we have

$$\frac{dx}{x} = \frac{(b^2 - c^2) d\lambda}{(a^2 + \lambda) \Delta_\lambda G_\lambda}.$$

Thus

$$x e^{-\int_\lambda^x \frac{(b^2 - c^2) d\lambda}{(a^2 + \lambda) \Delta_\lambda G_\lambda}} = \text{constant}$$

is one first integral. A second integral cannot readily be found

directly; but, observing that at a great distance from the origin the stream lines are circles, being the intersection of the planes $x = \text{constant}$ with spheres having the origin for centre, we are led to assume as another integral

$$x^2\theta(\lambda) + y^2\phi(\lambda) + z^2\psi(\lambda) = \text{constant},$$

where θ , ϕ , and ψ are unknown functions of λ .

To justify this assumption, we must have

$$2 \left[x\theta(\lambda) dx + y\phi(\lambda) dy + z\psi(\lambda) dz \right] \\ + \left[x^2\theta'(\lambda) + y^2\phi'(\lambda) + z^2\psi'(\lambda) \right] d\lambda \equiv 0,$$

when dx , dy , dz , $d\lambda$ have the ratios given by the differential equations above.

Remembering that

$$\frac{1}{p_1^2} = \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2},$$

we get

$$\frac{2\theta(\lambda)x^2(b^2 - c^2)}{\Delta_\lambda^3} + \phi(\lambda) \left[\frac{2(b^2 - c^2)y^2}{(b^2 + \lambda)^2(c^2 + \lambda)\Delta_\lambda} \right. \\ \left. + (B_\lambda - C_\lambda + K) \left(\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right) \right] \\ + \psi(\lambda) \left[\frac{2(b^2 - c^2)z^2}{(b^2 + \lambda)(c^2 + \lambda)^2\Delta_\lambda} \right. \\ \left. + (B_\lambda - C_\lambda - K) \left(\frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right) \right] \\ + \frac{G_\lambda}{(b^2 + \lambda)(c^2 + \lambda)} \left[x^2\theta'(\lambda) + y^2\phi'(\lambda) + z^2\psi'(\lambda) \right] \equiv 0.$$

Here the coefficients of x^2 , y^2 , z^2 must all vanish. This gives

$$\frac{(b^2 + \lambda)(c^2 + \lambda)}{(a^2 + \lambda)^2} \left[\phi(\lambda)(B_\lambda - C_\lambda + K) + \psi(\lambda)(B_\lambda - C_\lambda - K) \right] \\ + \frac{2(b^2 - c^2)\theta(\lambda)}{(a^2 + \lambda)\Delta_\lambda} + \theta'(\lambda)G_\lambda = 0,$$

$$\frac{\phi(\lambda)}{b^2 + \lambda} \left[\frac{2(b^2 - c^2)}{\Delta_\lambda} + (c^2 + \lambda)(B_\lambda - C_\lambda + K) \right] \\ + \frac{\psi(\lambda)(c^2 + \lambda)}{b^2 + \lambda} (B_\lambda - C_\lambda - K) + \phi'(\lambda)G_\lambda = 0,$$

$$\phi(\lambda) \frac{(b^2 + \lambda)}{c^2 + \lambda} [B_\lambda - C_\lambda + K] \\ + \frac{\psi(\lambda)}{c^2 + \lambda} \left[\frac{2(b^2 - c^2)}{\Delta_\lambda} + (b^2 + \lambda)(B_\lambda - C_\lambda - K) \right] + \psi'(\lambda) G_\lambda = 0.$$

Putting $\phi(\lambda) = \frac{\phi_1(\lambda)}{b^2 + \lambda}$ and $\psi(\lambda) = \frac{\psi_1(\lambda)}{c^2 + \lambda}$,

we easily find that the last two equations become

$$(\phi_1 - \psi_1)(B_\lambda - C_\lambda - K) = \frac{d\phi_1}{d\lambda} G_\lambda, \\ (\phi_1 - \psi_1)(B_\lambda - C_\lambda + K) = -\frac{d\psi_1}{d\lambda} G_\lambda.$$

Adding, we obtain

$$2(B_\lambda - C_\lambda)(\phi_1 - \psi_1) = \left(\frac{d\phi_1}{d\lambda} - \frac{d\psi_1}{d\lambda} \right) G_\lambda.$$

But $\frac{dG_\lambda}{d\lambda} = 2(B_\lambda - C_\lambda) - \frac{b^2 - c^2}{(a^2 + \lambda)\Delta_\lambda}.$

Hence $\frac{\phi_1' - \psi_1'}{\phi_1 - \psi_1} = \frac{1}{G_\lambda} \left[\frac{dG_\lambda}{d\lambda} + \frac{b^2 - c^2}{(a^2 + \lambda)\Delta_\lambda} \right].$

Integrating, we have

$$\phi_1 - \psi_1 = A_0 G_\lambda e^{\int_0^\lambda \frac{(b^2 - c^2)d\lambda}{(a^2 + \lambda)\Delta_\lambda G_\lambda}},$$

where A_0 is an arbitrary constant. Hence

$$\frac{d\phi_1}{d\lambda} = 2(B_\lambda - C_\lambda - K) H_\lambda, \\ \frac{d\psi_1}{d\lambda} = -2(B_\lambda - C_\lambda + K) H_\lambda,$$

where H_λ denotes $e^{\int_0^\lambda \frac{(b^2 - c^2)d\lambda}{(a^2 + \lambda)\Delta_\lambda G_\lambda}}$, the arbitrary constant being supposed absorbed in the integral. Hence

$$\phi_1 = G_\lambda H_\lambda - 2K \int_0^\lambda H d\lambda, \\ \psi_1 = -G_\lambda H_\lambda - 2K \int_0^\lambda H d\lambda.$$

Substituting the values of ϕ and ψ , i.e., of $\frac{\phi_1}{b^2+\lambda}$ and $\frac{\psi_1}{c^2+\lambda}$, in the first equation above, we get, since

$$\begin{aligned} \frac{1}{H_\lambda} \frac{dH_\lambda}{d\lambda} &= \frac{b^2-c^2}{(a^2+\lambda) \Delta_\lambda G_\lambda}, \\ \frac{d\theta_\lambda}{d\lambda} + \frac{2}{H_\lambda} \frac{dH_\lambda}{d\lambda} \theta_\lambda \\ &= -\frac{1}{G_\lambda (a^2+\lambda)^2} \left[G_\lambda H_\lambda \{ (B_\lambda - C_\lambda)(b^2-c^2) - K(b^2+c^2+2\lambda) \} \right. \\ &\quad \left. + 2K \{ (B_\lambda - C_\lambda)(b^2+c^2+2\lambda) - K(b^2-c^2) \} \int H d\lambda \right] \\ &= -\frac{1}{(a^2+\lambda)^2} \left[H_\lambda \{ (B_\lambda - C_\lambda)(b^2-c^2) - K(b^2+c^2+2\lambda) \} \right. \\ &\quad \left. + 2K \left\{ 1 - \frac{2(b^2-c^2)}{\Delta_\lambda G_\lambda} \right\} \int H d\lambda \right]. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{d\lambda} [H_\lambda^2 \theta(\lambda)] \\ &= -\frac{H_\lambda^3}{(a^2+\lambda)} \left[\{ (B_\lambda - C_\lambda)(b^2-c^2) - K(b^2+c^2+2\lambda) \} \right. \\ &\quad \left. + 2K \left\{ \frac{H_\lambda^2}{(a^2+\lambda)^2} - \frac{2H_\lambda}{a^2+\lambda} \frac{dH_\lambda}{d\lambda} \right\} \int H d\lambda \right]. \end{aligned}$$

Integrating and dividing across by H_λ^2 , we get

$$\begin{aligned} \theta(\lambda) &= -\frac{2K}{a^2+\lambda} \int_0^\lambda H_\lambda d\lambda + \frac{1}{H_\lambda^2} \int_0^\lambda \frac{H_\lambda^3}{(a^2+\lambda)^2} \\ &\quad \times \{ (B_\lambda - C_\lambda)(b^2-c^2) + K(2a^2-b^2-c^2) \} d\lambda. \end{aligned}$$

Also

$$\phi(\lambda) = \frac{1}{b^2+\lambda} \left[G_\lambda H_\lambda - 2K \int_0^\lambda H d\lambda \right],$$

$$\psi(\lambda) = -\frac{1}{c^2+\lambda} \left[G_\lambda H_\lambda + 2K \int_0^\lambda H d\lambda \right].$$

Thus the second integral is

$$x^2 \theta(\lambda) + y^2 \phi(\lambda) + z^2 \psi(\lambda) = \text{constant};$$

θ , ϕ , and ψ having the above values; and the limits of the integrals having been so chosen that when λ is indefinitely increased our second integral becomes a sphere, and when zero the ellipsoid itself.

If an ellipsoidal solid is moving parallel to the axis of x with

velocity V through a viscous incompressible fluid, the component velocities at any point of the fluid are given by

$$u = \alpha \left(x \frac{dH_\lambda}{dx} - H_\lambda + \beta \frac{d^2\Omega}{dx^2} \right),$$

$$v = \alpha \left(x \frac{dH_\lambda}{dy} + \beta \frac{d^2\Omega}{dx dy} \right),$$

$$w = \alpha \left(x \frac{dH_\lambda}{dz} + \beta \frac{d^2\Omega}{dx dz} \right),$$

where $\beta = -\frac{1}{2}a^2$ and $\alpha = \frac{-2V}{a^2A + 2H}$.

The differential equations of the relative stream lines are

$$\frac{dx}{u-V} = \frac{dy}{v} = \frac{dz}{w}.$$

Substituting and reducing, we get

$$\begin{aligned} \frac{dx}{H_\lambda - H + \frac{a^2}{2}(A_\lambda - A) + \frac{\lambda}{(a^2 + \lambda)} \Delta_\lambda \frac{p_1^2 x^3}{a^2 + \lambda}} \\ = \frac{dy}{\frac{\lambda}{(a^2 + \lambda)} \Delta_\lambda \frac{p_1^2 xy}{(b^2 + \lambda)}} = \frac{dz}{\frac{\lambda}{(a^2 + \lambda)} \Delta_\lambda \frac{p_1^2 xz}{c^2 + \lambda}}. \end{aligned}$$

Each of these expressions

$$\begin{aligned} &= \frac{\Sigma \frac{x dx}{a^2 + \lambda}}{\frac{x}{a^2 + \lambda} \left[H_\lambda - H + \frac{a^2}{2}(A_\lambda - A) + \frac{\lambda}{\Delta_\lambda} \right]} \\ &= \frac{d\lambda}{\frac{2p_1^2 x}{a^2 + \lambda} \left[H_\lambda - H + \frac{a^2}{2}(A_\lambda - A) + \frac{\lambda}{\Delta_\lambda} \right]}. \end{aligned}$$

Hence $\frac{\frac{dy}{y}}{\frac{\lambda}{(b^2 + \lambda)} \Delta_\lambda} = \frac{\frac{dz}{z}}{\frac{\lambda}{(c^2 + \lambda)} \Delta_\lambda} = \frac{d\lambda}{2 \left[H_\lambda - H + \frac{a^2}{2}(A_\lambda - A) + \frac{\lambda}{\Delta_\lambda} \right]}$

and we have

$$\frac{dy}{y} + \frac{dz}{z} - \frac{\left(\frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) \frac{\lambda d\lambda}{\Delta_\lambda}}{2 \left[H_\lambda - H + \frac{a^2}{2}(A_\lambda - A) + \frac{\lambda}{\Delta_\lambda} \right]} = 0.$$

Integrating gives

$$yz \left\{ 2 (H_\lambda - H) + a^2 (A_\lambda - A) + \frac{2\lambda}{\Delta_\lambda} \right\} = \text{constant},$$

that is,
$$yz \left\{ \int_0^\lambda \left(\frac{a^2}{a^2 + \psi} + 1 \right) \frac{d\psi}{\Delta_\psi} - \frac{2\lambda}{\Delta_\lambda} \right\} = \text{constant}. \quad (15)$$

Letting θ_λ denote $2 (H_\lambda - H) + a^2 (A_\lambda - A) + \frac{2\lambda}{\Delta_\lambda}$, we have also the pair of first integrals

$$ye^{\int_\lambda^\infty \frac{\lambda d\lambda}{(b^2 + \lambda)\Delta_\lambda \theta_\lambda}} = \text{constant},$$

$$ze^{\int_\lambda^\infty \frac{\lambda d\lambda}{(c^2 + \lambda)\Delta_\lambda \theta_\lambda}} = \text{constant}.$$

The equation (15), which assumes a simple form, is clearly the product of these integrals. Neither of them, however, separately admits of further reduction, unless the ellipsoid is one of revolution.

When the ellipsoid is rotating in viscous incompressible fluid, round the axis of x , with small angular velocity ω , the component velocities at any point, parallel to the principal axes of the ellipsoid, are given by (cf. § 1)

$$u = 2\sigma \frac{(b^2 - c^2) p_1^2 \lambda x y z}{\Delta_\lambda^3},$$

$$v = 2\sigma \frac{(b^2 - c^2) p_1^2 \lambda y^2 z}{(b^2 + \lambda)^2 (c^2 + \lambda) \Delta_\lambda} - \sigma (b^2 B_\lambda + c^2 C_\lambda) z,$$

$$w = 2\sigma \frac{(b^2 - c^2) p_1^2 \lambda y z^2}{(b^2 + \lambda) (c^2 + \lambda)^2 \Delta_\lambda} + \sigma (b^2 B_\lambda + c^2 C_\lambda) y,$$

where

$$\sigma = \frac{\omega}{b^2 B + c^2 C}.$$

The differential equations of the stream lines, relative to the moving solid, are

$$\frac{dx}{u} = \frac{dy}{v + \omega z} = \frac{dz}{w - \omega y}.$$

that is,

$$\begin{aligned} \frac{dx}{\frac{2p_1^2 \lambda x y z (b^2 - c^2)}{\Delta_\lambda^3}} &= \frac{dy}{\frac{2p_1^2 \lambda y^2 z (b^2 - c^2)}{(b^2 + \lambda)^2 (c^2 + \lambda) \Delta_\lambda} - \left(b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} \right) z} \\ &= \frac{dz}{\frac{2p_1^2 \lambda y z^2 (b^2 - c^2)}{(b^2 + \lambda) (c^2 + \lambda)^2 \Delta_\lambda} + \left(b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} \right) y}. \end{aligned}$$

Each of these expressions

$$\begin{aligned} & \Sigma \frac{x dx}{a^2 + \lambda} \\ & \equiv \frac{(b^2 - c^2) yz}{(b^2 + \lambda)(c^2 + \lambda)} \left\{ b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} + \frac{2\lambda}{\Delta_\lambda} \right\} \\ & \equiv \frac{d\lambda}{\frac{2p^2 yz (b^2 - c^2)}{(b^2 + \lambda)(c^2 + \lambda)} \left\{ b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} + \frac{2\lambda}{\Delta_\lambda} \right\}}, \\ \text{and also} \quad & \equiv \frac{x dx + y dy + z dz}{\frac{2p^2 yz \lambda (b^2 - c^2)}{(b^2 + \lambda)(c^2 + \lambda) \Delta_\lambda}}. \end{aligned}$$

Hence, removing common factors,

$$\frac{\frac{dx}{x}}{\frac{\lambda}{(a^2 + \lambda) \Delta_\lambda}} = \frac{x dx + y dy + z dz}{\frac{\lambda}{\Delta_\lambda}} = \frac{d\lambda}{b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} + \frac{2\lambda}{\Delta_\lambda}}.$$

But
$$\frac{d}{d\lambda} \left[b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} + \frac{2\lambda}{\Delta_\lambda} \right] = - \frac{\lambda}{(a^2 + \lambda) \Delta_\lambda}.$$

Thus one first integral is

$$x \left[b^2 B_\lambda + c^2 C_\lambda - \frac{\omega}{\sigma} + \frac{2\lambda}{\Delta_\lambda} \right] = \text{constant},$$

that is,
$$x \left\{ \int_0^\lambda \left[\frac{b^2}{b^2 + \psi} + \frac{c^2}{c^2 + \psi} \right] \frac{d\psi}{\Delta_\psi} - \frac{2\lambda}{\Delta_\lambda} \right\} = \text{constant}.$$

The other first integral is evidently

$$x^2 + y^2 + z^2 + 2 \int_\lambda^\infty \frac{\lambda d\lambda}{\Delta_\lambda [b^2 (B - B_\lambda) + c^2 (C - C_\lambda)] - 2\lambda} = \text{constant}.$$