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ON THE PARTIAL CORRELATION RATIO.

PART I. THEORETICAL.

By L. ISSERLIS, B.A.

§ 1. The theory of non-linear regression in the case of two correlated variables is due to Prof. Karl Pearson*. He shows that regression ceases to be linear when the correlation ratio η differs sensibly from the correlation coefficient r and establishes criteria for parabolic, cubic and higher forms of regression.

The present paper deals with the regression surface of three correlated variables x, y, z , where, though the regression of z on x, y cannot be adequately represented by an equation of type

$$\frac{\bar{z}_{xy} - \bar{z}}{\sigma_z} = \gamma_s \frac{x - \bar{x}}{\sigma_x} + \gamma'_s \frac{y - \bar{y}}{\sigma_y} \dots\dots\dots(1),$$

the regression of z on x for a constant y and of z on y for a constant x is linear. A large proportion of the non-linear cases that occur in practice fall into this class. It will be remembered that $\bar{z}_{xy}$ in (1) denoting the mean of the array of z 's for a given x and y the coefficients γ_s, γ'_s are partial regression coefficients and it will appear that just as it is necessary to introduce the correlation ratio η for an adequate description of non-linear regression of two variables, there must be introduced multiple or partial η 's for the description of such regression in the case of more than two variables.

We recall the definition and principal properties of ${}_x\eta_y$ —the correlation ratio of y on x, σ_{a_y} being the square root mean weighted square standard deviations of the arrays of y :

$$(1 - \eta^2) \sigma_y^2 = \sigma_{a_y}^2 = \frac{S(n_x \sigma_{n_x}^2)}{N} = \frac{SS\{n_{xy}(y - \bar{y}_x)^2\}}{N} \dots\dots\dots(2),$$

$$\eta^2 = \frac{\sigma_{m_y}^2}{\sigma_y^2} = \frac{S\{n_x(\bar{y}_x - \bar{y})^2\}}{N\sigma_y^2} \dots\dots\dots(3),$$

and
$$N(\eta^2 - r^2) \sigma_y^2 = S\{n_x(y_{n_x} - Y)^2\} \dots\dots\dots(4).$$

* *Drapers' Company Research Memoirs. Mathematical Contributions to the Theory of Evolution.*
XIV. "On the General Theory of Skew Correlation and Non-linear Regression." 1905.

Here we are dealing with N pairs of two characters A and B . n_x of these have the character x of A . $\bar{y}_x$ is the mean of this x -array of B 's. σ_{n_x} is the standard deviation of this array, σ_{n_y} is the weighted standard deviation of the means of the arrays and Y is the value of y given by the regression straight line, i.e.

$$Y = \bar{y} + r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \dots\dots\dots(5).$$

This is the "best fitting" straight line (in the Gaussian sense) to the means of the arrays, and is the regression line when the regression is linear.

§ 2. Consider now three correlated characters A, B, C . If N combinations of A, B, C are taken, we may denote by n_x the number of these which have the character $A = x$ and by n_{xy} the number in which A has the value x and B has the value y . Let $\bar{x}, \bar{y}, \bar{z}$ be the mean values of the total population, and let $\bar{z}_{xy}$ be the mean of z for a given x and y . The frequency of $A = x, B = y, C = z$ is n_{xyz} .

We define the correlation ratio of z on x and y , which may be denoted by $_{xy}H_z$, or if no confusion is likely to arise by H_z , by the equation

$$\sigma_z^2 [1 - _{xy}H_z^2] = \frac{SSS \{n_{xyz} (z - \bar{z}_{xy})^2\}}{N} \dots\dots\dots(6).$$

The triple sum in the definition can be written

$$\begin{aligned} & SSS \{n_{xyz} (z - \bar{z} + \bar{z} - \bar{z}_{xy})^2\} \\ &= SS \{n_{xy} (\bar{z} - \bar{z}_{xy})^2\} + 2SS \{(\bar{z} - \bar{z}_{xy}) \times S \{n_{xyz} (z - \bar{z})\} + SSS \{n_{xyz} (z - \bar{z})^2\} \\ &= SS \{n_{xy} (\bar{z} - \bar{z}_{xy})^2\} - 2SS \{n_{xy} (\bar{z} - \bar{z}_{xy})^2\} + N\sigma_z^2. \end{aligned}$$

Hence
$$_{xy}H_z^2 = \frac{SS \{n_{xy} (\bar{z} - \bar{z}_{xy})^2\}}{N\sigma_z^2} \dots\dots\dots(7)$$

This is a generalisation of the property of $_{xy}r_z$ given by (3).

Further, the "best fitting" plane to the means $\bar{z}_{xy}$ is given by

$$\frac{z - \bar{z}}{\sigma_z} = \gamma_3 \frac{x - \bar{x}}{\sigma_x} + \gamma_3' \frac{y - \bar{y}}{\sigma_y} \dots\dots\dots(8),$$

where

$$\gamma_3 = \frac{r_{xz} - r_{yz}r_{xy}}{1 - r_{xy}^2} \dots\dots\dots(9),$$

$$\gamma_3' = \frac{r_{yz} - r_{xz}r_{xy}}{1 - r_{xy}^2} \dots\dots\dots(10).$$

Let $_{xy}R_z$ denote as usual the maximum correlation of z with any linear function of x and y , then

$$_{xy}R_z^2 = \gamma_3 r_{zx} + \gamma_3' r_{zy} \dots\dots\dots(11),$$

$$= \frac{r_{yz}^2 + r_{zx}^2 - 2r_{yz}r_{xz}r_{xy}}{1 - r_{xy}^2} \dots\dots\dots(12).$$

Subtract (11) from (7) after replacing r_{zx} and r_{zy} by the appropriate sums, and we obtain

$$\begin{aligned} & [{}_{xy}H_z^2 - {}_{xy}R_z^2] N\sigma_z^2 \\ &= SS \left[n_{xy} (\bar{z} - \bar{z}_{xy})^2 - S \left\{ n_{xyz} \frac{\sigma_z}{\sigma_x} (z - \bar{z}) (x - \bar{x}) \gamma_3 \right\} - S \left\{ n_{xyz} \frac{\sigma_z}{\sigma_y} (z - \bar{z}) (y - \bar{y}) \gamma_3' \right\} \right] \\ &= SS \left[n_{xy} (\bar{z} - \bar{z}_{xy})^2 - n_{xy} \frac{\sigma_z}{\sigma_x} (\bar{z}_{xy} - \bar{z}) (x - \bar{x}) \gamma_3 - n_{xy} \frac{\sigma_z}{\sigma_y} (\bar{z}_{xy} - \bar{z}) (y - \bar{y}) \gamma_3' \right] \end{aligned}$$

Using (8) this can be written

$$\begin{aligned} [{}_{xy}H_z^2 - {}_{xy}R_z^2] N\sigma_z^2 &= SS \{ n_{xy} (\bar{z}_{xy} - \bar{z}) (\bar{z}_{xy} - Z) \} \\ &= SS \{ n_{xy} (\bar{z}_{xy} - Z)^2 \} + SS \{ n_{xy} (Z - \bar{z}) (\bar{z}_{xy} - Z) \} \dots (13). \end{aligned}$$

But $SS \{ n_{xy} (Z - \bar{z}) (\bar{z}_{xy} - Z) \}$

$$\begin{aligned} &= SS \left\{ n_{xy} \left(\gamma_3 \frac{x - \bar{x}}{\sigma_x} \sigma_z + \gamma_3' \frac{y - \bar{y}}{\sigma_y} \sigma_z \right) (\bar{z}_{xy} - \bar{z} - \gamma_3 \frac{x - \bar{x}}{\sigma_x} \sigma_z - \gamma_3' \frac{y - \bar{y}}{\sigma_y} \sigma_z) \right\} \\ &= N\sigma_z (\gamma_3 \sigma_z r_{zx} - \gamma_3^2 \sigma_z - \gamma_3 \gamma_3' r_{xy} \sigma_z + \gamma_3' \sigma_z r_{yz} - \gamma_3 \gamma_3' r_{xy} \sigma_z - \gamma_3'^2 \sigma_z) \\ &= N\sigma_z^2 \{ \gamma_3 (r_{zx} - \gamma_3 - \gamma_3' r_{xy}) + \gamma_3' (r_{yz} - \gamma_3' - \gamma_3 r_{xy}) \} \dots \dots \dots (14). \end{aligned}$$

Using the values of γ_3 and γ_3' given by (9) and (10) we see immediately that

$$r_{zx} - \gamma_3 - \gamma_3' r_{xy} = r_{yz} - \gamma_3' - \gamma_3 r_{xy} \equiv 0.$$

Hence (13) becomes

$$({}_{xy}H_z^2 - {}_{xy}R_z^2) N\sigma_z^2 = SS \{ n_{xy} (\bar{z}_{xy} - Z)^2 \} \dots \dots \dots (15).$$

This is the generalisation of (4). We deduce that ${}_{xy}H_z = {}_{xy}R_z$ if and only if the regression is strictly linear, that otherwise ${}_{xy}H_z^2 > {}_{xy}R_z^2$ and by (6) that ${}_{xy}H_z^2 < 1$.

§ 3. These properties and definitions can be extended to the case of m variables $x_1, x_2, \dots, x_m$. We now use ${}_{2\dots m}\bar{x}_1$ for the mean of x_1 when $x_2, x_3, \dots, x_m$ are given and denote by S a summation extending to the variables $x_1, x_2, \dots, x_m$.

If we define the correlation ratio of x_1 on $x_2, x_3, \dots, x_m$ by the equation

$$N\sigma_1^2 (1 - {}_{2,3\dots m}H_1^2) = S (x_1 - {}_{2,3\dots m}\bar{x}_1)^2 n_{1\dots m} \dots \dots \dots (16).$$

We can deduce in the same way as in § 2, the relation

$$N\sigma_1^2 {}_{2,3\dots m}H_1^2 = S \{ (\bar{x}_1 - {}_{2,3\dots m}\bar{x}_1)^2 n_{2\dots m} \} \dots \dots \dots (17).$$

In order to generalise (15) we recall that the "best fitting" linear function of the variables $x_2, x_3, \dots, x_m$ to the mean ${}_{2,3\dots m}\bar{x}_1$ is

$$\frac{X_1}{\sigma_1} = \frac{\bar{x}_1}{\sigma_1} - b_{12} \frac{x_2 - \bar{x}_2}{\sigma_2} - b_{13} \frac{x_3 - \bar{x}_3}{\sigma_3} - \dots - b_{1m} \frac{x_m - \bar{x}_m}{\sigma_m} \dots \dots \dots (18),$$

where $b_{it} = \frac{R_{it}}{R_{11}}$ and R_{pq} is the minor with its proper sign of the element in the p th row and q th column of the determinant

$$R = \begin{vmatrix} 1 & r_{12} & \dots & r_{1m} \\ r_{21} & 1 & \dots & r_{2m} \\ \dots & \dots & \dots & \dots \\ r_{m1} & r_{m2} & \dots & 1 \end{vmatrix} \dots \dots \dots (19),$$

while the maximum correlation between any linear function of $x_2, x_3, \dots, x_m$ and x_1 is ${}_{2,3,\dots,m}R_1$ where

$$-{}_{2,3,\dots,m}R_1^2 = \frac{R - R_{11}}{R_{11}} = b_{12}r_{12} + b_{13}r_{13} \dots + b_{1m}r_{1m} \dots \dots \dots (20),$$

$$= b_{12} \frac{S \{n_{12}(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\}}{N\sigma_1\sigma_2} + b_{13} \frac{S \{n_{13}(x_1 - \bar{x}_1)(x_3 - \bar{x}_3)\}}{N\sigma_1\sigma_3} \\ + \dots + b_{1m} \frac{S \{n_{1m}(x_1 - \bar{x}_1)(x_m - \bar{x}_m)\}}{N\sigma_1\sigma_m} \dots \dots \dots (21).$$

Subtract (21) from (17), noting that $n_{12} = S \{n_{12\dots m}\}$, etc., we obtain*

$$\begin{aligned} & ({}_{2,3,\dots,m}H_1^2 - {}_{2,3,\dots,m}R_1^2) N\sigma_1^2 \\ &= S \{n_{2\dots m}(\bar{x}_1 - {}_{2\dots m}\bar{x}_1)^2\} + S \left\{ n_{12} \frac{\sigma_1}{\sigma_2} b_{12} (x_1 - \bar{x}_1)(x_2 - \bar{x}_2) \right\} \\ & \quad + \dots + S \left\{ n_{1m} \frac{\sigma_1}{\sigma_m} b_{1m} (x_1 - \bar{x}_1)(x_m - \bar{x}_m) \right\} \\ &= S \{n_{2\dots m}(\bar{x}_1 - {}_{2\dots m}\bar{x}_1)^2\} + S \left\{ n_2 \frac{\sigma_1}{\sigma_2} b_{12} (x_2 - \bar{x}_2)({}_{2\dots m}\bar{x}_1 - \bar{x}_1) \right\} - \dots \\ & \quad + S \left\{ n_m \frac{\sigma_1}{\sigma_m} b_{1m} (x_m - \bar{x}_m)({}_{2\dots m}\bar{x}_1 - \bar{x}_1) \right\} \\ &= S \left[n_{2\dots m}(\bar{x}_1 - {}_{2\dots m}\bar{x}_1)^2 + n_{2\dots m} \frac{\sigma_1}{\sigma_2} b_{12} (x_2 - \bar{x}_2)({}_{2\dots m}\bar{x}_1 - \bar{x}_1) - \dots \right] \\ &= S \left[n_{2\dots m}({}_{2\dots m}\bar{x}_1 - \bar{x}_1) \left\{ {}_{2\dots m}\bar{x}_1 - \bar{x}_1 + b_{12} \frac{\sigma_1}{\sigma_2} (x_2 - \bar{x}_2) - \dots \right. \right. \\ & \quad \left. \left. + b_{1m} \frac{\sigma_1}{\sigma_m} (x_m - \bar{x}_m) \right\} \right] \dots \dots \dots (22) \end{aligned}$$

using (18) this equation becomes

$$\begin{aligned} ({}_{2,3,\dots,m}H_1^2 - {}_{2,3,\dots,m}R_1^2) N\sigma_1^2 &= S \{n_{2\dots m}({}_{2\dots m}\bar{x}_1 - \bar{x}_1)({}_{2\dots m}\bar{x}_1 - X_1)\} \\ &= S \{n_{2\dots m}({}_{2\dots m}\bar{x}_1 - X_1)^2\} + S \{n_{2\dots m}(X_1 - \bar{x}_1)({}_{2\dots m}\bar{x}_1 - X_1)\} \\ & \quad \dots \dots \dots (23). \end{aligned}$$

* By an extension of the notation described at the beginning of this section $S_{3\dots m}$ denotes a summation with regard to the variables $x_3, x_4, \dots, x_m$; $n_{1,2,\dots,m}$ is the frequency of a particular combination of the characters $x_1, x_2, \dots, x_m$ while n_{12} is the frequency of the combination x_1, x_2 .

$$\text{But } S_{2\dots m} \{n_{2\dots m} (X_1 - \bar{x}_1) ({}_{2\dots m}\bar{x}_1 - X_1)\} \\ = S_{2\dots m} \left\{ n_{2\dots m} \left(-b_{12} \frac{x_2 - \bar{x}_2}{\sigma_2} \sigma_1 - b_{13} \frac{x_3 - \bar{x}_3}{\sigma_3} \sigma_1 - \dots \right) \left({}_{2\dots m}\bar{x}_1 - \bar{x}_1 + b_{12} \frac{x_2 - \bar{x}_2}{\sigma_2} \sigma_1 + \dots \right) \right\} \quad (24),$$

$$\text{and } N\sigma_1\sigma_2r_{12} = S_{1\dots m} \{n_{1\dots m} (x_1 - \bar{x}_1) (x_2 - \bar{x}_2)\} \\ = S_{2\dots m} \{n_{2\dots m} (x_2 - \bar{x}_2) ({}_{2\dots m}\bar{x}_1 - \bar{x}_1)\}$$

with similar values for r_{13} , r_{23} , etc.

$\therefore$ the right hand side of (24)

$$= -b_{12}\sigma_1^2r_{12} - b_{12}^2\sigma_1^2 - b_{13}b_{12}\sigma_1^2r_{23} \dots \\ - \sigma_1^2b_{13}r_{13} - b_{13}b_{12}\sigma_1^2r_{23} - b_{13}^2\sigma_1^2 - b_{13}b_{14}\sigma_1^2 \dots \\ - \dots \\ = \sigma_1^2b_{12}(-r_{12} - b_{12} - b_{13}r_{23} - b_{14}r_{24} - \dots - b_{1m}r_{2m}) \\ + \sigma_1^2b_{13}(-r_{13} - b_{13} - b_{14}r_{34} - \dots) \\ + \dots \dots \dots (25).$$

Each line in (25) is identically zero from the definition of the b 's and the properties of the determinant in (19).

$$\text{Hence } {}_{2\dots m}H_1^2 - {}_{2\dots m}R_1^2 = \frac{S_{2\dots m} \{n_{2\dots m} ({}_{2\dots m}\bar{x}_1 - X_1)^2\}}{N\sigma_1^2} \dots \dots \dots (26),$$

so that the fundamental properties proved by Professor Pearson in connection with the correlation ratio η , hold for the generalised H defined in this section. In particular equation (26) shows that a necessary and sufficient condition for linear regression in multiple correlation of m variables is that

$${}_{2,3,\dots m}H_1^2 = {}_{2,3,\dots m}R_1^2.$$

For in this case the mean value of any array of x_1 will lie on the "best fitting" m dimensional plane.

§ 4. The regression surface of z on xy being assumed of any particular type the constants in the equation may be determined (i) by the method of least squares, i.e. by making the sum of the squares of the deviations $\bar{z}_{xy} - \phi(xy)$ a minimum, $z = \phi(x, y)$ being the regression surface, or (ii) by giving such values to the constants that the correlation between z and $\phi(x, y)$ shall be a maximum. When $\phi(x, y)$ is of the second degree the two methods lead to identical equations for the determination of the coefficients.

The same equations are also obtained if the surface be "fitted" to the means by the method of moments. There is, however, a distinction to be observed. The equation $z = \phi(x, y)$ when the regression surface is of specified degree contains a definite number of constants and the first two methods will give exactly as many independent equations as there are constants to be determined. The method of moments will give as many equations as we please if sufficiently high moments

are used, including of course the equations given by the "least squares" method or maximum correlation method. Even without introducing high moments, when there are three variable characters new equations may be obtained by the method of moments, by combinations of characters which do not arise in the other methods. The method of moments is most convenient for our purpose, but we shall only employ those equations which can also be justified by (say) the method of least squares.

For convenience let the origin be taken at the mean of the three characters so that $\bar{x} = \bar{y} = \bar{z} = 0$, let $q_{x^s y^t z^u}$ denote

$$\frac{SSS \{n_{xyz} x^s y^t z^u\}}{N \sigma_x^s \sigma_y^t \sigma_z^u} = \frac{p_{s,t,u}}{\sigma_x^s \sigma_y^t \sigma_z^u} \dots\dots\dots (27).$$

With this notation r_{xy} and $q_{x^1 y^1 z^0}$ are identical; when z does not appear in the product, it is sufficient to write $q_{x^s y^t}$.

The most reasonable next approximation to make when a linear function $\phi(x, y)$ does not adequately represent the statistics is $\phi(x, y) = a$ quadratic function of x, y .

Let
$$\frac{z}{\sigma_z} = d + \frac{ax}{\sigma_x} + \frac{by}{\sigma_y} + \frac{cxy}{\sigma_x \sigma_y} + \frac{ex^2}{\sigma_x^2} + \frac{fy^2}{\sigma_y^2} \dots\dots\dots (28).$$

Multiply (28) by n_{xyz} , sum for all values of x, y, z and divide by N

$$0 = d + cr_{xy} + e + f \dots\dots\dots (29).$$

Multiply (28) in turn by $\frac{n_{xyz}}{N}$ times $\frac{x}{\sigma_x}, \frac{y}{\sigma_y}, \frac{xy}{\sigma_x \sigma_y}, \frac{x^2}{\sigma_x^2}, \frac{y^2}{\sigma_y^2}$ and sum as before,

and we obtain

$$r_{xy} = a + br_{xy} + cq_{x^2 y} + eq_{x^3} + fq_{xy^2} \dots\dots\dots (30),$$

$$r_{yz} = b + ar_{xy} + cq_{xy^2} + eq_{x^2 y} + fq_{y^3} \dots\dots\dots (31),$$

$$q_{xyz} = dr_{xy} + aq_{x^2 y} + bq_{xy^2} + cq_{x^2 y^2} + eq_{x^3 y} + fq_{xy^3} \dots\dots\dots (32),$$

$$q_{x^2 z} = d + aq_{x^3} + bq_{x^2 y} + cq_{x^3 y} + eq_{x^4} + fq_{x^2 y^2} \dots\dots\dots (33),$$

$$q_{y^2 z} = d + aq_{xy^2} + bq_{y^3} + cq_{xy^3} + eq_{x^2 y^2} + fq_{y^4} \dots\dots\dots (34).$$

Actual numerical fitting shows that in many cases e and f are small compared with c^* . This is the case when the regression of z on x for a constant y and of z on y for a constant x is linear. We shall therefore confine ourselves in the present preliminary paper to the case where we may write

$$\frac{z}{\sigma_z} = d + \frac{ax}{\sigma_x} + \frac{by}{\sigma_y} + \frac{cxy}{\sigma_x \sigma_y} \dots\dots\dots (35).$$

Here for constant x or y the regression of z on y or of z on x is linear.

* Cf. *Census of Scotland*, 1911, Vol. III. p. XLVII. where Mr G. Rae obtains by moments the regression of fertility on age of husband and wife. Let W = age of wife, H = age of husband, C = number of children in completed marriages. He finds

$$C_{WH} = 20.149493 - 0.555812W - 0.173804H - 0.002846W^2 - 0.003494H^2 + 0.012675WH.$$

See also the paper by E. M. Elderton in the current part of this Journal, pp. 291—295.

Equations (29) to (32) become, when the regression surface is given by (35),

$$0 = d + cr_{xy} \dots\dots\dots(36),$$

$$r_{xy} = a + br_{xy} + cq_{x^2y} \dots\dots\dots(37),$$

$$r_{yz} = b + ar_{xy} + cq_{xy^2} \dots\dots\dots(38),$$

$$q_{xyz} = dr_{xy} + aq_{x^2y} + bq_{xy^2} + cq_{x^2y^2} \dots\dots\dots(39),$$

$$= -cr^2_{xy} + aq_{x^2y} + bq_{xy^2} + cq_{x^2y^2} \text{ by (36).}$$

Solving these equations we obtain

$$\begin{array}{c} a \\ \left| \begin{array}{ccc} r_{xy} & q_{x^2y} & r_{xz} \\ 1 & q_{xy^2} & r_{yz} \\ q_{xy^2} & q_{x^2y^2} - r^2_{xy} & q_{xyz} \end{array} \right| \end{array} = \begin{array}{c} b \\ \left| \begin{array}{ccc} q_{x^2y} & 1 & r_{xz} \\ q_{xy^2} & r_{xy} & r_{yz} \\ q_{x^2y^2} - r^2_{xy} & q_{x^2y} & q_{xyz} \end{array} \right| \end{array}$$

$$= \begin{array}{c} c \\ \left| \begin{array}{ccc} 1 & r_{xy} & r_{xz} \\ r_{xy} & 1 & r_{yz} \\ q_{x^2y} & q_{xy^2} & q_{xyz} \end{array} \right| \end{array} = \begin{array}{c} 1 \\ \left| \begin{array}{ccc} 1 & r_{xy} & q_{x^2y} \\ r_{xy} & 1 & q_{xy^2} \\ q_{x^2y} & q_{xy^2} & q_{x^2y^2} - r^2_{xy} \end{array} \right| \end{array} \dots\dots(40),$$

we have already denoted the partial regression coefficients by γ_3 and γ_3' so that

$$\gamma_3 = \frac{r_{xz} - r_{yz}r_{xy}}{1 - r^2_{xy}} \text{ and } \gamma_3' = \frac{r_{yz} - r_{xz}r_{xy}}{1 - r^2_{xy}} \dots\dots\dots(9) \text{ and } (10).$$

In addition let

$$\frac{r_{xy}q_{xy^2} - q_{x^2y}}{1 - r^2_{xy}} = \theta \dots\dots\dots(41),$$

and

$$\frac{r_{xy}q_{x^2y} - q_{xy^2}}{1 - r^2_{xy}} = \phi \dots\dots\dots(42).$$

After some reductions the determinants in (40) yield

$$a = \gamma_3 + c\theta \dots\dots\dots(43),$$

$$b = \gamma_3' + c\phi \dots\dots\dots(44),$$

$$c = \frac{\theta r_{xz} + \phi r_{yz} + q_{xyz}}{\theta q_{x^2y} + \phi q_{xy^2} + q_{x^2y^2} - r^2_{xy}} \dots\dots\dots(45),$$

also

$$d = -c\gamma_{xy} \dots\dots\dots(36).$$

Note that

$$\gamma_3 q_{x^2y} + \gamma_3' q_{xy^2} \equiv -(\theta r_{xz} + \phi r_{yz}) \dots\dots\dots(46),$$

and

$$\gamma_3 r_{zx} + \gamma_3' r_{zy} = {}_{xy}R_x^2 \text{ (cf. eqn. 11).}$$

$$\S 5. \text{ By definition } (1 - {}_{xy}H_z^2) = \frac{S \{n_{xyz}(z - \bar{z}_{xy})^2\}}{N\sigma_z^2},$$

$\therefore$ using (35)

$$\begin{aligned} (1 - {}_{xy}H_z^2) &= S \left\{ \frac{n_{xyz}}{N} \left(\frac{z}{\sigma_z} - d - \frac{ax}{\sigma_x} - \frac{by}{\sigma_y} - \frac{cxy}{\sigma_x\sigma_y} \right)^2 \right\} \\ &= 1 - 2S \left\{ \frac{z}{\sigma_z} \left(d + \frac{ax}{\sigma_x} + \frac{by}{\sigma_y} + \frac{cxy}{\sigma_x\sigma_y} \right) \frac{n_{xyz}}{N} \right\} \\ &\quad + d^2 + a^2 + b^2 + c^2 q_{x^2y^2} + 2dcr_{xy} + 2ab r_{xy} + 2acq_{x^2y} + 2bcq_{xy^2}, \end{aligned}$$

$$\therefore -{}_{xy}H_z^2 = -2ar_{zx} - 2br_{yz} - 2cq_{xyz} + a^2 + b^2 + d^2 + c^2q_{x^2y} \\ + 2cdr_{xy} + 2abr_{xy} + 2acq_{x^2y} + 2bcq_{xy^2} \dots \dots \dots (47),$$

$$= a(-r_{zx} + a + br_{xy} + cq_{x^2y}) \\ + b(-r_{yz} + b + ar_{xy} + cq_{xy^2}) \\ + c(-q_{xyz} + dr_{xy} + aq_{x^2y} + bq_{xy^2} + cq_{x^2y^2}) \\ + d(d + cr_{xy}) \\ - ar_{zx} - br_{yz} - cq_{xyz} \dots \dots \dots (48).$$

The first four terms vanish by eqns. (36)...(39),

$$\therefore {}_{xy}H_z^2 = ar_{zx} + br_{zy} + cq_{xyz} \dots \dots \dots (49).$$

If we now insert the values of a, b, c from (43)...(45) in (49)

$${}_{xy}H_z^2 = (\gamma_3 + c\theta)r_{zx} + (\gamma_3' + c\phi)r_{zy} + cq_{xyz} \\ = \gamma_3 r_{zx} + \gamma_3' r_{zy} + c(\theta r_{zx} + \phi r_{zy} + q_{xyz}) \\ = {}_{xy}R_z^2 + \frac{(\theta r_{zx} + \phi r_{zy} + q_{xyz})^2}{q_{x^2y}\theta + q_{xy^2}\phi + q_{x^2y^2} - r_{xy}^2},$$

$$\text{or } {}_{xy}H_z^2 - {}_{xy}R_z^2 = \frac{\{q_{xyz}(1 - r_{xy}^2) - q_{xy}(r_{zx}r_{xy} - r_{yz}) - q_{x^2y}(r_{yz}r_{xy} - r_{xz})\}^2}{\left[(q_{x^2y^2} - r_{xy}^2) - \left\{\frac{q_{x^2y}^2 + q_{xy^2}^2 - 2q_{x^2y}q_{xy^2}r_{xy}}{1 - r_{xy}^2}\right\}\right](1 - r_{xy}^2)} \dots (50).$$

It follows from (50) and (15) that

$$q_{x^2y^2} - r_{xy}^2 > \frac{q_{x^2y}^2 + q_{xy^2}^2 - 2q_{x^2y}q_{xy^2}r_{xy}}{1 - r_{xy}^2} \dots \dots \dots (51).$$

If we eliminate q_{xyz} (which is a triple moment troublesome to calculate) between equations (45) and (49) we have

$${}_{xy}H_z^2 = ar_{zx} + br_{yz} + cq_{xyz} \\ = ar_{zx} + br_{yz} + c^2(q_{x^2y^2} - r_{xy}^2) + acq_{x^2y} + bcq_{xy^2} \\ = r_{zx}(\gamma_3 + c\theta) + r_{zy}(\gamma_3' + c\phi) + c^2(q_{x^2y^2} - r_{xy}^2) \\ + (\gamma_3 c + c^2\theta)q_{x^2y} + (\gamma_3' c + c^2\phi)q_{xy^2} \\ = c^2[q_{x^2y^2} - r_{xy}^2 + \theta q_{x^2y} + \phi q_{xy^2}] + c[\theta r_{zx} + \phi r_{zy} + \gamma_3 q_{x^2y} + \gamma_3' q_{xy^2}] \\ + \gamma_3 r_{zx} + \gamma_3' r_{zy} \\ = {}_{xy}R_z^2 + c^2[q_{x^2y^2} - r_{xy}^2 + \theta q_{x^2y} + \phi q_{xy^2}] \text{ by (46) and (11),} \\ \therefore {}_{xy}H_z^2 - {}_{xy}R_z^2 = c^2 \left[q_{x^2y^2} - r_{xy}^2 - \frac{q_{x^2y}^2 + q_{xy^2}^2 - 2q_{x^2y}q_{xy^2}r_{xy}}{1 - r_{xy}^2} \right]$$

$$\text{Hence } c^2 = \frac{{}_{xy}H_z^2 - {}_{xy}R_z^2}{q_{x^2y^2} - r_{xy}^2 - \frac{q_{x^2y}^2 + q_{xy^2}^2 - 2q_{x^2y}q_{xy^2}r_{xy}}{1 - r_{xy}^2}} \dots \dots \dots (52).$$

This value of c^2 is positive by (51).

Equation (52) shows that ${}_{xy}H_z^2 = {}_{xy}R_z^2$ is a necessary condition for linear regression, which we have already proved in equation (26).

The regression surface of z on x, y is, with the values we have now obtained for the constants

$$\begin{aligned} \frac{Z - \bar{z}}{\sigma_z} = & \frac{r_{xz} - r_{yz}r_{xy}}{1 - r_{xy}^2} \frac{x - \bar{x}}{\sigma_x} + \frac{r_{yz} - r_{xz}r_{xy}}{1 - r_{xy}^2} \frac{y - \bar{y}}{\sigma_y} \\ & + \frac{\sqrt{x_{yy}H_z^2 - x_{yl}R_z^2}}{\sqrt{q_{x^2y^2} - r_{xy}^2 - \frac{q_{xy}^2 + q_{x^2y} - 2q_{x^2y}q_{xy}r_{xy}}{1 - r_{xy}^2}}} \left\{ \frac{r_{xy}q_{xy}^2 - q_{x^2y}}{1 - r_{xy}^2} \frac{x - \bar{x}}{\sigma_x} + \frac{r_{xy}q_{x^2y} - q_{xy}^2}{1 - r_{xy}^2} \frac{y - \bar{y}}{\sigma_y} \right. \\ & \left. + \frac{(x - \bar{x})(y - \bar{y})}{\sigma_x \sigma_y} - r_{xy} \right\} \dots \dots \dots (53). \end{aligned}$$

The terms in the first line give the ordinary regression plane. In most cases the regression does not differ widely from linearity so that $x_{yy}H_z^2 - x_{yl}R_z^2$ is small.

§ 6. We must now get some idea of the relative magnitudes of $q_{x^2y^2}$, q_{x^2y} , $q_{x^2y^2}$ and moments of lower order.

First with regard to q_{x^2y} which is equal to $\frac{\bar{p}_{21}}{\sigma_x^2 \sigma_y}$,

$$q_{x^2y} = \frac{S(n_{xy}x^2y)}{N\sigma_x^2\sigma_y} = \frac{S(n_x\bar{y}_xx^2)}{N\sigma_x^2\sigma_y} \dots \dots \dots (54).$$

If the regression of y on x be linear

$$q_{x^2y} = S \left(\frac{n_x \frac{\sigma_y}{\sigma_x} x^3 r_{xy}}{N\sigma_x^2\sigma_y} \right) = r_{xy} \sqrt{\beta_1},$$

so that q_{x^2y} is zero if the regression is linear and the frequency of x symmetrical. In fact $q_{x^2y} - r_{xy} \sqrt{\beta_1} = 0$ is the same as Pearson's criterion for linear regression given by $\bar{\epsilon} = 0$ (*Skew Correlation and Non-linear Regression*, p. 30, Eqn. (lxix)).

We may obtain a good approximation for q_{x^2y} by considering the regression of y on x to be parabolic. This is a natural assumption to make if the regression surfaces of x on z, y and y on z, x be also of the hyperboloid type we are discussing for the regression of z on x, y .

For with origin at mean we may write

$$\frac{\bar{x}_{yz}}{\sigma_x} = e + \frac{fy}{\sigma_y} + \frac{gz}{\sigma_z} + \frac{hyz}{\sigma_y \sigma_z}.$$

Hence keeping y constant and summing for the z 's

$$\frac{\bar{x}_y}{\sigma_x} = e + \frac{fy}{\sigma_y} + \frac{g\bar{z}_y}{\sigma_z} + \frac{hy\bar{z}_y}{\sigma_y \sigma_z}.$$

But
$$\frac{\bar{z}_y}{\sigma_z} = r_{yz} \frac{y}{\sigma_y} \pm \frac{\sqrt{y\eta_z^2 - r_{yz}^2}}{\sqrt{\beta_2'' - \beta_1'' - 1}} \left\{ \frac{y^2}{\sigma_{y^2}} - \sqrt{\beta_1''} \frac{y}{\sigma_y} - 1 \right\}^*.$$

* See Pearson: *l.c.* Eqn. (lxv) (where Y_{yx} is a misprint for X_p), and β_2'' , β_1'' refer to the distribution of z .

$\therefore \frac{\bar{x}_y}{\sigma_x}$ is a quadratic expression in y if we remember that h being of order $\sqrt{{}_yH_x^2 - {}_yR_x^2}$ is of the same order as $\sqrt{{}_y\eta_x^2 - r^2}^*$.

But the relation $\phi_2({}_x\eta_y^2 - r^2_{xy}) - \bar{\epsilon} = 0$ is satisfied when the regression of y on x is parabolic†.

$$\text{Here} \quad \phi_2 = \beta_2 - \beta_1 - 1, \quad \bar{\epsilon} = \epsilon - r_{xy} \sqrt{\beta_1}, \quad \epsilon = q_{x^2y}^\ddagger.$$

$$\text{Hence} \quad q_{x^2y} - r_{xy} \sqrt{\beta_1} = \sqrt{{}_x\eta_y^2 - r^2_{xy}} \sqrt{\beta_2 - \beta_1 - 1} \dots\dots\dots(55).$$

$$\text{Similarly} \quad q_{xy^2} - r_{xy} \sqrt{\beta_1'} = \sqrt{{}_y\eta_x^2 - r^2_{xy}} \sqrt{\beta_2' - \beta_1' - 1} \dots\dots\dots(56).$$

The use of these approximations will save the direct calculation of q_{x^2y} and q_{xy^2} provided we can determine the signs to be attached to $\sqrt{{}_x\eta_y^2 - r^2_{xy}}$ and $\sqrt{{}_y\eta_x^2 - r^2_{xy}}$. This is often easily done by inspection of the regression curve whose equation is

$$\frac{\bar{y}_x}{\sigma_y} = r \frac{x}{\sigma_x} \pm \sqrt{\frac{{}_x\eta_y^2 - r^2_{xy}}{\beta_2 - \beta_1 - 1}} \left\{ \frac{x^2}{\sigma_x^2} - \sqrt{\beta_1} \frac{x}{\sigma_x} - 1 \right\} \S.$$

We can approximate to $q_{x^2y^2}$ as follows||:

$$q_{x^2y^2} = \frac{SS(n_{xy}x^2y^2)}{N\sigma_x^2\sigma_y^2} = \frac{S\{n_x x^2(\sigma_{Y_x}^2 + (Y_x - \bar{y})^2)\}}{N\sigma_x^2\sigma_y^2},$$

where Y_x is the value given by the regression straight line and $\sigma_{Y_x}^2 \times n_x$ the second moment of the array of y 's for a given x , about the point Y_x .

$$\begin{aligned} \text{But} \quad \frac{S\{n_x x^2(Y_x - \bar{y})^2\}}{N\sigma_x^2\sigma_y^2} &= S(n_x x^4 r^2_{xy} \sigma_y^2) / \sigma_x^2 \\ &= \beta_2 r^2_{xy}. \end{aligned}$$

$$\text{Thus} \quad q_{x^2y^2} = \frac{S(n_x x^2 \sigma_{Y_x}^2)}{N\sigma_x^2\sigma_y^2} + \beta_2 r^2_{xy},$$

$$\text{Similarly} \quad q_{x^2y^2} = \frac{S(n_y y^2 \sigma_{X_y}^2)}{N\sigma_x^2\sigma_y^2} + \beta_2' r^2_{xy};$$

or, so far *without approximation*

$$q_{x^2y^2} = \frac{S(n_x x^2 \sigma_{Y_x}^2) + S(n_y y^2 \sigma_{X_y}^2)}{2N\sigma_x^2\sigma_y^2} + \frac{1}{2}(\beta_2 + \beta_2') r^2_{xy}.$$

* It is noteworthy that the hypothesis that regression of z on x, y although of 2nd degree is such that regression of z on x for a constant y is linear leads to the result that the total regression of z on x is *parabolic*.

† Pearson, *l.c.* p. 28, Eqn. (lxiii).

‡ Pearson, *l.c.* Eqns. (li), (xlv) and (lxiii).

§ Pearson, *l.c.* Eqn. (lxv).

|| It can be found fairly directly by tabling to the squares of the variates, when we need a simple product moment. In a later part of this paper some comparisons of actual and approximate values for numerical cases will be found.

Now the mean values of σ_{Yx}^2 and σ_{Xy}^2 are known to be

$$\sigma_y^2(1 - r_{xy}^2) \text{ and } \sigma_x^2(1 - r_{xy}^2),$$

and the deviations from these are usually somewhat irregular. Rarely can we do anything better than assume them to vary with a *slight* linear variation from the mean. For example

$$\sigma_{Yx}^2 = \sigma_y^2(1 - r_{xy}^2)(1 + \lambda x),$$

where λ is small. In such a case

$$\frac{S(n_x \bar{x}^2 \sigma_{Yx}^2)}{N \sigma_x^2 \sigma_y^2} = (1 - r_{xy}^2)(1 + \lambda \sqrt{\beta_1}),$$

or to a fair degree of approximation*, we may put

$$q_{x^2y^2} = 1 - r_{xy}^2 + \frac{1}{2}(\beta_2 + \beta_2') r_{xy}^2,$$

and thus write

$$\begin{aligned} q_{x^2y^2} - r_{xy}^2 &= 1 + \frac{1}{2}(\beta_2 + \beta_2' - 4) r_{xy}^2 \\ &= 1 + r_{xy}^2 + \frac{1}{2}(\beta_2 - 3 + \beta_2' - 3) r_{xy}^2. \end{aligned}$$

The latter part of this expression vanishes if the frequency of the x and y variates be mesokurtic. It can of course be retained if desired but its product with $\sqrt{{}_{xy}H_z^2 - {}_{xy}R_z^2}$ will usually be of the second order. If we write

$$\psi = \frac{1}{2}(\beta_2 - 3 + \beta_2' - 3),$$

we find the approximate regression surface

$$\begin{aligned} \frac{Z - \bar{z}}{\sigma_z} &= \frac{r_{xz} - r_{yz}r_{xy}}{1 - r_{xy}^2} \frac{(x - \bar{x})}{\sigma_x} + \frac{r_{yz} - r_{xz}r_{xy}}{1 - r_{xy}^2} \frac{(y - \bar{y})}{\sigma_y} \\ &+ \sqrt{\frac{{}_{xy}H_z^2 - {}_{xy}R_z^2}{1 + (1 + \psi)r_{xy}^2}} \left\{ \frac{(x - \bar{x})}{\sigma_x} \frac{(y - \bar{y})}{\sigma_y} - r_{xy} \right\} \dots\dots(57). \end{aligned}$$

This equation (57) enables us to express approximately the multiple ${}_{xy}H_z$ in terms of the simple ${}_y\eta_z$, ${}_x\eta_z$, ${}_x\eta_y$, ${}_y\eta_x$.

§ 7. To obtain this connection between the multiple ${}_{xy}H_z$ and the simple η 's we may proceed as follows:

$$\frac{\bar{z}_{xy}}{\sigma_z} = d + \frac{ax}{\sigma_x} + \frac{by}{\sigma_y} + \frac{cxy}{\sigma_x\sigma_y}, \text{ origin at mean.}$$

Hence keeping y constant and summing for the x 's

$$\frac{\bar{z}_y}{\sigma_z} = d + \frac{a\bar{x}_y}{\sigma_x} + \frac{by}{\sigma_y} + \frac{c\bar{x}_y y}{\sigma_x\sigma_y} \dots\dots\dots(58).$$

$$\text{But } {}_y\eta_z^2 = \frac{SSS \{n_{xyz} (\bar{z}_y - \bar{z})^2\}}{N\sigma_z^2} = \frac{SSS (n_{xyz} \bar{z}_y^2)}{N\sigma_z^2} \text{ if } \bar{z} = 0,$$

$$\text{and } {}_{xy}H_z^2 = \frac{SSS \{n_{xyz} (\bar{z}_{xy} - \bar{z})^2\}}{N\sigma_z^2} = \frac{SSS \{n_{xyz} \bar{z}_{xy}^2\}}{N\sigma_z^2},$$

* I.e. we are neglecting terms of the second order as $\lambda\sqrt{\beta_1}$.

$$\begin{aligned}
\therefore {}_{xy}H_z^2 - {}_y\eta_z^2 &= \frac{SSS}{x\ y\ z} \left[\frac{n_{xyz}}{N} \left(\frac{a(x - \bar{x}_y)}{\sigma_x} + \frac{cy(x - \bar{x}_y)}{\sigma_y \sigma_y} \right) \left(2d + a \frac{x + \bar{x}_y}{\sigma_x} + 2 \frac{by}{\sigma_y} \right. \right. \\
&\quad \left. \left. + \frac{cy}{\sigma_y} \frac{x + \bar{x}}{\sigma_x} \right) \right] \\
&= SS_{xy} \left\{ \frac{n_{xy}}{N} \left(\frac{x - \bar{x}_y}{\sigma_x} \right) \left(a + \frac{cy}{\sigma_y} \right) \left[2 \left(d + \frac{by}{\sigma_y} \right) + \left(a + \frac{cy}{\sigma_y} \right) \left(\frac{x + \bar{x}_y}{\sigma_x} \right) \right] \right\} \\
&= 2S_y \left\{ \left(a + \frac{cy}{\sigma_y} \right) \left(d + \frac{by}{\sigma_y} \right) S_x \frac{x - \bar{x}_y}{\sigma_x} \frac{n_{xy}}{N} \right\} + SS_{xy} \left\{ \frac{x^2 - \bar{x}_y^2}{\sigma_x^2} \left(a + \frac{cy}{\sigma_y} \right)^2 \frac{n_{xy}}{N} \right\} \\
&= 0 + S_y \left\{ \left(a + \frac{cy}{\sigma_y} \right)^2 \left(S_x \frac{n_{xy}}{N} \frac{x^2 - \bar{x}_y^2}{\sigma_x^2} \right) \right\}
\end{aligned}$$

Now $S_x \{n_{xy}x^2\} = S_x \{(x - \bar{x}_y + \bar{x}_y)^2 n_{xy}\} = n_y (\sigma_{xy}^2 + 0 + \bar{x}_y^2),$

$$\begin{aligned}
\therefore {}_{xy}H_z^2 - {}_y\eta_z^2 &= S_y \left\{ \left(a + \frac{cy}{\sigma_y} \right)^2 \frac{\sigma_{xy}^2}{\sigma_x^2} \frac{n_y}{N} \right\} \\
&= S_y \left\{ \left(a + \frac{cy}{\sigma_y} \right)^2 (1 - {}_y\eta_x^2) \frac{n_y}{N} \right\} \\
&= (1 - {}_y\eta_x^2) S_y \left\{ \frac{n_y}{N} \left(a^2 + \frac{2acy}{\sigma_y} + \frac{c^2 y^2}{\sigma_y^2} \right) \right\}, \\
\therefore {}_{xy}H_z^2 - {}_y\eta_z^2 &= (1 - {}_y\eta_x^2) (a^2 + c^2) \dots \dots \dots (59).
\end{aligned}$$

Similarly

$${}_{xy}H_z^2 - {}_x\eta_z^2 = (1 - {}_x\eta_y^2) (b^2 + c^2) \dots \dots \dots (60).$$

Remembering that $a = \gamma_s + c\theta$, $b = \gamma_s' + c\phi$ we get from (59) and (60)

$$\begin{aligned}
\gamma_s' \phi \left(\frac{{}_{xy}H_z^2 - {}_y\eta_z^2}{1 - {}_y\eta_x^2} \right) - \gamma_s \theta \left(\frac{{}_{xy}H_z^2 - {}_x\eta_z^2}{1 - {}_x\eta_y^2} \right) &= \gamma_s \gamma_s' (\gamma_s \phi - \gamma_s' \theta) \\
&\quad + c^2 \{ \gamma_s' \phi - \gamma_s \theta - \theta \phi (\gamma_s \phi - \gamma_s' \theta) \} \dots \dots (61).
\end{aligned}$$

From the values of γ_s , γ_s' , θ , ϕ in (9), (10), (41), (42) we obtain easily

$$\gamma_s \phi - \gamma_s' \theta = \frac{r_{yz} q_{x^2 y} - r_{xz} q_{xy^2}}{1 - r_{xy}^2},$$

$$\gamma_s' \phi - r_s \theta = \frac{r_{xz} q_{x^2 y} - r_{yz} q_{xy^2}}{1 - r_{xy}^2},$$

$$\gamma_s' \phi - \gamma_s \theta - \theta \phi (\gamma_s \phi - \gamma_s' \theta)$$

$$= \frac{r_{xz} q_{x^2 y} - r_{yz} q_{xy^2}}{1 - r_{xy}^2} - \frac{(r_{xy} q_{xy^2} - q_{x^2 y})(r_{xy} q_{x^2 y} - q_{xy^2})(r_{yz} q_{x^2 y} - r_{xz} q_{xy^2})}{(1 - r_{xy}^2)^3}$$

and c^2 is given by (52). Hence (61) may be written

$$\begin{aligned}
({}_{xy}H_z^2 - {}_{xy}R_z^2) &\left\{ \frac{(r_{yz} - r_{xz} r_{xy})(r_{xy} q_{xy^2} - q_{x^2 y})}{(1 - r_{xy}^2)^2 (1 - {}_y\eta_x^2)} - \frac{(r_{xz} - r_{yz} r_{xy})(r_{xy} q_{xy^2} - q_{x^2 y})}{(1 - r_{xy}^2)^2 (1 - {}_x\eta_y^2)} \right. \\
&\quad - \frac{(r_{xz} q_{x^2 y} - r_{yz} q_{xy^2})}{1 - r_{xy}^2} - \frac{(r_{xy} q_{xy^2} - q_{x^2 y})(r_{xy} q_{x^2 y} - q_{xy^2})(r_{yz} q_{x^2 y} - r_{xz} q_{xy^2})}{(1 - r_{xy}^2)^3} \\
&\quad \left. - \frac{(q_{x^2 y}^2 - r_{xy}^2)}{(q_{x^2 y}^2 - r_{xy}^2) - \frac{(q_{x^2 y}^2 + q_{xy^2}^2 - 2q_{xy^2} q_{x^2 y} r_{xy})}{1 - r_{xy}^2}} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2})y\eta_z^2}{(1 - r_{xy}^2)^2(1 - y\eta_x^2)} - \frac{(r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})x\eta_z^2}{(1 - r_{xy}^2)^2(1 - y\eta_x^2)} \\
&\quad + \frac{(r_{xz} - r_{yz}r_{xy})(r_{yz} - r_{xz}r_{xy})(r_{yz}q_{x^2y} - r_{xz}q_{xy^2})}{(1 - r_{xy}^2)^3} \\
&- {}_{xy}R_z^2 \left\{ \frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2})}{(1 - r_{xy}^2)^2(1 - y\eta_x^2)} - \frac{(r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})}{(1 - r_{xy}^2)^2(1 - x\eta_y^2)} \right\} \dots\dots(62).
\end{aligned}$$

In general the square of any correlation ratio (ordinary or generalised) differs little from the square of the corresponding correlation coefficient; also we have seen in (55) that an approximate value for q_{x^2y} is

$$r_{xy}\sqrt{\beta_1} - \sqrt{x\eta_y^2 - r_{xy}^2}\sqrt{\beta_2 - \beta_1 - 1}.$$

Now β_1 is itself in general small, so that without making any assumption as to the relative order of magnitude of β_1 and $x\eta_y^2 - r_{xy}^2$ we may safely treat q_{x^2y} and q_{xy^2} as small quantities.

Thus it appears that (62) is an equation in which all the terms involved are small, and a certain amount of care is required in deducing from it an approximation to the value of ${}_{xy}H_z^2 - {}_{xy}R_z^2$.

We have

$$\begin{aligned}
\frac{1}{1 - y\eta_x^2} &\equiv \frac{1}{1 - r_{xy}^2} + \frac{y\eta_x^2 - r_{xy}^2}{(1 - y\eta_x^2)(1 - r_{xy}^2)} = \frac{1}{1 - r_{xy}^2} + \xi_1 \text{ say} \\
\frac{1}{1 - x\eta_y^2} &\equiv \frac{1}{1 - r_{xy}^2} + \frac{x\eta_y^2 - r_{xy}^2}{(1 - x\eta_y^2)(1 - r_{xy}^2)} = \frac{1}{1 - r_{xy}^2} + \xi_1' \\
\frac{y\eta_z^2}{1 - y\eta_x^2} &= r_{zy}^2 \left(\frac{1}{1 - r_{xy}^2} + \xi_1 \right) + (y\eta_z^2 - r_{zy}^2) \left(\frac{1}{1 - r_{xy}^2} + \xi_1 \right) \\
&= \frac{r_{zy}^2}{1 - r_{xy}^2} + \lambda_1 + \lambda_2 \dots\dots\dots(64),
\end{aligned}$$

$$\text{where } \lambda_1 = \frac{r_{zy}^2(y\eta_x^2 - r_{xy}^2)}{(1 - y\eta_x^2)(1 - r_{xy}^2)} + \frac{y\eta_z^2 - r_{zy}^2}{1 - r_{xy}^2} \text{ and } \lambda_2 = \frac{(y\eta_z^2 - r_{zy}^2)(y\eta_x^2 - r_{xy}^2)}{(1 - y\eta_x^2)(1 - r_{xy}^2)},$$

$$\text{while } \frac{x\eta_z^2}{1 - x\eta_y^2} = \frac{r_{zx}^2}{1 - r_{xy}^2} + \lambda_1' + \lambda_2' \dots\dots\dots(64)',$$

where λ_1', λ_2' may be obtained from λ_1, λ_2 by an interchange of x and y in the suffixes.

$$\text{Finally } \frac{1}{q_{x^2y^2} - r_{xy}^2 - \frac{q_{x^2y}^2 + q_{xy^2}^2 - 2q_{xy^2}q_{x^2y}r_{xy}}{1 - r_{xy}^2}} \equiv \frac{1}{q_{x^2y^2} - r_{xy}^2} + \kappa_2 \dots\dots\dots(65),$$

where

$$\kappa_2 = \frac{q_{x\eta_y^2}^2 + q_{x^2y}^2 - 2q_{x\eta_y^2}q_{x^2y}r_{xy}}{(1 - r_{xy}^2)(q_{x^2y^2} - r_{xy}^2) \left(q_{x^2y^2} - r_{xy}^2 - \frac{q_{x^2y}^2 + q_{xy^2}^2 - 2q_{xy^2}q_{x^2y}r_{xy}}{1 - r_{xy}^2} \right)}.$$

The suffixes of $\xi_1, \xi_1', \lambda_1, \lambda_2, \lambda_1', \lambda_2'$ and κ_2 denote the order of smallness of these terms.

We make the corresponding substitutions in equation (62), including the value $\frac{r_{yz}^2 + r_{zx}^2 - 2r_{zx}r_{yz}r_{xy}}{1 - r_{xy}^2}$ for xyR_z^2 on the right-hand side and obtain the following accurate formula for $xyH_z^2 - xyR_z^2$:

$$\begin{aligned}
 (xyH_z^2 - xyR_z^2) & \left[\frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2})}{(1 - r_{xy}^2)^2} \left(\frac{1}{1 - r_{xy}^2} + \xi_1 \right) \right. \\
 & \quad \left. - \frac{(r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})}{(1 - r_{xy}^2)^2} \left(\frac{1}{1 - r_{xy}^2} + \xi_1' \right) \right. \\
 & \quad \left. - \left\{ \frac{r_{xz}q_{x^2y} - r_{yz}q_{xy^2}}{1 - r_{xy}^2} - \frac{(r_{yz}q_{x^2y} - r_{xz}q_{xy^2})(r_{xy}q_{xy^2} - q_{x^2y})(r_{xy}q_{x^2y} - q_{xy^2})}{(1 - r_{xy}^2)^3} \right\} \right. \\
 & \quad \left. \times \left(\frac{1}{q_{x^2y^2} - r_{xy}^2} + \kappa_2 \right) \right] \\
 & = \frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2})}{(1 - r_{xy}^2)^2} \left(\frac{r_{zy}^2}{1 - r_{xy}^2} + \lambda_1 + \lambda_2 \right) \\
 & \quad - \frac{(r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})}{(1 - r_{xy}^2)^2} \left(\frac{r_{zz}^2}{1 - r_{xy}^2} + \lambda_1' + \lambda_2' \right) \\
 & \quad + \frac{(r_{yz}q_{x^2y} - r_{xz}q_{xy^2})(r_{xz} - r_{yz}r_{xy})(r_{yz} - r_{xz}r_{xy})}{(1 - r_{xy}^2)^3} \\
 & \quad - \frac{r_{yz}^2 + r_{zx}^2 - 2r_{zx}r_{yz}r_{xy}}{(1 - r_{xy}^2)} \left[\frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2})}{(1 - r_{xy}^2)^2} \left(\frac{1}{1 - r_{xy}^2} + \xi_1 \right) \right. \\
 & \quad \left. - \frac{(r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})}{(1 - r_{xy}^2)^2} \left(\frac{1}{1 - r_{xy}^2} + \xi_1' \right) \right] \dots (66).
 \end{aligned}$$

The right-hand side of (66) apparently contains terms of the first order, while on the left the lowest order occurring is the second. But the coefficient of q_{x^2y} in the first order terms is

$$\begin{aligned}
 & \frac{1}{(1 - r_{xy}^2)^3} \{ (r_{yz} - r_{xz}r_{xy})r_{xy}r_{zy}^2 + (r_{xz} - r_{yz}r_{xy})r_{zy}^2 + r_{yz}(r_{xz} - r_{yz}r_{xy})(r_{yz} - r_{xz}r_{xy}) \} \\
 & - \frac{1}{(1 - r_{xy}^2)^4} \{ (r_{yz} - r_{xz}r_{xy})r_{xy} + (r_{xz} - r_{yz}r_{xy}) \} (r_{yz}^2 + r_{zx}^2 - 2r_{yz}r_{zx}r_{xy}),
 \end{aligned}$$

and vanishes identically. Similarly the coefficient of q_{xy^2} is zero and thus equation (66) is a relation between terms of second and higher orders.

A first approximation then may be obtained by equating second order terms. This gives

$$\begin{aligned}
 (xyH_z^2 - xyR_z^2) & \left[\frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2}) - (r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})}{(1 - r_{xy}^2)^3} \right. \\
 & \quad \left. - \frac{r_{xz}q_{x^2y} - r_{yz}q_{xy^2}}{(1 - r_{xy}^2)(q_{x^2y^2} - r_{xy}^2)} \right] \\
 & = \frac{(r_{yz} - r_{xz}r_{xy})(r_{xy}q_{x^2y} - q_{xy^2})}{(1 - r_{xy}^2)^2} \left[\lambda_1 - \frac{r_{yz}^2 + r_{zx}^2 - 2r_{zx}r_{yz}r_{xy}}{1 - r_{xy}^2} \xi_1 \right] \\
 & \quad - \frac{(r_{xz} - r_{yz}r_{xy})(r_{xy}q_{xy^2} - q_{x^2y})}{(1 - r_{xy}^2)^2} \left[\lambda_1' - \frac{r_{yz}^2 + r_{zx}^2 - 2r_{zx}r_{yz}r_{xy}}{1 - r_{xy}^2} \xi_1' \right] \dots (67).
 \end{aligned}$$

The coefficient of $_{xy}H_z^2 - _{xy}R_z^2$ on the left reduces to

$$\frac{1}{(1 - r_{xy}^2)^2} [q_{x^2y}r_{xz} - q_{xy^2}r_{yz}] \left(1 - \frac{1 - r_{xy}^2}{q_{x^2y^2} - r_{xy}^2}\right),$$

$$\lambda_1 - \frac{r_{yz}^2 + r_{xz}^2 - 2r_{xz}r_{yz}r_{xy}}{1 - r_{xy}^2} \xi_1 = \frac{y\eta_z^2 - r_{zy}^2}{1 - r_{xy}^2} - \xi_1 \frac{(r_{yz}r_{xy} - r_{xz})^2}{1 - r_{xy}^2}$$

$$= \frac{y\eta_z^2 - r_{zy}^2}{1 - r_{xy}^2} - \frac{(y\eta_x^2 - r_{xy}^2)}{1 - y\eta_x^2} \left(\frac{r_{yz}r_{xy} - r_{xz}}{1 - r_{xy}^2}\right)^2 \dots\dots(68).$$

Similarly

$$\lambda_1' - \frac{r_{yz}^2 + r_{xz}^2 - 2r_{xz}r_{yz}r_{xy}}{1 - r_{xy}^2} \xi_1' = \frac{x\eta_z^2 - r_{zx}^2}{1 - r_{xy}^2} - \frac{x\eta_y^2 - r_{xy}^2}{1 - x\eta_y^2} \left(\frac{r_{xz}r_{xy} - r_{yz}}{1 - r_{xy}^2}\right)^2 \dots(69).$$

We shall still be correct to second order terms if when using (68) and (69) in (67) we replace $1 - y\eta_x^2$ and $1 - x\eta_y^2$ occurring in denominators by $1 - r_{xy}^2$; so that

$$(_{xy}H_z^2 - _{xy}R_z^2) \frac{q_{x^2y^2} - 1}{q_{x^2y^2} - r_{xy}^2}$$

$$= \frac{r_{xy}q_{x^2y} - q_{xy^2}}{q_{x^2y}r_{xz} - q_{xy^2}r_{yz}} \cdot \frac{r_{yz} - r_{xz}r_{xy}}{1 - r_{xy}^2} \left[(y\eta_z^2 - r_{zy}^2) - \left(\frac{r_{yz}r_{xy} - r_{xz}}{1 - r_{xy}^2}\right)^2 (y\eta_x^2 - r_{xy}^2) \right]$$

$$+ \frac{r_{xy}q_{xy^2} - q_{x^2y}}{q_{xy^2}r_{yz} - q_{x^2y}r_{xz}} \cdot \frac{r_{xz} - r_{yz}r_{xy}}{1 - r_{xy}^2} \left[(x\eta_z^2 - r_{zx}^2) - \left(\frac{r_{xz}r_{xy} - r_{yz}}{1 - r_{xy}^2}\right)^2 (x\eta_y^2 - r_{xy}^2) \right]$$

$$\dots\dots(70).$$

§ 8. This result is of importance, as it shows that the heavy labour of the direct calculation of the generalised correlation ratio can be replaced by the calculation of four simple correlation ratios.

The coefficients involved are the ordinary coefficients of linear regression denoted above by γ_s , γ_s' and expressions involving product moments of orders 3 and 4. To these latter we may approximate by the methods of § 6.

A good approximation for $\frac{q_{x^2y^2} - 1}{q_{x^2y^2} - r_{xy}^2}$ is $\frac{2r_{xy}^2}{1 + r_{xy}^2}$. If greater accuracy be needed we may use

$$\frac{q_{x^2y^2} - 1}{q_{x^2y^2} - r_{xy}^2} = \frac{(\frac{1}{2}(\beta_2 + \beta_2' - 6) + 2)r_{xy}^2}{1 + r_{xy}^2 + \frac{1}{2}(\beta_2 + \beta_2' - 6)r_{xy}^2}$$

We saw in § 6 (equation 55) that

$$q_{x^2y} = r_{xy} \sqrt{\beta_1} + \sqrt{x\eta_y^2 - r_{xy}^2} \sqrt{\beta_2 - \beta_1 - 1},$$

$$q_{xy^2} = r_{xy} \sqrt{\beta_1'} + \sqrt{y\eta_x^2 - r_{xy}^2} \sqrt{\beta_2' - \beta_1' - 1},$$

approximately: β_1 and β_1' , which are zero in normal correlation, will in general be very small compared with $x\eta_y^2 - r_{xy}^2$ and $y\eta_x^2 - r_{xy}^2$ so that

$$\frac{r_{xy}q_{x^2y} - q_{xy^2}}{q_{x^2y}r_{xz} - q_{xy^2}r_{yz}} = \frac{r_{xy} \sqrt{(\beta_2 - 1)(x\eta_y^2 - r_{xy}^2)} - \sqrt{(\beta_2' - 1)(y\eta_x^2 - r_{xy}^2)}}{r_{xz} \sqrt{(\beta_2 - 1)(x\eta_y^2 - r_{xy}^2)} - r_{yz} \sqrt{(\beta_2' - 1)(y\eta_x^2 - r_{xy}^2)}},$$

and

$$\frac{r_{xy}q_{xy^2} - q_{x^2y}}{q_{xy^2}r_{yz} - q_{x^2y}r_{xz}} = \frac{r_{xy} \sqrt{(\beta_2' - 1)(y\eta_x^2 - r_{xy}^2)} - \sqrt{(\beta_2 - 1)(x\eta_y^2 - r_{xy}^2)}}{r_{yz} \sqrt{(\beta_2' - 1)(y\eta_x^2 - r_{xy}^2)} - \sqrt{(\beta_2 - 1)(x\eta_y^2 - r_{xy}^2)}}.$$

In the important case $\beta_1 = \beta'_1 = 0$ and ${}_y\eta_x = {}_x\eta_y = r_{xy}$ these approximations fail, and so does the process by which (70) was obtained as θ and ϕ vanish and (61) is indeterminate.

We must then fall back on equation (59)

$$\begin{aligned} {}_{xy}H_z^2 - {}_y\eta_z^2 &= (a^2 + c^2)(1 - {}_y\eta_x^2) \\ &= (\gamma_3^2 + c^2)(1 - r_{xy}^2) \text{ since } \theta = 0 \text{ and } {}_y\eta_x = r_{xy} \\ &= \gamma_3^2(1 - r_{xy}^2) + \frac{({}_{xy}H_z^2 - {}_{xy}R_z^2)(1 - r_{xy}^2)}{q_{x^2y^2} - r_{xy}^2}(1 + \kappa_2), \end{aligned}$$

$$\text{or } ({}_{xy}H_z^2 - {}_{xy}R_z^2) \left(1 - \frac{1 - r_{xy}^2}{q_{x^2y^2} - r_{xy}^2} \right) = {}_y\eta_z^2 - {}_{xy}R_z^2 + \gamma_3^2(1 - r_{xy}^2),$$

neglecting 3rd order terms.

The right-hand side reduces to ${}_y\eta_z^2 - r_{yz}^2$, so that

$$\begin{aligned} {}_{xy}H_z^2 - {}_{xy}R_z^2 &= \left(\frac{q_{x^2y^2} - r_{xy}^2}{q_{x^2y^2} - 1} \right) ({}_y\eta_z^2 - r_{yz}^2) \dots\dots\dots(71) \\ &= \frac{2r_{xy}^2}{1 + r_{xy}^2} ({}_y\eta_z^2 - r_{yz}^2) \text{ approximately; } \end{aligned}$$

of course in these circumstances (60) would lead to the value

$$\frac{2r_{xy}^2}{1 + r_{xy}^2} ({}_x\eta_z^2 - r_{zx}^2) \dots\dots\dots(72),$$

showing that if $\beta_1 = \beta'_1 = 0$ and if $({}_x\eta_y^2 - r_{yx}^2) - ({}_y\eta_x^2 - r_{xy}^2)$ is of higher order than the first then $({}_x\eta_z^2 - r_{zx}^2) - ({}_y\eta_z^2 - r_{zy}^2)$ is also of higher order than the first.

We shall now seek relations between the six correlation ratios of three "hyperbolic" variates.

From (59) and (60) we get, on eliminating ${}_{xy}H_z^2$,

$$\begin{aligned} {}_y\eta_z^2 - {}_x\eta_z^2 &= b^2 - a^2 + c^2({}_y\eta_x^2 - {}_x\eta_y^2)^2 + a^2{}_y\eta_x^2 - b^2{}_x\eta_y^2 \\ &= \gamma_3'^2 - \gamma_3^2 + \gamma_3^2{}_y\eta_x^2 - \gamma_3'^2{}_x\eta_y^2 + 2(\gamma_3\theta{}_y\eta_x^2 - \gamma_3'\phi{}_x\eta_y^2) \\ &\quad + c^2(\theta^2 - \phi^2 + {}_y\eta_x^2 - {}_x\eta_y^2 + \theta^2{}_y\eta_x^2 - \phi^2{}_x\eta_y^2) \\ &= (\gamma_3'^2 - \gamma_3^2)(1 - r_{xy}^2) + \gamma_3^2({}_y\eta_x^2 - r_{xy}^2) - \gamma_3'^2({}_x\eta_y^2 - r_{xy}^2) \\ &\quad - 2c(\gamma\theta - \gamma'\phi)(1 - r_{xy}^2) \\ &\quad + \{2c\gamma_3\theta({}_y\eta_x^2 - r_{xy}^2) - 2c\gamma_3'\phi({}_x\eta_y^2 - r_{xy}^2) \\ &\quad + c^2(\theta^2 - \phi^2 + {}_y\eta_x^2 - {}_x\eta_y^2 + \theta^2{}_y\eta_x^2 - \phi^2{}_x\eta_y^2)\}. \end{aligned}$$

The terms in the second line are second order terms. Neglecting these and noting that

$$(\gamma_3'^2 - \gamma_3^2)(1 - r_{xy}^2) \equiv r_{yz}^2 - r_{zx}^2 \text{ and } (\gamma_3\theta - \gamma_3'\phi)(1 - r_{xy}^2) \equiv r_{yz}q_{xy^2} - r_{xz}q_{x^2y},$$

we obtain the following equation for c :

$$\begin{aligned} ({}_y\eta_z^2 - r_{zy}^2) - ({}_x\eta_z^2 - r_{zx}^2) - \gamma_3^2({}_y\eta_x^2 - r_{xy}^2) + \gamma_3'^2({}_x\eta_y^2 - r_{xy}^2) \\ = 2c(r_{xz}q_{x^2y} - r_{yz}q_{xy^2}) \dots\dots\dots(73). \end{aligned}$$

Now $(x_y H_z^2 - x_y R_z^2) \frac{q_{x^2 y^2} - 1}{q_{x^2 y^2} - r_{xy}^2} = 2c^2 \gamma_{xy}^2$ if we neglect second order terms. If we use the value of c given by (73) in (70), and adopt the notation ${}_x U_z$ for ${}_x \eta_z^2 - r_{xz}^2$, we have

$$\begin{aligned} & r_{xy}^2 [{}_y U_z - {}_x U_z - \gamma_3^2 {}_y U_x + \gamma_3'^2 {}_x U_y]^2 \\ &= 2(r_{xz} q_{x^2 y} - r_{yz} q_{xy^2}) \{ (r_{xy} q_{x^2 y} - q_{xy^2}) \gamma_3' ({}_y U_z - \gamma_3^2 {}_y U_x) \\ &\quad - (r_{xy} q_{xy^2} - q_{x^2 y}) \gamma_3 ({}_x U_z - \gamma_3'^2 {}_x U_y) \} \dots\dots (74). \end{aligned}$$

Let us write ${}_x \chi_z$ for ${}_x U_z - \gamma_3'^2 {}_x U_y$ and ${}_y \chi_z$ for ${}_y U_z - \gamma_3^2 {}_y U_x$, then (74) becomes

$$({}_x \chi_z - {}_y \chi_z)^2 r_{xy}^2 = 2(r_{xz} q_{x^2 y} - r_{yz} q_{xy^2}) \{ \gamma_3' (r_{xy} q_{x^2 y} - q_{xy^2}) {}_y \chi_z - \gamma_3 (r_{xy} q_{xy^2} - q_{x^2 y}) {}_x \chi_z \} \dots\dots (75)$$

(75) is a relation between second order terms and it is sufficient to use equation (55) for $q_{x^2 y}$ with β_1 replaced by zero and β_2 by 3, so that

$$\begin{aligned} q_{x^2 y} &= \sqrt{2 {}_x U_y}, \\ q_{xy^2} &= \sqrt{2 {}_y U_x}, \\ \therefore ({}_x \chi_z - {}_y \chi_z)^2 r_{xy}^2 &= 4(r_{xz} \sqrt{{}_x U_y} - r_{yz} \sqrt{{}_y U_x}) \{ \gamma_3' {}_y \chi_z (\sqrt{{}_x U_y} - \sqrt{{}_y U_x}) \\ &\quad - \gamma_3 {}_x \chi_z (\sqrt{{}_y U_x} - \sqrt{{}_x U_y}) \} \dots\dots (76). \end{aligned}$$

This identity between ${}_x \eta_z$, ${}_y \eta_z$, ${}_x \eta_y$ and ${}_y \eta_x$ is symmetrical in x, y . Two more such identities may be obtained by interchanging the letters x, y, z in cyclic order*. There are therefore three identities between the six correlation ratios:

$${}_y \eta_x, \quad {}_x \eta_y, \quad {}_y \eta_z, \quad {}_z \eta_y, \quad {}_x \eta_z, \quad {}_z \eta_x.$$

I have not so far succeeded in reducing them to simpler forms, although possibly such exist. In special cases simplifications result. These are illustrated in the following section.

§ 9. We defined γ_3, γ_3' the regression coefficients of z on x, y by the equations

$$\gamma_3 = \frac{r_{xz} - r_{yz} r_{xy}}{1 - r_{xy}^2} \dots\dots\dots (9), \quad \gamma_3' = \frac{r_{yz} - r_{xz} r_{xy}}{1 - r_{xy}^2} \dots\dots\dots (10).$$

Let us now introduce the remaining regression coefficients

$$\left. \begin{aligned} \gamma_1 &= \frac{r_{yx} - r_{zx} r_{yz}}{1 - r_{yz}^2}, & \gamma_1' &= \frac{r_{zx} - r_{yx} r_{yz}}{1 - r_{yz}^2} \\ \gamma_2 &= \frac{r_{yz} - r_{xz} r_{xy}}{1 - r_{xz}^2}, & \gamma_2' &= \frac{r_{xy} - r_{yz} r_{zx}}{1 - r_{xz}^2} \end{aligned} \right\} \dots\dots\dots (77).$$

It will simplify the algebra if we use $\lambda^2, \mu^2, \nu^2, \lambda'^2, \mu'^2, \nu'^2$ for ${}_y U_x, {}_z U_y, {}_x U_z, {}_z U_x, {}_x U_y, {}_y U_z$ respectively and P, Q, R, P', Q', R' for ${}_y \chi_x, {}_z \chi_y, {}_x \chi_z, {}_z \chi_x, {}_x \chi_y, {}_y \chi_z$ respectively so that

$$\left. \begin{aligned} P &= \lambda^2 - \gamma_1'^2 \nu'^2, & Q &= \mu^2 - \gamma_2'^2 \lambda'^2, & R &= \nu^2 - \gamma_3'^2 \mu'^2 \\ P' &= \lambda'^2 - \gamma_1^2 \mu^2, & Q' &= \mu'^2 - \gamma_2^2 \nu^2, & R' &= \nu'^2 - \gamma_3^2 \lambda^2 \end{aligned} \right\} \dots\dots\dots (78).$$

* We are supposing here that the regression surfaces of x on y, z and of y on z, x are also hyperboloids of type similar to (35).

The three identities connecting the six simple η 's become in this notation

$$(R - R')^2 r_{xy}^2 = 4 (r_{xz} \mu' - r_{yz} \lambda) \{ \gamma_3' R' (r_{xy} \mu' - \lambda) - \gamma_3 R (r_{xy} \lambda - \mu') \} \dots (79),$$

$$(P - P')^2 r_{yz}^2 = 4 (r_{yx} \nu' - r_{zx} \mu) \{ \gamma_1' P' (r_{yz} \nu' - \mu) - \gamma_1 P (r_{yz} \mu - \nu') \} \dots (80),$$

$$(Q - Q')^2 r_{zx}^2 = 4 (r_{zx} \lambda' - r_{xy} \nu) \{ \gamma_2' Q' (r_{zx} \lambda' - \nu) - \gamma_2 Q (r_{zx} \nu - \lambda') \} \dots (81).$$

(i) We can deduce from (79) that if the correlations of both x and y on z be linear and equal, then the correlation ratio of x on y and y on x , i.e. ${}_x\eta_y$ and ${}_y\eta_x$ are equal. Thus in biparental correlation, if the regression of the child on each parent be linear, then the correlation ratio of the father on the mother is equal to that of the mother on the father.

For under the conditions stated

$${}_x\eta_z = {}_y\eta_z = r_{zx} = r_{zy}.$$

Hence $\gamma_3 = \gamma_3' = \frac{r_{zx}}{1 + r_{xy}}$ and (79) becomes

$$\gamma_3^4 [\lambda^2 - \mu'^2]^2 r_{xy}^2 = 4 r_{yz} (\mu' - \lambda) \{ -\gamma_3^3 \lambda^2 (r_{xy} \mu' - \lambda) + \gamma_3^3 \mu'^2 (r_{xy} \lambda - \mu') \},$$

or
$$\gamma_3 (\lambda - \mu')^2 (\lambda + \mu')^2 r_{xy}^2 = 4 r_{yz} (\lambda - \mu')^2 \{ r_{xy} \lambda \mu' - \lambda^2 - \lambda \mu' - \mu'^2 \},$$

or
$$(\lambda - \mu')^2 \left[\frac{r_{xy}^2}{1 + r_{xy}} (\lambda + \mu')^2 - 4 (r_{xy} \lambda \mu' - \lambda^2 - \lambda \mu' - \mu'^2) \right] = 0,$$

which reduces to

$$(\lambda - \mu')^2 \left\{ \frac{[(r_{xy} + 2)^2 \lambda + (2 - r_{xy}^2) \mu']^2 + 4 \mu'^2 (2 r_{xy} + 3) (r_{xy} + 1)^2}{(r_{xy} + 1) (r_{xy} + 2)^2} \right\} = 0.$$

But r_{xy} is numerically < 1 . $\therefore$ the factor in curved brackets is positive. Hence $\lambda = \mu'$, i.e. ${}_yU_x = {}_xU_y$. $\therefore {}_y\eta_x = {}_x\eta_y$.

(ii) An interesting deduction from the identities (79)–(81) is the following: "If any four of the six regression lines that occur in the mutual variation of three variables are linear, so are the other two."

We have to prove that if any four of the six quantities $\lambda, \mu, \nu, \lambda', \mu', \nu'$ vanish, then the remaining two vanish as well.

First let $\lambda = \mu = \nu = \lambda' = 0$.

(79) gives
$$r_{xy}^2 (\gamma_3' \mu'^2 + \nu'^2)^2 = 4 r_{xz} \mu' \{ \gamma_3' \nu'^2 r_{xy} \mu' - \gamma_3 \gamma_3' \mu'^3 \},$$

(80)
$$\gamma_1'^4 \nu'^4 r_{yz}^2 = -4 r_{xy} \nu'^4 \gamma_1' \gamma_1'^2,$$

(81)
$$\mu'^4 r_{xz}^2 = 0.$$

From (80)
$$\gamma_1'^2 \nu'^4 [4 \gamma_1' r_{yx} + \gamma_1'^2 r_{yz}^2] = 0.$$

But
$$4 \gamma_1' r_{xy} + \gamma_1'^2 r_{yz}^2 \equiv \left(\frac{r_{yz} r_{xz} + r_{yz}^2 r_{xy} - 2 r_{xy}}{1 - r_{yz}} \right)^2 > 0.$$

$$\therefore \nu' = 0,$$

and

$$\mu' = 0,$$

and these satisfy (79).

There are three cases of this type.

The case $\lambda' = \mu' = \nu' = \lambda = 0$ is proved in the same way. There are three cases of this type as well.

Next take the three cases of type

$$\lambda = \mu = 0,$$

$$\lambda' = \mu' = 0.$$

Equations (79)–(81) become

$$(79) \quad R = R', \text{ i.e. } \nu^2 = \nu'^2,$$

$$(80) \quad \gamma_1'^4 \nu'^4 r_{yz}^2 = 4r_{xy} \nu' \{ \gamma_1 \gamma_1'^2 \nu'^2 (-\nu') \},$$

$$(81) \quad \gamma_2^4 \nu^4 r_{xz}^2 = 4(-r_{xy} \nu) \{ -\gamma_2' \gamma_2^2 \nu^2 (-\nu) \},$$

$$(80) \text{ leads to } (\gamma_1'^2 r_{yz}^2 + 4r_{xy} \gamma_1) \gamma_1'^2 \nu'^4 = 0.$$

We have already shown that the first factor is positive,

$$\therefore \nu' = 0,$$

and hence

$$\nu = 0,$$

and these values satisfy (81).

The three cases of type $\lambda = \mu = \mu' = \nu' = 0$ lead to

$$(79) \quad \nu^4 r_{xy}^2 = 0,$$

$$(80) \quad \lambda'^4 r_{yz}^2 = 0,$$

$$(81) \quad (\gamma_2^2 \nu^2 - \gamma_2'^2 \lambda'^2) r_{yz}^2 = 4(r_{yz} \lambda' - r_{xy} \nu) \{ -\gamma_2' \gamma_2^2 \nu^2 (r_{xz} \lambda' - \nu) + \gamma_2 \gamma_2'^2 \lambda'^2 (r_{xz} \nu - \lambda') \},$$

whence $\lambda' = \nu = 0$.

There remain the three cases of type

$$\lambda = \nu = 0,$$

$$\mu' = \nu' = 0.$$

Here (79) is satisfied identically.

$$(80) \text{ becomes } (\lambda'^2 - \gamma_1^2 \mu^2)^2 r_{yz}^2 = -4r_{yz} \mu \{ \gamma_1' (\lambda'^2 - \gamma_1^2 \mu^2) (-\mu) \},$$

$$(81) \quad (\mu^2 - \gamma_2'^2 \lambda'^2)^2 r_{xz}^2 = 4r_{yz} \lambda' \{ -\gamma_2 (\mu^2 - \gamma_2'^2 \lambda'^2) (-\lambda') \},$$

which reduce to

$$(\lambda'^2 - \gamma_1^2 \mu^2) \{ r_{yz}^2 \lambda'^2 - (\gamma_1^2 r_{yz}^2 + 4\gamma_1' r_{xz}) \mu^2 \} = 0,$$

$$(\mu^2 - \gamma_2'^2 \lambda'^2) \{ r_{xz}^2 \mu^2 - (\gamma_2'^2 r_{xz}^2 + 4\gamma_2 r_{yz}) \lambda'^2 \} = 0.$$

The only common solution of these equations is

$$\lambda' = \mu = 0.$$

We have thus accounted for all the fifteen possible cases.

(iii) *Three regression curves linear.*

In six cases out of the possible 20 cases the linearity of three only of the regression curves involves the linearity of the remaining three.

Let $\lambda = \mu' = 0$ and either ν or $\nu' = 0$.

It follows from (79) that $R = R'$, i.e. $\nu = \nu'$.

$\therefore$ both ν and ν' are zero. We have now four linear regression curves, $\therefore$ all six are linear.

Let $\mu = \nu' = 0$ and $\mu' = 0$.

Since $\mu = \nu' = 0$, $\therefore P = P'$ or $\lambda = \lambda'$, so that

$$\begin{aligned} P &= \lambda^2, & Q &= -\gamma_2'^2 \lambda^2, & R &= \nu^2, \\ P' &= \lambda^2, & Q' &= \mu'^2 - \gamma_2'^2 \nu^2, & R' &= -\gamma_3'^2 \lambda^2. \end{aligned}$$

$$(79) \text{ becomes } (\nu^2 + \gamma_3'^2 \lambda^2)^2 r_{xy}^2 = 4r_{yz}\gamma_3\lambda^2 (r_{xy}\nu^2 - \gamma_3\gamma_3'\lambda^2),$$

$$(80) \text{ becomes}$$

$$(\gamma_2'^2 \nu^2 - \gamma_2'^2 \lambda^2)^2 r_{yz}^2 = 4(r_{yz}\lambda - r_{xy}\nu)\gamma_2\gamma_2'\nu^2 [(\gamma_2\nu - \gamma_2'\lambda) + r_{xz}(\gamma_2'\nu - \gamma_2\lambda)].$$

The first reduces to

$$\left\{ \frac{\lambda^2 \gamma_3 (r_{xz} + r_{yz} r_{xy}) r_{xy} - 2r_{yz}}{1 - r_{xy}^2} + r_{xy} \nu^2 \right\}^2 = 0,$$

and the second to

$$(\gamma_2\nu^2 - \gamma_2'\lambda^2)^2 r_{xz}^2 + 4\gamma_2\gamma_2' (r_{yz}\lambda - r_{xy}\nu)^2 \nu^2 = 0.$$

Hence either $\lambda = \nu = 0$ or there must be a very special relation between r_{xy} , r_{yz} , r_{zx} .

If instead of $\mu' = 0$ we take $\nu = 0$ we get similar results, i.e. in general the vanishing μ , ν' , μ' or μ , ν' , ν involves that of λ , ν , λ' or λ , μ' , λ' .

This accounts for six more cases.

There are eight left. Of these six are typified by

$$\mu' = \nu' = 0 = \nu \quad \text{or} \quad \mu' = \nu' = 0 = \mu,$$

and lead to the same conclusions.

The remaining two are

$$\lambda = \mu = \nu = 0 \quad \text{or} \quad \lambda' = \mu' = \nu' = 0.$$

The first supposition, $\lambda = \mu = \nu = 0$ gives

$$\begin{aligned} P &= -\gamma_1'^2 \nu'^2, & Q &= -\gamma_2'^2 \lambda'^2, & R &= -\gamma_3'^2 \mu'^2, \\ P' &= \lambda'^2, & Q' &= \mu'^2, & R' &= \nu'^2, \end{aligned}$$

leading to

$$(79) \quad (\nu'^2 + \gamma_3'^2 \mu'^2)^2 r_{yx}^2 = 4r_{xz}\mu'^2\gamma_3' \{r_{xy}\nu'^2 - \gamma_3\gamma_3'\mu'^2\},$$

$$(80) \quad (\lambda'^2 + \gamma_1'^2 \nu'^2)^2 r_{yz}^2 = 4r_{xy}\nu'^2\gamma_1' \{r_{yz}\lambda'^2 - \gamma_1\gamma_1'\nu'^2\}$$

$$(81) \quad (\mu'^2 + \gamma_2'^2 \lambda'^2)^2 r_{zx}^2 = 4r_{yz}\lambda'^2\gamma_2' \{r_{xz}\mu'^2 - \gamma_2\gamma_2'\lambda'^2\},$$

which give $\lambda' = \mu' = \nu' = 0$ or a very special condition to be satisfied by the correlation coefficients r_{xy} , r_{yz} , r_{zx} .

We may conclude then that in general the *linearity of any three of the six regression lines involves that of the remaining three.*

(iv) If the regression surface of z on x, y reduces to a plane, the regression curves of x on y and y on x reduce to straight lines.

We have as in § 7

$$\frac{\bar{z}_y}{\sigma_z} = d + a \frac{\bar{x}_y}{\sigma_x} + \frac{by}{\sigma_y} + c \frac{\bar{x}_y y}{\sigma_x \sigma_y} \dots \dots \dots (58).$$

But
$$\frac{\bar{x}_y}{\sigma_x} = r_{xy} \frac{y}{\sigma_y} \pm \sqrt{\frac{y\eta_x^2 - r_{xy}^2}{\beta_2 - \beta_1 - 1}} \left\{ \frac{y^2}{\sigma_y^2} - \sqrt{\beta_1} \frac{y}{\sigma_y} - 1 \right\},$$

$$\begin{aligned} \therefore \frac{\bar{z}_y}{\sigma_z} &= d \pm a \sqrt{\frac{y\eta_x^2 - r_{xy}^2}{\beta_2 - \beta_1 - 1}} + \frac{y}{\sigma_y} \left\{ b + ar_{xy} \mp (a\sqrt{\beta_1} + c) \sqrt{\frac{y\eta_x^2 - r_{xy}^2}{\beta_2 - \beta_1 - 1}} \right\} \\ &+ \frac{y^2}{\sigma_y^2} \left\{ \left(cr_{xy} \mp c \sqrt{\frac{\beta_1(y\eta_x^2 - r_{xy}^2)}{\beta_2 - \beta_1 - 1}} \right) \pm a \sqrt{\frac{y\eta_x^2 - r_{xy}^2}{\beta_2 - \beta_1 - 1}} \right\} \\ &+ \text{terms of higher order.} \end{aligned}$$

Now the regression of z on y for a constant x is linear. Therefore the coefficient of y^2 is zero. To first order terms we may put $\beta_1 = 0$ and $\beta_2 = 3$, and thus

$$cr_{xy} \pm a \sqrt{\frac{y\eta_x^2 - r_{xy}^2}{2}} = 0.$$

Similarly

$$cr_{xy} \pm b \sqrt{\frac{x\eta_y^2 - r_{xy}^2}{2}} = 0,$$

But c vanishes when $_{xy}H_z^2 = _{xy}R_z^2$ by (52).

Hence if

$$_{xy}H_z^2 = _{xy}R_z^2,$$

it follows that

$$_x\eta_y = _y\eta_x = r_{xy}.$$

We thus see that if the three generalised correlation ratios $_{xy}H_z$, $_{yz}H_x$, $_{zx}H_y$ are equal to $_{xy}R_z$, $_{yz}R_x$, $_{zx}R_y$ respectively, the six correlation ratios $_x\eta_y$, $_y\eta_x$, $_x\eta_z$, $_z\eta_x$, $_z\eta_y$, $_y\eta_z$ reduce to the corresponding correlation coefficients r_{xy} , r_{zx} , r_{yz} and that the "linearity" of the three regression surfaces involves the linearity of the six regression lines.