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“A Practical Method of Tracing Some Useful Curves.”

By ARTHUR HINDHAUGH SHIELD, M. Inst. C.E.

WHILE the method to be described is not mathematically correct, it is sufficiently accurate for most practical purposes, and has been used by the Author for many years in the general work of an engineer's office. With ordinary care and skill a parabola, hyperbola, or logarithmic curve can be drawn with the same approach to accuracy as a circle is drawn with the usual drawing instruments. Further, it can be drawn much more quickly and more neatly than by determining points in the curve by computation or by intersections, and subsequently tracing the curve through these points.

To bring the method to be described within the limits of pure geometry it would be sufficient to assume, as a new postulate, that a line of continuously varying direction can be drawn if its direction at every part of its course is known. If the changes in direction are made at immeasurably small intervals the line becomes a perfect curve. If they are made at intervals sufficiently close to be imperceptible to the eye the line is a curve sufficiently good for practical use.

A measure of the direction of a curve at any point in its course is given by the differential coefficient. The values of this function $\frac{dy}{dx}$ are tabulated for all curves in common use, and there is no need to consider here the means by which they are obtained. It is sufficient to note that the differential coefficient expresses the ratio which an indefinitely small change in the value of y bears to the corresponding indefinitely small change in the value of x .

It thus defines or measures the slope of the curve at any point. For instance, if at a point in a curve $\frac{dy}{dx} = \frac{1}{3}$ the slope of the curve or, more correctly, the slope of a line tangential to the curve at that point is 1 in 3.

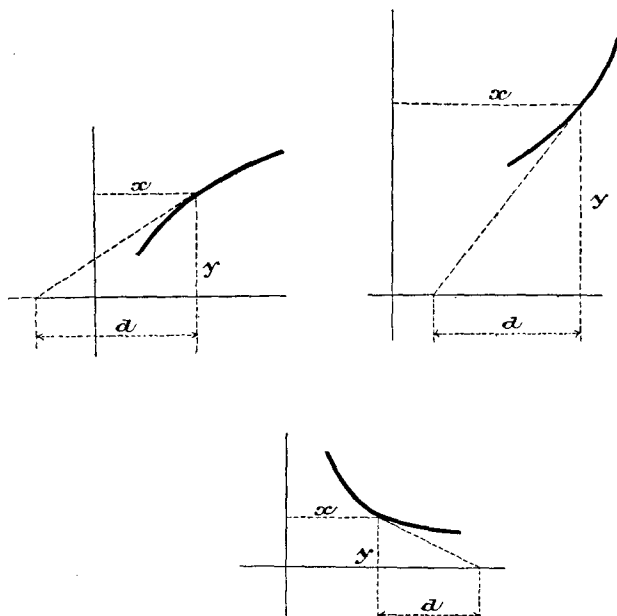
To make practical use of this law for drawing the curve it is necessary to have a comparatively distant point exactly and readily located to which to direct the ruler. For this purpose the quantity¹

$$\frac{y}{\frac{dy}{dx}}$$

is taken, and for convenience may be denoted by the symbol d .

In *Figs. 1*, d is shown in three examples varying with the position and character of the curve.

Figs. 1.

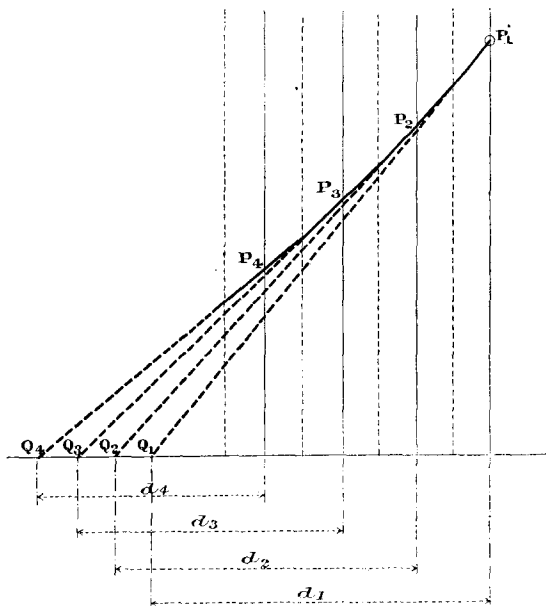


The practical use of the method is limited to those curves in which the expression takes a simple or easily computed form. To make use of this quantity d for drawing a curve, a number of lines are taken at short distances apart and a point P_1 on the curve,

¹ This is known in conic sections as the sub-tangent.

as shown in *Fig. 2*, having been found by computation or otherwise, tangents are drawn in the direction of points Q_1 Q_2 Q_3 , etc., on the x axis, located at distances d_1 d_2 d_3 respectively, from the vertical line passing through the point (P_1 P_2 P_3) through which the tangent is drawn. Each tangent is drawn from a point midway between two vertical lines to a point midway between the next pair. The result is a series of tangents which, if the lines are taken sufficiently close together, is indistinguishable from a smooth curve. The error involved is in the assumption that the envelop-

Fig. 2.



ing tangents intersect midway between the points at which they are tangential to the curve. This would only be the case if the radius of curvature were constant, as in the circle. The error may be kept within any assigned limits by having the lines sufficiently close together.

As already stated, the application of the method is limited to curves for which the values of d can be readily determined mentally or by a short computation for plotting or counting off upon the sectional paper usually employed. The curves for which it is capable

of such ready determination include, however, all those of the general form—

$$y = cx^n;$$

for which

$$\frac{dy}{dx} = cnx^{n-1},$$

and

$$d = \frac{x}{n};$$

also

$$y = e^x,$$

for which

$$\frac{dy}{dx} = e^x,$$

and

$$d = 1;$$

also

$$y = a^x;$$

for which

$$\frac{dy}{dx} = a^x \log_e a,$$

and

$$d = \frac{1}{\log_e a}.$$

The method is, therefore, applicable to many cases of everyday occurrence, of which the simplest and most frequently required is the plotting in diagram form of such an expression as—

$$y = cx^2.$$

The procedure in respect to curves of this form was described by the Author under the description of “Practical Curve Tracing for Graphic Computation” in *The Engineer*,¹ but, as this issue is now out of print, may with advantage be repeated with some modifications which experience has shown to be desirable.

To provide a simple example, let it be assumed that it is required to describe the parabola $y = 0.7854x^2$ showing the areas of circles up to $x = 10$. These are, of course, to be found in any pocket-book, but the same procedure applies to a diagram of the area or weight per foot of any other section which may not be in a pocket-book.

The value 78.54 is computed for $x = 10$ and plotted on a piece of squared paper having conveniently 100 divisions each way in the portion used (*Fig. 3*).² The value of

$$\frac{dy}{dx} = 0.7854 \times 2x;$$

therefore

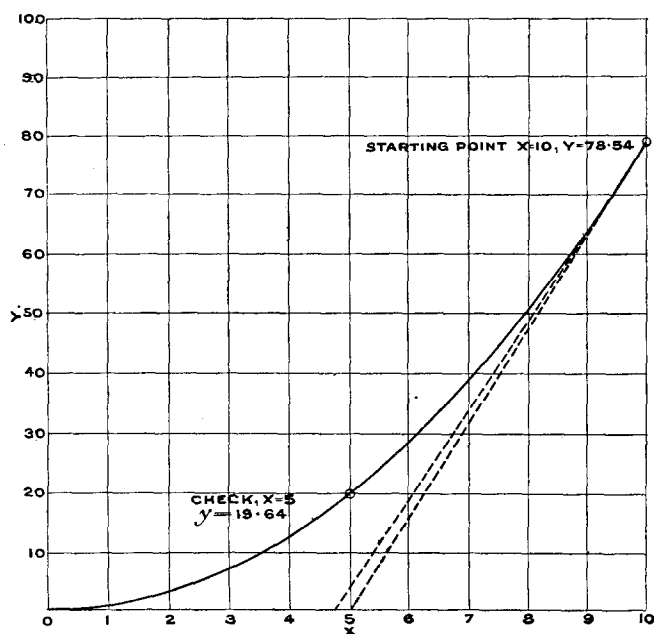
$$d = \frac{1}{2}x.$$

¹ Vol. lxxix (1895), p. 137.

² Owing to the difficulty of printing in one colour a sufficient number of lines to illustrate the squared paper used in the preparation of this diagram and *Figs. 4, 5, 7, 8 and 9*, the groundwork of these diagrams must be regarded as a conventional representation of paper having a sufficient number of lines.

The tangent through the plotted value (marked "starting-point") is therefore directed to $5 (\frac{1}{2}x)$ on the x axis and drawn as far as the vertical through 9.75 . Without moving the pencil the ruler is shifted to pass through 4.75 on the axis and the tangent continued through 9.50 to 9.25 , and so on, always directing on the half of the value through which the tangent passes. The last tangent is drawn through 0.50 to 0.25 directed on 0.25 . If a better finish is wanted, shorter steps should be taken from 1 to 0 , i.e., on reaching 1.00 go to 0.9 only, and then directing on 0.4 through 0.8 to 0.7 ,

Fig. 3.



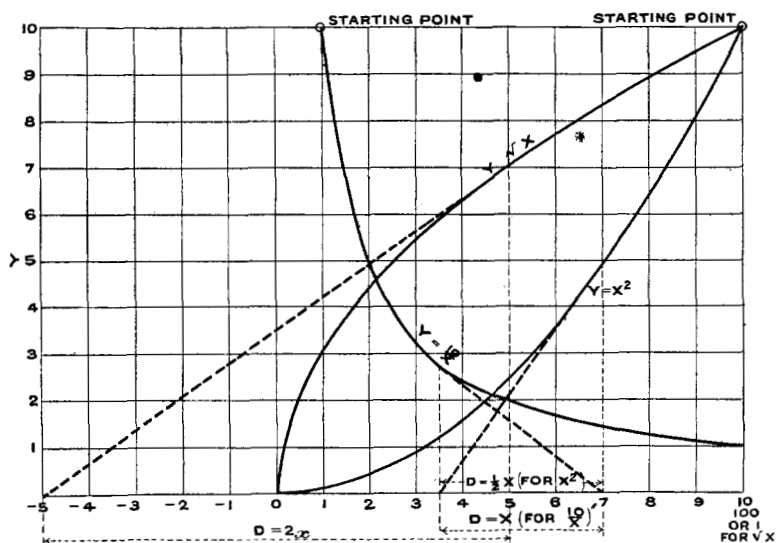
and directing on 0.3 through 0.6 to 0.5 . Longer or shorter steps may be taken throughout, but it should be remembered that the first is a half step. Experience will show the necessary closeness of the lines to secure the best results. If too many lines are taken, the process is not only tedious but the attempt to obtain greater accuracy may be defeated by the cumulative effect of the tendency to a slight upward shift of the pencil at the moment of changing the ruler. To avoid this the ruler should always be as light as possible. One or two intermediate values, such, for instance, as $x = 5$, $y = 19.64$, are then examined as a check on the work.

For other curves the values of $\frac{dy}{dx}$ and d are:—

y	$\frac{dy}{dx}$	d
x^2	$2x$	$\frac{1}{2}x$
x^3	$3x^2$	$\frac{1}{3}x$
x^N	Nx^{N-1}	$\frac{x}{N}$
$x^{\frac{1}{2}}$ or \sqrt{x}	$\frac{x^{-\frac{1}{2}}}{2}$	$2x$
x^{-1} or $\frac{1}{x}$	$-x^{-2}$	$-x$
x^{-N} or $\frac{1}{x^N}$	$-Nx^{-(N+1)}$	$-\frac{x}{N}$

Note that when d is positive it is set backwards or to the left along the axis. When it is negative it is set forwards.

Fig. 4.



A general example giving the form of these curves and the position of d is given in Fig. 4.

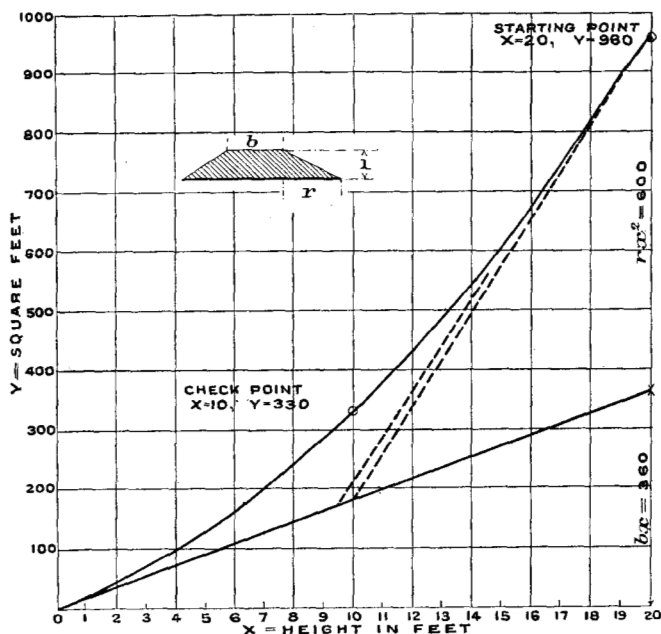
For the following cases d is a constant :—

y	$\frac{dy}{dx}$	d
e^x	e^x	1
10^x	$10^x \log_e 10$	0.4324

In all other cases it must be noted that the value of x used in computing or counting off d is that corresponding with the centre of the short tangent which is being drawn.

It is not necessary that the axis of x should be at right angles to the axis of y . The curve of $y = cx + c_1x^2$ can, therefore, be drawn

Fig. 5.



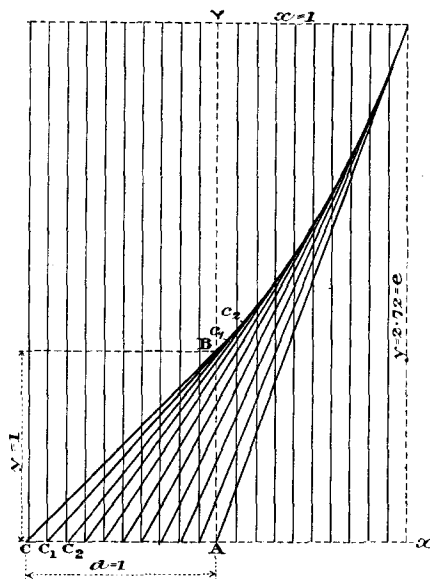
by first plotting a value of cx and drawing the straight line which represents cx and using it as a base to which to direct the tangents in drawing c_1x^2 . The ordinates are vertical and parallel to the y axis, and the measurement of d is horizontal, and not along the inclined line cx . A common example of the cases to which this

procedure is applicable is the curve giving the cross section of an embankment—

$$y = bx + rx^2,$$

in which b is the breadth at top and r the ratio of slope to height, and y is the area of its cross section for height x . This is shown in *Fig. 5*. Taking paper suited to a maximum value of y 1,000 square feet and $x = 20$ feet; bx is computed for $x = 20$ and $b = 18$ and plotted as 360 over $x = 20$. This point is joined to 0 by a straight line. The value $rx^2 = 600$ for $x = 20$ is then plotted

Fig. 6

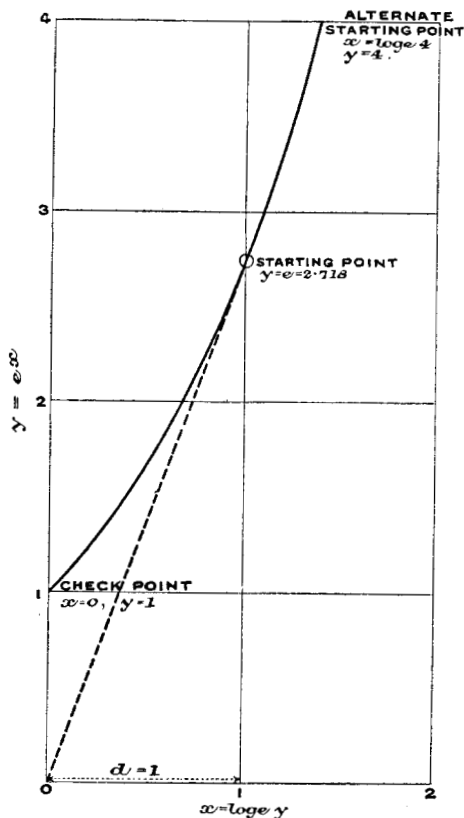


above bx , and the curve is drawn as shown with the tangents directed to the line bx .

The last two cases, $y = e^x$ and $y = 10^x$, are not of so great practical use as the preceding; they are very easily drawn because d is constant. The curve $y = e^x$, however, is interesting from the facility with which the value of e can be determined from unity with a pencil, ruler and squared paper. For if $x = 0$, $e^x = 1$. This gives a starting-point B on AY (*Fig. 6*), d is 1 (AC). Then, drawing from C through B to c_1 , and from C_1 through c_1 to c_2 , and so on as far as $x = 1$, the value of y will be found as near 2.718 as

the skill of the draughtsman can make it and the scale permit its reading. To draw a curve from a low value to a higher is, however, merely interesting from an academic point of view. For practical purposes a curve is always drawn from a higher to a lower value in order to minimize the error, and the curve of e^x is drawn

Fig. 7.

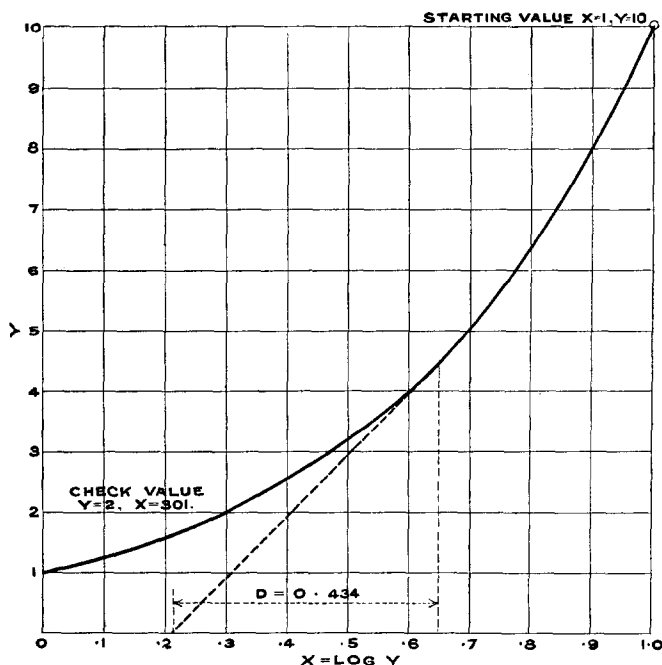


downward from the known value of e for $x = 1$ or some higher value of y , such as $y = 10$, for $x = \log_e 10$, or $y = 4$, for $x = \log_e 4$ (Fig. 7). The value $y = 1$ for $x = 0$ is always available for a check point.

The curve on Fig. 8 is drawn for $y = 10^x$, in which case d is 0.434, or the modulus of the common logarithms. This cannot be

counted off on the squared paper, but being constant the necessary directing points can readily be set off with dividers before commencing to draw the curve.

Fig. 8.



If x and y are reversed or the paper is turned through a right angle after drawing the curve,

$$y = e^x$$

becomes

$$x = \log_e y,$$

and

$$y = 10^x$$

becomes

$$x = \log y.$$

These curves may therefore be used for constructing the logarithmic scales required for the rectification of graphic representations of products or powers.

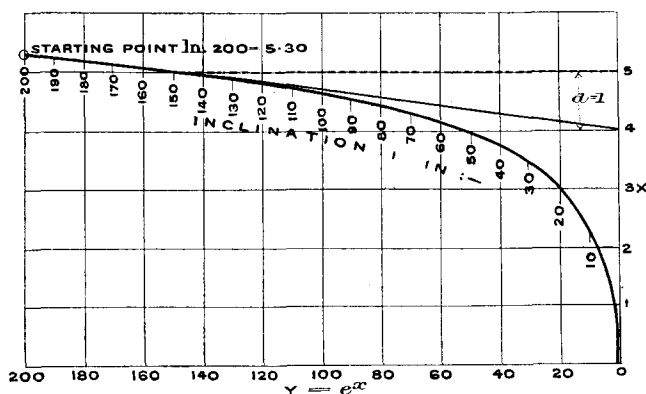
The curve $y = e^x$ has also the property that at any point y the slope is 1 in y , and if drawn on tracing paper or celluloid or cut in

stiff paper or card it may be used as a fairly accurate means of measuring the slope of a line. For this purpose the curve is marked with the values of y , and the point is noted at which the line of which the slope is required is tangent to the curve, when the latter is slid over or against it with its base (the y axis) guided by a straight-edge or parallel ruler.

It can be constructed to a distorted scale and used to read off at sight the approximate gradients from a road or railway section or profile drawn, as is usual, to a distorted scale. In *Fig. 9* is shown an example in which the distortion is 20 to 1.

The usefulness of this method is not confined to the drawing-office, but is also applicable to setting out on the ground the cubic parabola $y = cx^3$ used for transition curves on railways, and to the

Fig. 9.



production in patterns, templates, or stencils of curves for artistic and other work. It is a well-known principle in art, having its origin in nature, that a curve is more beautiful than a straight line, and a curve which continually changes its curvature is more beautiful than a curve of uniform curvature. Such a line of changing curvature can readily be given to a bracket by setting it out on the pattern or template by the method given for $y = \frac{c}{x}$ or any similar curve, and will be much better in appearance than the usual approximation by arcs of circles.

The Paper is accompanied by nine diagrams, from which the Figures in the text have been prepared.