Note on Beams fixed at the Ends

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Supplementary Observations.

Date.	Time.	Temp.	Elongation.
April 25.....	9.30 P.M.	49.0	24.5
" 25.....	10.0	48.5	23.0
" 26.....	10.45	50.8	22.5
" 27.....	9.20	63.7	23.0
" 28.....	10.20	54.5	25.0
" 29.....	9.20	50.0	24.5
" 29.....	11.0	48.6	25.5
" 30.....	11.20	45.8	23.5
May 4.....	10.0	48.6	40.0
" 4.....	11.0	56.8	86.0
" 5.....	11.0	51.8	118.0
" 6.....	7.45	51.0	124.0
" 6.....	9.20	52.5	124.5
" 8.....	9.15	59.0	124.0
" 10.....	8.50 A.M.	54.7	143.5

IX. *Note on Beams fixed at the Ends.*

By W. E. AYRTON, *F.R.S.*, and JOHN PERRY, *F.R.S.**

I. A HORIZONTAL beam with vertical loads, fixed at the ends, being given, to find the bending-moment everywhere: this is a problem the solution of which has not hitherto been put in an elementary form. A knowledge of the bending-moment everywhere leads, of course, to a complete knowledge of the strength and stiffness of the beam. The problem is quite soluble by a method which is obvious to any one who has worked out the theorem of three moments; but even for advanced students the work is tedious; and practically the answers known for the two cases—(1) when a uniform beam is loaded merely in the middle, (2) when a uniform beam is loaded uniformly—are regarded as roughly applicable in all cases which occur in practice.

If M is the bending-moment at a section, I the moment of inertia of the section about its neutral line, and E Young's modulus of elasticity for the material, then $\frac{M}{EI}$ is the curvature of the beam. If OO' is a short length of the beam, $\frac{M}{EI} OO'$

* Read June 11, 1887.

is the angle which the originally parallel sections at O and O' now make with one another. Hence, if we divide the beam into a great number of parts, and if M is taken in the middle of each division, the moment of inertia there being I , the sum of all such terms as $\frac{M}{EI} \cdot OO'$ (if OO' is one of the elementary lengths) gives the angle between the two end sections of the beam.

This principle, that $\sum \frac{M}{EI} \cdot OO'$ gives the angle between the two end sections of the beam, is, of course, well known. It is the basis of the theory of flat spiral springs. As applicable to beams, it may be put in the words of Professor Cotterill ('Applied Mechanics,' § 169, p. 334) :—" . . . The angle (i) between two tangents to the deflection-curve of a beam is proportional to the area of the curve of bending moments [had he been speaking of non-uniform beams, Mr. Cotterill would have said the area of the curve whose ordinate is $\frac{M}{EI}$] intercepted between two ordinates at the points considered." It is also the basis of Prof. Fuller's beautifully simple method of dealing with arch-ribs fixed at the ends ; and, indeed, it was when putting Prof. Fuller's method before our students that we discovered the following simple application of the principle to beams. In spite of the apparent rashness of such a statement about such a well-ventilated subject as the elasticity of beams, we believe our method to be quite new.

II. In the case of a horizontal beam fixed at the ends, the angle between the end sections remains equal to nothing ; and hence, in this case,

$$\sum \frac{M}{I} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if the beam were merely supported at the ends, from a given system of loading it is easy to find what the bending-moment everywhere would be, either by numerical calculation or a link-polygon method. Let us suppose the diagram of this bending-moment m to be known. Then the bending-moment M of a beam fixed at the ends with the same loading, is $m - c$, where c is a constant ; so that it is only necessary from the condition (1) to determine this constant c .

(1) is

$$\sum \frac{m-c}{I} = 0, \quad \text{or} \quad \sum \frac{m}{I} = \sum \frac{c}{I},$$

$$\text{or} \quad c = \frac{\sum \frac{m}{I}}{\sum \frac{1}{I}}. \quad . \quad . \quad . \quad . \quad (2)$$

The rule then is, knowing m and I at every point :—Divide the beam into any number, n , of equal parts ; find $\frac{m}{I}$ at the middle of each part, and add all the values together : this is the numerator in (2). Find $\frac{1}{I}$ at the middle of each part, and add the values together : this is the denominator in (2); so that c is known. Subtract c from every value of m , or diminish all the ordinates of the m diagram by the amount c , and we have the M diagram ; that is, the real diagram of bending-moment of the beam fixed at the ends, with any distribution of loading and any variation in cross section.

III. If the cross section is the same everywhere, it is obvious that (2) becomes

$$c = \frac{1}{n} \sum m, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

if the beam was divided into n parts. In fact, c is the average value of m for the whole span. The application of (3) to beams of uniform section, fixed at the ends with one load in the middle or with a uniformly distributed load, is very simple.

IV. The condition (1) is applicable when I at every cross section of the beam is previously settled in any arbitrary manner. Let us give to I such a value that the beam shall be of uniform strength everywhere, that is that

$$\frac{M}{I} y = f_c \text{ or } f_t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where y is the greatest distance of any point in the section from the neutral line on the compression- or tension-side, and f_c and f_t are the constant maximum stresses in compression or tension to which the material is subjected in every section.

f_c may generally be taken as equal to f_t ; and if d be the depth of the beam, y may generally be taken as equal to $\frac{1}{2}d$. Hence (4) becomes

$$\frac{M}{I}d = \pm 2f, \quad \dots \dots \dots (5)$$

the + sign being taken when M is positive, and the - sign when M is negative.

Hence (1) becomes $\sum \frac{1}{\pm d} = 0, \quad \dots \dots \dots (6)$

the negative sign being taken from the ends of the beam to the points of inflexion, and the positive sign being taken between the two points of inflexion.

Now to satisfy (6) we have merely to solve the following

problem :—In the figure E A C G E is a diagram whose ordinates represent the value of $\frac{1}{d}$, or the reciprocal of the depth of the beam, which may be arbitrarily fixed; E F G E is a diagram of the values of m , easily drawn when the loading is known. We are required to find two points, P and Q, such that

$$\text{area ET} + \text{area QC} = \text{area PU}$$

and

$$PR = QS.$$

When found, these points P and Q are the *points of inflexion*, and PR or QS is what we called c . That is, $m - PR$ is the real bending-moment M at every place; and, knowing the bending-moment and depth d , it is easy to find I , as the condition of uniform strength is given in (5), or

$$I = \pm \frac{M \cdot l}{2f},$$

I being of course always positive.

We have found it very easy to solve this problem by trial. First find the area EC . Choose two points P', Q' , whose ordinates $P'R'$ and $Q'S'$ are equal, and such that the area $P'U'$ seems to the eye nearly half the area EC . Measure the area $P'U'$, subtract from $\frac{1}{2}EC$, divide by about $4Q'U'$ or $4P'T'$ or by the mean value of these two. This will give an approximation to the error in the positions of P' and Q' . With an easy exercise of one's judgment, it will probably not be necessary to make a second approximation.

V. If the depth of the beam is constant, the problem of finding the diagram of bending-moment for a beam of uniform strength is of course solved by finding two points, P and Q , whose distance asunder is half the length EG of the beam, and at which the ordinates PR and QS are equal. Subtract PR or QS from every ordinate of the m diagram EFG , and we have the M diagram.

It will be seen that, with any system of loading and any variation of section in a beam, it is quite easy to get the diagram of bending-moment; and also that, with any system of loading and any variation of depth of beam, it is quite easy to get diagrams of bending-moment and moment of inertia for beams of uniform strength. Many other applications of the method are obvious.

X. *Note on Comparing Capacities.*

By E. C. RIMINGTON*.

LET two condensers, of capacities K_1 and K_2 , be connected to the arms A and D of a Wheatstone-bridge, as shown in woodcut; and let

$$AC = BD,$$

so that there is a balance for permanent currents.

Let $x+y$, x , z , and w be the cyclic currents at some moment after the battery-circuit is broken, and when the condensers are discharging; and let q_1 be

* Read June 25, 1887.